# On the Dual of a Mixed $H_2/l_1$ Optimisation Problem

Jun Wu[†], Sheng Chen[‡] and Jian Chu[†]

[†] National Key Laboratory of Industrial Control Technology
Institute of Advanced Process Control
Zhejiang University
Hangzhou, 310027, P. R. China

[‡] School of Electronics and Computer Science
University of Southampton
Highfield, Southampton SO17 1BJ, U.K.
E-mail: sqc@ecs.soton.ac.uk

ABSTRACT

The general discrete-time single-input single-output (SISO) mixed $H_2/l_1$ control problem is considered in this paper. It is found that the existing results of duality theory cannot directly be applied to this infinite dimensional optimisation problem. By means of two finite dimensional approximate problems to which the duality theory can be applied, the dual of this mixed $H_2/l_1$ control problem is verified to be the limit of the duals of these two approximate problems.

**Keywords**: Optimal control, mixed $H_2/l_1$ control, duality theory.

## I. INTRODUCTION

Most controller synthesis problems can be formulated as follows: Given a plant $\hat{P}$, design a controller $\hat{C}$ such that the closed-loop system is stable and satisfies some given (optimal) performance criteria. When the optimal performance criterion is the $H_\infty$-norm ($H_2$-norm or $l_1$-norm respectively) of the closed-loop transfer function based on the Youla parameterisation [3] — a parameterisation of the class of controllers which stabilise the plant, the controller synthesis problem can be changed into the $H_\infty$-norm ($H_2$-norm or $l_1$-norm respectively) model matching problem — the problem of finding the optimal stable free parameter which minimises the $H_\infty$-norm ($H_2$-norm or $l_1$-norm respectively) of a map of the free parameter. Consequently, $H_\infty$ control design [16], $H_2$ control design [15] and $l_1$ control design [10] have been introduced respectively. Mixed performance controls, such as mixed $H_2/H_\infty$ control [4], mixed $l_1/H_\infty$ control [9], mixed $l_1/H_2$ control [6] and mixed $H_2/l_1$ control [11], have been the pole of attraction for many researchers lately. Mixed performance control can directly accommodate realistic situations where a system must satisfy several different performance constraints. Based on the Youla parameterisation, a mixed performance control problem can be changed into a special optimisation problem with two kinds of norms.

The topic of this paper is mixed $H_2/l_1$ control. A discrete-time SISO mixed $H_2/l_1$ control problem was addressed by Voulgaris [11], [12] through minimising the $H_2$-norm of the closed-loop map while maintaining its $l_1$-norm at a prescribed level. Based on the duality theory, a finite step method was presented to solve for exactly this mixed $H_2/l_1$ optimisation problem. A more general class of discrete-time SISO mixed $H_2/l_1$ control problems, in which Voulgaris' problem [11], [12] is a special case, was addressed in [13]. This class of problems consider minimising the $H_2$-norm of the closed-loop map while maintaining the $l_1$-norm of another closed-loop map at a prescribed level. A methodology using finite-dimensional quadratic programming was presented in [14] to obtain converging lower and upper bounds to this class of mixed $H_2/l_1$ control problems. This methodology was also developed to approximately solve for a multi-input multi-output (MIMO) square mixed $H_2/l_1$ control problem [8]. It was shown in [8] that this methodology without zero interpolation avoids some problems presented in methods which employ zero interpolation techniques. Using zero interpolation techniques, a definition of the MIMO multi-block mixed $H_2/l_1$ control problem was given by Elia & Dahleh [2], and the dual of this MIMO multi-block mixed $H_2/l_1$ control problem was derived in [2]. An upper approximation methodology was also presented in [7] for the MIMO multi-block mixed $H_2/l_1$ control problem in which the related approximate problem is to minimize a positive linear combination of the squared $H_2$ norms and the $l_1$ norms over all the stabilizing controllers.

A mixed $H_2/l_1$ control problem is intrinsically an infinite dimensional quadratic programming problem. For various minimisation problems, such as linear programming, quadratic programming, convex programming and approximation theory, one comes across a remarkable phenomenon, which is very useful in concrete applications. There exists an associated maximisation problem called dual, involving a different variable, which attains the same optimal value as the original problem called primal. Moreover, the value of the variable for which the maximum is attained in the dual problem can be interpreted as the so-called "shadow price" [1]. Abstractly speaking, there is a duality correspondence between the primal problem in the vector space and a dual problem in the normed dual of the constraint space [1], [5]. The constraint space is the space where the image of the constraint operator lies. An obvious observation on duality principle is that the minimum distance from a point to a convex set is equal to the maximum of the distances from the point to the hyperplanes separating the point and the convex set. This observation is of course completely trivial. However, it turns out to be surprisingly useful in concrete applications where the primal problem has nonlinear constraints or infinite dimension while the dual problem has linear constraints or finite dimension. This often leads to a simpler indirect method of solving the primal problem, in which the dual problem is solved for first.

The duality relationship between a mixed $H_2/l_1$ control problem and its dual uncovers finite-dimensional structures in the optimal solution, when such finite-dimensional structures exist, and provides finite-dimensional approximations when the problem does not appear to have a finite-dimensional structure [12], [2]. Consequently, the duality theory plays an important role in the research on $H_2/l_1$ problems. Elia & Dahleh [2] developed the dual of the MIMO multi-block mixed $H_2/l_1$ control problem based on zero interpolation techniques. Nevertheless, for a mixed $H_2/l_1$ control problem defined by zero interpolation techniques, the task of determining the optimal controller still remains, even if the optimal closed-loop map has been determined [8]. The closed-loop map needs to satisfy the zero interpolation conditions exactly to guarantee that the correct cancellations take place while solving for the controller. Therefore, an extremely high accurate closed-loop map is required in order to determine correct pole and zero cancellation. However, numerical errors are always present which may result in a controller structure different from the optimal one. All of these difficulties motivates us to develop the dual of a mixed $H_2/l_1$ control problem without zero interpolation.

