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An Integrated Algorithm for Stochastic Steady-State System Optimization and Parameter Estimation

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This paper proposes an algorithm of integrated system optimization and parameter estimation (ISOPE) for stochastic steady-state systems and gives a thorough analysis for the linear–quadratic Gaussian (LQG) problem with unknown parameters under a classical information structure. A simple simulation example is used to illustrate the algorithm.

1. Introduction

DURING the last two decades there has been an increasing interest in maintaining an industrial process at its optimal operating condition. A common method is to calculate optimal values of feedback-controller set points by utilizing a steady-state mathematical model of the process. Often, values of parameters in the model are unknown and have to be estimated. The estimation is achieved by making use of output and input data which are referred to as real-time information. In many situations this real-time information is likely to be corrupted by process or measurement noise.

In situations where the noise level is low, a deterministic ISOPE approach (Roberts, 1979; Brdyś, Chen, & Roberts, 1986) can be used to provide the correct steady-state optimum operating condition of the real process. Sometimes, simple filter techniques can also be employed to attenuate the influence of the noise (Roberts & Ellis, 1981). This ISOPE technique has successfully been applied to some pilot-scale processes (Ellis & Roberts, 1985; Stevenson, Brdyś, & Roberts, 1985).

However, noise levels in many industrial plants are too high for deterministic approaches to have practical usefulness. In such situations, it is preferable to employ stochastic approaches. The parameter estimation and system optimization of a stochastic system, in general, are dependent on each other. Such a dependent relationship is much more complicated than that of the deterministic case. For simplicity, a two-step recursive method may be used: that is, in one step, estimating the model parameters by some standard estimation method such as the least-squares (LS) estimation method based on known data, and in the other step, minimizing (or maximizing) the performance index with the given model. The solution obtained using this method provides the so-called certainty-equivalence control and is generally suboptimal (Bar-Shalom & Tse, 1976).

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In general, deterministic optimization results in an open-loop control strategy. A closed loop control strategy which uses the real-time information optimally can be obtained only by stochastic optimization (Bar-Shalom & Tse, 1976). In this paper, an analysis of stochastic steady-state optimization and parameter estimation is given. A new algorithm of stochastic integrated system optimization and parameter estimation (SISOPE) for the LQG problem is presented. This algorithm can be viewed as a modification of the existing two-step method.

2. Description of stochastic steady-state optimization and parameter estimation

It is assumed that a steady-state system can be described by a set of equations:

$$\boldsymbol{y}(k) = \boldsymbol{h}\big(\boldsymbol{\alpha}(k), \boldsymbol{u}(k)\big) + \boldsymbol{\xi}(k), \tag{2.1}$$

where k is the sampling instant, $\boldsymbol{y}(k) \in \mathbb{R}^m$ and $\boldsymbol{u}(k) \in \mathbb{R}^n$ are the output and input (set point) vectors, respectively; $\boldsymbol{\alpha}(k) \in \mathbb{R}^q$ is the parameter vector which may change slowly, $\boldsymbol{\xi}(k)$ is an m-dimensional stochastic process (noise) and $\boldsymbol{h}(\cdot) : \mathbb{R}^q \times \mathbb{R}^n \to \mathbb{R}^m$ is a measurable mapping.

There may exist a set of