The general discrete-time SISO mixed $H_2/l_1$ control problem is addressed in this paper, which minimises the $H_2$-norm of the closed-loop map while maintaining the $l_1$-norm of another closed-loop map at a prescribed level. It is found that the existing results of duality theory cannot directly be applied to this infinite dimensional optimisation problem. Two finite dimensional approximate problems are constructed to which the duality theory can be applied. These two approximate optimisation problems converge to the original infinite dimensional optimisation problem from lower and upper sides, respectively. The dual of the general discrete-time SISO mixed $H_2/l_1$ control problem is then shown to be the limit of the duals of the two constructed approximate problems.

## II. Notation and mathematical preliminaries

Let $R$ denote the field of real numbers, $R^m$ denote the space of $m$-dimensional real vectors, and $Z_+$ the nonnegative integers. A causal SISO linear-time-invariant (LTI) transfer function $\hat{G}$ can be described as

$$\hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots, \quad G(k) \in R,\ \forall k \in Z_+. \tag{1}$$

As $\hat{G}$ can be represented uniquely by its impulse response sequence $[G(0) \quad G(1)\ G(2) \quad \cdots \quad ]^T$, $\hat{G}$ and its impulse response sequence are not differentiated in notation throughout this paper. Define

$$l_e = \left\{ \hat{G} \,\middle|\, \hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots, \right. \left. G(k) \in R,\ \forall k \in Z_+ \right\}, \tag{2}$$

$$l_\infty = \left\{ \hat{G} \in l_e \,\middle|\, \sup_k |G(k)| < \infty \right\}, \tag{3}$$

$$l_2 = \left\{ \hat{G} \in l_e \,\middle|\, \sum_{k=0}^{\infty} (G(k))^2 < \infty \right\}, \tag{4}$$

and

$$l_1 = \left\{ \hat{G} \in l_e \left| \sum_{k=0}^{\infty} |G(k)| < \infty \right. \right\}. \tag{5}$$

For any $\hat{G} \in l_\infty$, the $l_\infty$-norm of $\hat{G}$ is given by

$$\|\hat{G}\|_\infty = \sup_k |G(k)| . \tag{6}$$

For any $\hat{G} \in l_2$, the $l_2$-norm of $\hat{G}$ is

$$\|\hat{G}\|_2 = \sqrt{\sum_{k=0}^{\infty} (G(k))^2}. \tag{7}$$

$\|\hat{G}\|_2$ is also the $H_2$-norm of $\hat{G}$. For any $\hat{G} \in l_1$, the $l_1$-norm of $\hat{G}$ is

$$\|\hat{G}\|_1 = \sum_{k=0}^{\infty} |G(k)| . \tag{8}$$

It is easily seen that $l_1 \subset l_2 \subset l_\infty$, and that $\forall \hat{G}_1,\ \hat{G}_2 \in l_1$, $\hat{G}_1\hat{G}_2 \in l_1$.

Let $X$ be a normed linear space. The space of all bounded linear functionals on $X$ is called the normed dual of $X$ and is denoted $X^*$. For any $x \in X$ and $x^* \in X^*$, $< x, x^* >$ denotes the value of the bounded linear functional $x^*$ at the point $x$. The norm of an element $x^* \in X^*$ is

$$\|x^*\| = \sup_{\|x\| \leq 1} < x, x^* > . \tag{9}$$

From standard functional analysis results [5], we have $(R^m)^* = R^m$, $(l_1)^* = l_\infty$ and $(l_2)^* = l_2$. For any $x \in R^{N+1}$ and $x^* \in R^{N+1}$,

$$< x, x^* >= \sum_{k=0}^{N} (x(k)x^*(k)) . \tag{10}$$

For any $x \in l_1$ and $x^* \in l_\infty$ (or for any $x \in l_2$ and $x^* \in l_2$),

$$< x, x^* >= \sum_{k=0}^{\infty} (x(k)x^*(k)) . \tag{11}$$

Given a convex cone $P$ in $X$, it is possible to define an ordering relation on $X$ as follows: $x_1 \geq x_2$ if and only if $x_1 - x_2 \in P$. The cone $P$ defining this relation on $X$ is called positive. Then it is natural to define a positive cone $P^\oplus$ inside $X^*$ in the following way: $P^\oplus = \{x^* \in X^* \,|< x, x^* >\geq 0,\ \forall x \in P\}$, this in turn defines an ordering relation on $X^*$. For any vector space in this paper, the positive cone which defines an ordering relation is the set consisting of elements with nonnegative point-wise components. Let $W$ be a vector space with positive cone. A mapping $F : X \to W$ is convex if $F(tx_1 + (1-t)x_2) \leq tF(x_1) + (1-t)F(x_2)$ for all $x_1$, $x_2$ in $X$ and $0 \leq t \leq 1$. The following result of duality theory can be found in [5].

*Lemma 1:* Let $f$ be a real-valued convex functional defined on a convex subset $\Omega$ of a vector space $X$, $G$ be a convex mapping of $X$ into a normed space $W$, and $H(x) = Ax - b$ is a map of $X$ into the finite dimensional normed space $Y$. Suppose that $A$ is linear, $0 \in Y$ is an interior point of $\{y \in Y \,| H(x) = y \text{ for some } x \in \Omega\}$ and there exists an $x_1 \in \Omega$ such that $G(x_1) < 0$ (i.e. $G(x_1)$ is an interior point of the positive cone of $W$) and $H(x_1) = 0$. Define the minimisation problem:

$$\mu = \inf_{\substack{x \in \Omega \\ G(x) \leq 0 \\ H(x) = 0}} f(x). \tag{12}$$

Assume that $\mu$ is finite. Then the dual problem is

$$\mu = \max_{\substack{w^* \in W^* \\ w^* \geq 0 \\ y^* \in Y^*}} \inf_{x \in \Omega} \left[ f(x) + < G(x), w^* > + < H(x), y^* > \right] . \tag{13}$$

## III. Mixed $H_2/l_1$ optimisation problem

The general discrete-time SISO mixed $H_2/l_1$ optimisation problem [14] can be stated as: Given $\hat{T}_1 \in l_1$, $\hat{T}_2 \in l_2$, $\hat{V}_1 = [V_1(0) \cdots V_1(m-1)\ 1]^T \in R^{m+1}$, $\hat{V}_2 = [V_2(0) \cdots V_2(n-1)\ 1]^T \in R^{n+1}$, and a constant $\gamma$, find $\hat{Q} \in l_1$ such that $\|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2$ is minimised and $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leq \gamma$.

This description of $H_2/l_1$ problem is without zero interpolation conditions. Once this $H_2/l_1$ problem has been solved for, the optimal controller can be determined directly from the optimal $\hat{Q}$. It should be pointed out that the notation $\hat{Q}$ does not appear in any $H_2/l_1$ problem defined with zero interpolation techniques, and there may exist some problems in determining the optimal controller by a $H_2/l_1$ problem defined with zero interpolation techniques, as mentioned in the introduction section.

For the above mixed $H_2/l_1$ optimisation problem, in order to make its feasible region nonempty, it is assumed that

$$\gamma > \inf_{\hat{Q} \in l_1} \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1. \tag{14}$$

In addition, we also assume that all the poles of $\hat{V}_1$ are inside the open unit disk in complex plane, i.e. for

$$\prod_{i=1}^{m} (\lambda - \lambda_i) = \lambda^m + V_1(m-1)\lambda^{m-1} + \cdots + V_1(0), \tag{15}$$

$|\lambda_i| < 1$, $\forall i \in \{1, \cdots, m\}$. Under these assumptions, we have the following lemma [14].

*Lemma 2:* Suppose that $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \le \gamma$. Then $\|\hat{Q}\|_1 \le L$, where

$$L = \frac{\|\hat{T}_1\|_1 + \gamma}{\prod_{i=1}^{m}(1 - |\lambda_i|)}. \tag{16}$$

Define $\xi = \left\{\hat{Q} \in l_1 \middle| \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \le \gamma,\ \|\hat{Q}\|_1 \le L\right\}$. From lemma 2, it is easy to see another description of the mixed $H_2/l_1$ optimisation problem:

$$\mu = \inf_{\hat{Q}\in\xi} \|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2^2. \tag{17}$$

Notice the fact that $\|\hat{Q}\|_1 \le L$ is equal to the linear constraints

$$\begin{array}{l} \hat{Q} = \hat{Q}_+ - \hat{Q}_-, \\ \hat{Q}_+ \ge 0, \quad \hat{Q}_- \ge 0, \\ \hat{Q}_+ + \hat{Q}_- \le L. \end{array} \tag{18}$$

Therefore, by additionally setting

$$\begin{array}{l} \hat{T}_1 - \hat{Q}\hat{V}_1 = \hat{\Psi} = \hat{\Psi}_+ - \hat{\Psi}_-, \\ \hat{\Psi}_+ \ge 0, \quad \hat{\Psi}_- \ge 0, \\ \hat{T}_2 - \hat{Q}\hat{V}_2 = \hat{\Phi}, \end{array} \tag{19}$$

the problem (17) can easily be transformed into

$$\begin{aligned} \mu \ &= \ \inf \|\hat{\Phi}\|_2^2. \\ \text{s.t. } & \hat{\Phi} = \hat{T}_2 - \hat{Q}_+\hat{V}_2 + \hat{Q}_-\hat{V}_2 \\ & \hat{\Psi}_+ - \hat{\Psi}_- = \hat{T}_1 - \hat{Q}_+\hat{V}_1 + \hat{Q}_-\hat{V}_1 \\ & \hat{\Psi}_+ + \hat{\Psi}_- \le \gamma, \quad \hat{Q}_+ + \hat{Q}_- \le L \\ & \hat{\Phi} \in l_2,\ 0 \le \hat{\Psi}_+ \in l_1,\ 0 \le \hat{\Psi}_- \in l_1 \\ & 0 \le \hat{Q}_+ \in l_1,\ 0 \le \hat{Q}_- \in l_1 \end{aligned} \tag{20}$$

To obtain the dual of the problem (17), we rewrite (20) in the form of (12). Let $X = l_2 \times l_1 \times l_1 \times l_1 \times l_1$, $Y = l_1 \times l_2$, $W = R^2$,

$$\Omega = \left\{ x = \begin{bmatrix} \hat{\Phi} \\ \hat{\Psi}_+ \\ \hat{\Psi}_- \\ \hat{Q}_+ \\ \hat{Q}_- \end{bmatrix} \middle| \begin{array}{c} \hat{\Phi} \in l_2 \\ 0 \le \hat{\Psi}_+ \in l_1 \\ 0 \le \hat{\Psi}_- \in l_1 \\ 0 \le \hat{Q}_+ \in l_1 \\ 0 \le \hat{Q}_- \in l_1 \end{array} \right\}, \tag{21}$$

$$f(x) = x^T \begin{bmatrix} I_\infty & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x, \tag{22}$$

$$G(x) = \begin{bmatrix} 0 & E_\infty^T & E_\infty^T & 0 & 0 \\ 0 & 0 & 0 & E_\infty^T & E_\infty^T \end{bmatrix} x - \begin{bmatrix} \gamma \\ L \end{bmatrix}, \tag{23}$$

$$\begin{aligned} H(x) &= \begin{bmatrix} 0 & I_\infty & -I_\infty & V_{1,\infty} & -V_{1,\infty} \\ I_\infty & 0 & 0 & V_{2,\infty} & -V_{2,\infty} \end{bmatrix} x - \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \end{bmatrix} \\ &= Ax - b, \end{aligned} \tag{24}$$

where

$$I_\infty = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix}, \tag{25}$$

$$E_\infty = [1 \quad 1 \quad \cdots]^T, \tag{26}$$

$$V_{1,\infty} = \begin{bmatrix} V_1(0) & & & 0 \\ \vdots & V_1(0) & & \\ V_1(m-1) & \vdots & \ddots & \\ 1 & V_1(m-1) & \ddots & \ddots \\ & 1 & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}, \tag{27}$$

$$V_{2,\infty} = \begin{bmatrix} V_2(0) & & & 0 \\ \vdots & V_2(0) & & \\ V_2(n-1) & \vdots & \ddots & \\ 1 & V_2(n-1) & \ddots & \ddots \\ & 1 & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}. \tag{28}$$

With these definitions, the mixed $H_2/l_1$ optimisation problem (17) becomes

$$\mu = \inf_{\substack{x\in\Omega \\ G(x)\le 0 \\ H(x)=0}} f(x) \tag{29}$$

which has the same form as (12). However, lemma 1 cannot be applied to (17). This is because here $Y = l_1 \times l_2$ is infinite dimensional which does not satisfy the conditions of lemma 1. In this paper, this difficulty in setting up the dual of the problem (17) is overcome by first considering some approximations of (17).

## IV. Two approximate problems and their duals

$\forall N \in Z_+$, define

$$\xi_{+N} = \left\{\hat{Q} \in R^{N+1} \middle| \hat{Q} \in \xi\right\}. \tag{30}$$

The variable $N$-truncation problem of (17) is constructed as

$$\mu_{+N} = \inf_{\hat{Q}\in\xi_{+N}} \|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2^2. \tag{31}$$

The mixed $H_2/l_1$ problem (17) can be approximated from upper side by the variable $N$-truncation problem (31), as stated in the following lemma [14].

*Lemma 3:* $\mu_{+0} \geq \mu_{+1} \geq \mu_{+2} \geq \cdots$ and $\lim_{N\to\infty} \mu_{+N} = \mu$.

$\forall N \in Z_+$, define the $N$-th truncation operator $\Gamma_N : l_e \to R^{N+1}$ as

$$\Gamma_N \hat{G} = G(0) + G(1)\lambda + \cdots + G(N)\lambda^N. \tag{32}$$

Similar to the processing for the problem (17) in section III, define $X = R^{5N+2m+n+5}$, $Y = R^{N+m+1} \times R^{N+n+1}$, $W = R^2$,

$$\Omega = \left\{ x = \begin{bmatrix} \hat{\Phi} \\ \hat{\Psi}_+ \\ \hat{\Psi}_- \\ \hat{Q}_+ \\ \hat{Q}_- \end{bmatrix} \middle| \begin{matrix} \hat{\Phi} \in R^{N+n+1} \\ 0 \leq \hat{\Psi}_+ \in R^{N+m+1} \\ 0 \leq \hat{\Psi}_- \in R^{N+m+1} \\ 0 \leq \hat{Q}_+ \in R^{N+1} \\ 0 \leq \hat{Q}_- \in R^{N+1} \end{matrix} \right\}, \tag{33}$$

$$f(x) = x^T \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \alpha(N), \tag{34}$$

$$G(x) = \begin{bmatrix} 0 & E_{N+m+1}^T & E_{N+m+1}^T & 0 & 0 \\ 0 & 0 & 0 & E_{N+1}^T & E_{N+1}^T \end{bmatrix} x - \begin{bmatrix} \gamma - \beta(N) \\ L \end{bmatrix} \tag{35}$$

and

$$H(x) = \begin{bmatrix} 0 & I & -I & V_{1,N+m+1} & -V_{1,N+m+1} \\ I & 0 & 0 & V_{2,N+n+1} & -V_{2,N+n+1} \end{bmatrix} x - \begin{bmatrix} \Gamma_{N+m}\hat{T}_1 \\ \Gamma_{N+n}\hat{T}_2 \end{bmatrix} = Ax - b, \tag{36}$$

where $I$ denotes the finite identity matrix with a proper dimension,

$$\alpha(N) = < \hat{T}_2 - \Gamma_{N+n}\hat{T}_2, \hat{T}_2 - \Gamma_{N+n}\hat{T}_2 >, \tag{37}$$

$$E_{N+m+1} = [1 \ \cdots \ 1]^T \in R^{N+m+1}, \tag{38}$$

$$\beta(N) = \|\hat{T}_1 - \Gamma_{N+m}\hat{T}_1\|_1, \tag{39}$$

$$V_{1,N+m+1} = \begin{bmatrix} V_1(0) & & 0 \\ \vdots & \ddots & \\ V_1(m-1) & \ddots & V_1(0) \\ 1 & \ddots & \vdots \\ & \ddots & V_1(m-1) \\ 0 & & 1 \end{bmatrix} \in R^{(N+m+1)\times(N+1)} \tag{40}$$

and

$$V_{2,N+n+1} = \begin{bmatrix} V_2(0) & & 0 \\ \vdots & \ddots & \\ V_2(n-1) & \ddots & V_2(0) \\ 1 & \ddots & \vdots \\ & \ddots & V_2(n-1) \\ 0 & & 1 \end{bmatrix} \in R^{(N+n+1)\times(N+1)}. \tag{41}$$

With these definitions, (31) can be expressed in the form of (12). It is easy to see that $f(x)$ so constructed is a convex functional, $\Omega$ is a convex subset, $G(x)$ is a convex map, $Y$ is finite dimensional, $A$ is linear, $\mu_{+N}$ is finite, $0 \in Y$ is an interior point of $\{y \in Y \,|H(x) = y \text{ for some } x \in \Omega\}$, there exists $x_1 \in \Omega$ with $G(x_1) < 0$ and $H(x_1) = 0$. Hence, lemma 1 can directly be applied to derive the dual of the optimisation problem (31):

$$\begin{aligned} \mu_{+N} = & \max_{\substack{y_1^* \in R^{N+m+1} \\ y_2^* \in R^{N+n+1} \\ 0 \leq w_1^* \in R \\ 0 \leq w_2^* \in R}} \inf_{x\in\Omega} \Big[ < \hat{\Phi}, \hat{\Phi} > + \alpha(N) \\ & + < \hat{\Psi}_+ - \hat{\Psi}_- - \Gamma_{N+m}\hat{T}_1 \\ & \quad + V_{1,N+m+1}(\hat{Q}_+ - \hat{Q}_-), y_1^* > \\ & + < \hat{\Phi} - \Gamma_{N+n}\hat{T}_2 \\ & \quad + V_{2,N+n+1}(\hat{Q}_+ - \hat{Q}_-), y_2^* > \\ & + < E_{N+m+1}^T(\hat{\Psi}_+ + \hat{\Psi}_-) - \gamma + \beta(N), w_1^* > \\ & + < E_{N+1}^T(\hat{Q}_+ + \hat{Q}_-) - L, w_2^* > \Big] \\ = & \max_{\substack{y_1^* \in R^{N+m+1} \\ y_2^* \in R^{N+n+1} \\ 0 \leq w_1^* \in R \\ 0 \leq w_2^* \in R}} \inf_{x\in\Omega} \Big[ < \hat{\Phi}, \hat{\Phi} > + < \hat{\Phi}, y_2^* > \\ & + < \hat{\Psi}_+, E_{N+m+1}w_1^* + y_1^* > \\ & + < \hat{\Psi}_-, E_{N+m+1}w_1^* - y_1^* > \\ & + < \hat{Q}_+, E_{N+1}w_2^* + V_{1,N+m+1}^T y_1^* \\ & \quad + V_{2,N+n+1}^T y_2^* > \\ & + < \hat{Q}_-, E_{N+1}w_2^* - V_{1,N+m+1}^T y_1^* \\ & \quad - V_{2,N+n+1}^T y_2^* > \\ & + \alpha(N) - < \gamma - \beta(N), w_1^* > - < L, w_2^* > \\ & - < \Gamma_{N+m}\hat{T}_1, y_1^* > - < \Gamma_{N+n}\hat{T}_2, y_2^* > \Big]. \end{aligned} \tag{42}$$

In (42), we are sure that

$$E_{N+m+1}w_1^* + y_1^* \geq 0. \tag{43}$$

The reason is that, if $E_{N+m+1}w_1^* + y_1^* < 0$, $\hat{\Psi}_+$ can be chosen as such a large positive number that $\mu_{+N} < 0$, which contradicts the fact $\mu_{+N} \geq 0$. Similarly,

$$E_{N+m+1}w_1^* - y_1^* \geq 0, \tag{44}$$

$$E_{N+1}w_2^* + V_{1,N+m+1}^T y_1^* + V_{2,N+n+1}^T y_2^* \geq 0, \tag{45}$$

$$E_{N+1}w_2^* - V_{1,N+m+1}^T y_1^* - V_{2,N+n+1}^T y_2^* \geq 0. \tag{46}$$

Denote

$$\begin{aligned} g(x) &= <\hat{\Phi},\hat{\Phi}> + <\hat{\Phi}, y_2^*> \\ &\quad + <\hat{\Psi}_+, E_{N+m+1}w_1^* + y_1^*> \\ &\quad + <\hat{\Psi}_-, E_{N+m+1}w_1^* - y_1^*> \\ &\quad + <\hat{Q}_+, E_{N+1}w_2^* + V_{1,N+m+1}^T y_1^* \\ &\qquad + V_{2,N+n+1}^T y_2^*> \\ &\quad + <\hat{Q}_-, E_{N+1}w_2^* - V_{1,N+m+1}^T y_1^* \\ &\qquad - V_{2,N+n+1}^T y_2^*> \\ &\quad +\alpha(N) - <\gamma - \beta(N), w_1^*> - <L, w_2^*> \\ &\quad - <\Gamma_{N+m}\hat{T}_1, y_1^*> \\ &\quad - <\Gamma_{N+n}\hat{T}_2, y_2^*>. \end{aligned} \tag{47}$$

Obviously, under the conditions (43)–(46), we have $\hat{\Psi}_+ = 0$, $\hat{\Psi}_- = 0$, $\hat{Q}_+ = 0$ and $\hat{Q}_- = 0$ when $\inf_{x\in\Omega} g(x)$ is achieved. Thus

$$\begin{aligned} \inf_{x\in\Omega} g(x) &= \inf_{\hat{\Phi}\in l_2} \Big[ <\hat{\Phi},\hat{\Phi}> + <\hat{\Phi}, y_2^*> + \alpha(N) \\ &\quad - <\gamma-\beta(N), w_1^*> - <L, w_2^*> \\ &\quad - <\Gamma_{N+m}\hat{T}_1, y_1^*> - <\Gamma_{N+n}\hat{T}_2, y_2^*> \Big] \\ &= \inf_{\hat{\Phi}\in l_2} \Big[ <\hat{\Phi} + \frac{y_2^*}{2}, \hat{\Phi} + \frac{y_2^*}{2}> - <\frac{y_2^*}{2}, \frac{y_2^*}{2}> \\ &\quad - <\gamma-\beta(N), w_1^*> + \alpha(N) - <L, w_2^*> \\ &\quad - <\Gamma_{N+m}\hat{T}_1, y_1^*> - <\Gamma_{N+n}\hat{T}_2, y_2^*> \Big] \\ &= -\frac{1}{4} <y_2^*, y_2^*> + \alpha(N) \\ &\quad - <\gamma-\beta(N), w_1^*> - <L, w_2^*> \\ &\quad - <\Gamma_{N+m}\hat{T}_1, y_1^*> \\ &\quad - <\Gamma_{N+n}\hat{T}_2, y_2^*>. \end{aligned} \tag{48}$$

Consequently, the dual of the variable $N$-truncation problem (31) is

$$\begin{aligned} \mu_{+N} &= \max \Big[ -\frac{1}{4} <y_2^*, y_2^*> - <\Gamma_{N+m}\hat{T}_1, y_1^*> \\ &\quad - <\gamma-\beta(N), w_1^*> - <L, w_2^*> \\ &\quad + \alpha(N) - <\Gamma_{N+n}\hat{T}_2, y_2^*> \Big]. \\ \text{s.t. } & -E_{N+m+1}w_1^* \le y_1^* \le E_{N+m+1}w_1^* \\ & -E_{N+1}w_2^* \le \big(V_{1,N+m+1}^T y_1^* \\ & + V_{2,N+n+1}^T y_2^*\big) \le E_{N+1}w_2^* \\ & y_1^* \in R^{N+m+1},\ y_2^* \in R^{N+n+1} \\ & 0 \le w_1^* \in R,\ 0 \le w_2^* \in R \end{aligned} \tag{49}$$

$\forall N \in Z_+$, define

$$\xi_{-N} = \left\{ \hat{Q} \in l_1 \;\middle|\; \|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 \le \gamma,\ \|\hat{Q}\|_1 \le L \right\}. \tag{50}$$

Obviously,

$$\xi_{-0} \supset \xi_{-1} \supset \xi_{-2} \supset \cdots \supset \xi. \tag{51}$$

The constraint $N$-truncation problem of (17) is constructed as

$$\mu_{-N} = \inf_{\hat{Q}\in\xi_{-N}} \|\Gamma_N(\hat{T}_2 - \hat{Q}\hat{V}_2)\|_2^2. \tag{52}$$

The mixed $H_2/l_1$ optimisation problem (17) can be approximated from lower side by this constraint $N$-truncation problem (52), as summarized in the following lemma [14].

*Lemma 4:* $\mu_{-0} \le \mu_{-1} \le \mu_{-2} \le \cdots$ and $\lim_{N\to\infty} \mu_{-N} = \mu$.

Using the same method for constructing the dual of the variable $N$-truncation problem, the dual of the constraint $N$-truncation problem (52) is obtained as:

$$\begin{aligned} \mu_{-N} = \quad & \max \Big[ -\frac{1}{4} <y_2^*, y_2^*> - <\gamma, w_1^*> \\ & - <L, w_2^*> - <\Gamma_N\hat{T}_1, y_1^*> \\ & - <\Gamma_N\hat{T}_2, y_2^*> \Big]. \\ \text{s.t. } & -E_{N+1}w_1^* \le y_1^* \le E_{N+1}w_1^* \\ & -E_{N+1}w_2^* \le \big(U_{1,N+1}^T y_1^* \\ & + U_{2,N+1}^T y_2^*\big) \le E_{N+1}w_2^* \\ & y_1^* \in R^{N+1},\ y_2^* \in R^{N+1} \\ & 0 \le w_1^* \in R,\ 0 \le w_2^* \in R \end{aligned} \tag{53}$$

Here

$$U_{1,N+1} = \begin{bmatrix} V_1(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ V_1(N) & \cdots & V_1(0) \end{bmatrix} \in R^{(N+1)\times(N+1)} \tag{54}$$

and

$$U_{2,N+1} = \begin{bmatrix} V_2(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ V_2(N) & \cdots & V_2(0) \end{bmatrix} \in R^{(N+1)\times(N+1)}. \tag{55}$$

## V. The dual of mixed $H_2/l_1$ problem

Define

$$D = \left\{ \omega = \begin{bmatrix} y_1^* \\ y_2^* \\ w_1^* \\ w_2^* \end{bmatrix} \;\middle|\; \begin{array}{c} y_1^* \in l_\infty,\ y_2^* \in l_2 \\ 0 \le w_1^* \in R,\ 0 \le w_2^* \in R \\ -E_\infty w_1^* \le y_1^* \le E_\infty w_1^* \\ -E_\infty w_2^* \le \big(V_{1,\infty}^T y_1^* \\ + V_{2,\infty}^T y_2^*\big) \le E_\infty w_2^* \end{array} \right\} \tag{56}$$

and

$$\begin{aligned} \varphi(\omega) &= -\frac{1}{4} <y_2^*, y_2^*> - <\gamma, w_1^*> - <L, w_2^*> \\ &\quad - <\hat{T}_1, y_1^*> - <\hat{T}_2, y_2^*>. \end{aligned} \tag{57}$$

Construct an infinite dimensional optimisation problem as:

$$\upsilon = \sup_{\omega \in D} \varphi(\omega). \tag{58}$$

$\forall N \in Z_+$, define $D_{+N}$ as

$$D_{+N} = \left\{ \omega \left| \begin{array}{c} y_1^* \in l_\infty,\ y_2^* \in l_2 \\ 0 \le w_1^* \in R,\ 0 \le w_2^* \in R \\ -E_\infty w_1^* \le y_1^* \le E_\infty w_1^* \\ -\Gamma_N \left(E_\infty w_2^*\right) \le \Gamma_N \left(V_{1,\infty}^T y_1^* \right. \\ \left. +V_{2,\infty}^T y_2^*\right) \le \Gamma_N \left(E_\infty w_2^*\right) \end{array} \right. \right\}. \tag{59}$$

The constraint $N$-truncation problem of (58) can be constructed as

$$\upsilon_{+N} = \sup_{\omega \in D_{+N}} \varphi(\omega). \tag{60}$$

The following proposition is a direct consequence of $D_{+N} \supset D$.

*Proposition 1:* For the optimisation problems (58) and (60), $\upsilon_{+N} \ge \upsilon$.

However,

$$\begin{aligned} &\sup_{\omega \in D_{+N}} \varphi(\omega) = \sup_{\omega \in D_{+N}} \left[ -\tfrac{1}{4} < \Gamma_{N+n} y_2^*, \Gamma_{N+n} y_2^* > \right. \\ &- < \gamma, w_1^* > -\tfrac{1}{4} < y_2^* - \Gamma_{N+n} y_2^*, y_2^* - \Gamma_{N+n} y_2^* > \\ &- < L, w_2^* > - < \Gamma_{N+m} \hat{T}_1, \Gamma_{N+m} y_1^* > \\ &- < \hat{T}_1 - \Gamma_{N+m} \hat{T}_1, y_1^* - \Gamma_{N+m} y_1^* > \\ &- < \Gamma_{N+n} \hat{T}_2, \Gamma_{N+n} y_2^* > \\ &\left. - < \hat{T}_2 - \Gamma_{N+n} \hat{T}_2, y_2^* - \Gamma_{N+n} y_2^* > \right] \\ &= \sup_{\omega \in D_{+N}} \left[ -\tfrac{1}{4} < \Gamma_{N+n} y_2^*, \Gamma_{N+n} y_2^* > \right. \\ &- < \gamma, w_1^* > - < L, w_2^* > \\ &- < (y_2^* - \Gamma_{N+n} y_2^*)/2 + (\hat{T}_2 - \Gamma_{N+n} \hat{T}_2), \\ &\quad (y_2^* - \Gamma_{N+n} y_2^*)/2 + (\hat{T}_2 - \Gamma_{N+n} \hat{T}_2) > \\ &+ < \hat{T}_2 - \Gamma_{N+n} \hat{T}_2, \hat{T}_2 - \Gamma_{N+n} \hat{T}_2 > \\ &- < \Gamma_{N+m} \hat{T}_1, \Gamma_{N+m} y_1^* > \\ &+ w_1^* \| \hat{T}_1 - \Gamma_{N+m} \hat{T}_1 \|_1 \\ &\left. - < \Gamma_{N+n} \hat{T}_2, \Gamma_{N+n} y_2^* > \right] \\ &= \sup_{\omega \in D_{+N}} \left[ -\tfrac{1}{4} < \Gamma_{N+n} y_2^*, \Gamma_{N+n} y_2^* > + \alpha(N) \right. \\ &- < \gamma - \beta(N), w_1^* > - < L, w_2^* > \\ &- < \Gamma_{N+m} \hat{T}_1, \Gamma_{N+m} y_1^* > \\ &\left. - < \Gamma_{N+n} \hat{T}_2, \Gamma_{N+n} y_2^* > \right] \end{aligned} \tag{61}$$

and the constraint $\omega \in D_{+N}$ can be changed into

$$\begin{array}{l} -E_{N+m+1} w_1^* \le \Gamma_{N+m} y_1^* \le E_{N+m+1} w_1^*, \\ -E_{N+1} w_2^* \le \left\{ V_{1,N+m+1}^T (\Gamma_{N+m} y_1^*) \right. \\ \quad \left. + V_{2,N+n+1}^T (\Gamma_{N+n} y_2^*) \right\} \le E_{N+1} w_2^*, \\ \Gamma_{N+m} y_1^* \in R^{N+m+1},\ \Gamma_{N+n} y_2^* \in R^{N+n+1}, \\ 0 \le w_1^* \in R,\ 0 \le w_2^* \in R. \end{array} \tag{62}$$

This verifies that the optimisation problem (60) is exactly the optimisation problem (49), which leads to the following proposition.

*Proposition 2:* For the optimisation problems (49) and (60), $\mu_{+N} = \upsilon_{+N}$.

$\forall N \in Z_+$, define

$$D_{-N} = \left\{ \omega \left| \begin{array}{c} y_1^* \in R^{N+1},\ y_2^* \in R^{N+1} \\ 0 \le w_1^* \in R,\ 0 \le w_2^* \in R \\ -E_\infty w_1^* \le y_1^* \le E_\infty w_1^* \\ -E_\infty w_2^* \le V_{1,\infty}^T y_1^* + V_{2,\infty}^T y_2^* \\ \le E_\infty w_2^* \end{array} \right. \right\}. \tag{63}$$

The variable $N$-truncation problem of (58) can be constructed as

$$\upsilon_{-N} = \sup_{\omega \in D_{-N}} \varphi(\omega). \tag{64}$$

The following proposition is the direct consequence of $D_{-N} \subset D$.

*Proposition 3:* For the optimisation problems (58) and (64), $\upsilon_{-N} \le \upsilon$.

In the same manner, the optimisation problem (64) can be transformed into the optimisation problem (53), and we have the following proposition.

*Proposition 4:* For the optimisation problems (53) and (64), $\mu_{-N} = \upsilon_{-N}$.

With lemmas 3 and 4, and propositions 1–4, the main result of this paper can be summarized in the following proposition.

*Proposition 5:* For the optimisation problems (17) and (58), $\mu = \upsilon$.

## VI. Conclusions

The dual problem sheds new lights on the mixed $H_2/l_1$ optimisation problem. It can be seen that the existing lemma 1 cannot be applied to the primal problem (17) directly. The idea in verifying the relation between the primal problem (17) and its dual problem (58) is to utilize their corresponding approximation problems (31) and (52) for which lemma 1 can be applied directly. It is interesting to notice that the variable truncation in the primal problem becomes the constraint truncation in the dual problem, and the constraint truncation in the primal problem becomes the variable truncation in the dual problem. The approach based on duality theory is useful in research on the mixed $H_2/l_1$ optimisation problem, as it is often that the dual problem can be solved for more easily than the primal problem.

## ACKNOWLEDGEMENTS

J. Wu and J. Chu wish to thank the support of the National Natural Science Foundation of China (Grants No.60374002 and No.60421002), 973 program of China (Grant No.2002CB312200) and program for New Century Excellent Talents in University (NCET-04-0547). S. Chen wishes to thank the support of the United Kingdom Royal Academy of Engineering.

## REFERENCES

[1] J. Brinkhuis, Introduction to duality in optimization theory, *J. Optimization Theory and Applications* 91 (1996) 523–542.
[2] N. Elia and M.A. Dahleh, Control design with multiple objectives, *IEEE Trans. Automatic Control* 42 (1997) 596–613.
[3] B.A. Francis, *A Course in $H_\infty$ Control Theory*, Springer-Verlag, Berlin, 1987.
[4] I. Kaminer, P.P. Khargonekar and M.A. Rotea, Mixed $H_2/H_\infty$ control for discrete time systems via convex optimization, *Automatica* 29 (1993) 57–70.
[5] D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley, New York, 1969.
[6] M.V. Salapaka, M. Dahleh and P. Voulgaris, Mixed objective control synthesis: optimal $l_1/H_2$ control, *SIAM J. Control and Optimization* 35 (1997) 1672–1689.
[7] M.V. Salapaka, M. Dahleh and P.G. Voulgaris, MIMO optimal control design: the interplay between the $H_2$ and the $l_1$ norms, *IEEE Trans. Automatic Control* 43 (1998) 1374–1388.
[8] M.V. Salapaka, M. Khammash and M. Dahleh, Solution of MIMO $H_2/l_1$ problem without zero interpolation, *SIAM J. Control and Optimization* 37 (1999) 1865–1873.
[9] M. Sznaier and J. Bu, On the properties of the solutions to mixed $l_1/H_\infty$ control problems, in: *Preprints 13th IFAC Congress*, San Francisco, USA, Vol.G, 1996, pp.249–254.
[10] M. Vidyasagar, Optimal rejection of persistent bounded disturbances, *IEEE Trans. Automatic Control* 31 (1986) 527–534.
[11] P. Voulgaris, Optimal $H_2/l_1$ control: the SISO case, in: *Proc. IEEE Int. Conf. Decision and Control*, Vol.4, 1994, pp.3181–3186.
[12] P. Voulgaris, Optimal $H_2/l_1$ control via duality theory, *IEEE Trans. Automatic Control* 40 (1995) 1881–1888.
[13] J. Wu and J. Chu, Mixed $H_2/l_1$ control for discrete time systems, in: *Preprints 13th IFAC Congress*, San Francisco, USA, Vol.G, 1996, pp.453–457.
[14] J. Wu and J. Chu, Approximation methods of scalar mixed $H_2/l_1$ problems for discrete-time systems, *IEEE Trans. Automatic Control* 44 (1999) 1869–1874.
[15] D.C. Youla, H.A. Jabr and J.J. Bongiorno, Modern Wiener-Hopf design of optimal controllers – part II: the multivariable case, *IEEE Trans. Automatic Control* AC-21 (1976) 319–338.
[16] G. Zames, Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms and approximate inverses, *IEEE Trans. Automatic Control* 26 (1981) 301–320.

**Jun Wu** received the BEng and PhD degrees both in industrial automation from Huazhong University of Science and Technology, China in 1989 and 1994, respectively. Since 1994, he has been a faculty member in Institute of Advanced Process Control, Zhejiang University, China, where he currently is a Professor. Dr Wu recent research works include finite-precision digital controller design, networked control, model reduction, robust control and optimization. He has published over 50 papers in journals and conference proceedings and he was awarded 1997/1998 IEE Heaviside Premium.

**Sheng Chen** obtained a BEng degree in control engineering from the East China Petroleum Institute, Dongying, China, in 1982, and a PhD degree in control engineering from the City University at London in 1986. He joined the School of Electronics and Computer Science at the University of Southampton, UK, in September 1999. He previously held research and academic appointments at the Universities of Sheffield, Edinburgh and Portsmouth, all in UK. Professor Chen holds a higher doctorate degree, DSc, from the University of Southampton. His recent research works include adaptive nonlinear signal processing, wireless communications, modelling and identification of nonlinear systems, neural network and machine learning, finite-precision digital controller design, evolutionary computation methods and optimization. He has published over 240 research papers. In the database of the world's most highly cited researchers in various disciplines, compiled by Institute for Scientific Information (ISI) of the USA, Dr Chen is on the list of the highly cited researchers in the engineering category, see www.ISIHighlyCited.com.

**Jian Chu** received the BEng and MSc degrees both in industrial automation from Zhejiang University, China in 1982 and 1984, respectively. As a PhD candidate of the joint education programme between Zhejiang University and Kyoto University Japan, he received the PhD degree in industrial automation from Zhejiang University in 1989. Since 1989, he has been a faculty member in Institute of Advanced Process Control, Zhejiang University, where he currently is the institute director and a Professor of Cheung Kong Scholars Programme. He is also the director of National Laboratory of Industrial Control Technology and a deputy director of National Engineering Research Center for Industrial Automation. His research areas include advanced process control, robust control and fieldbus control systems. He has published over 300 papers in journals and conference proceedings and he was awarded 1997/1998 IEE Heaviside Premium.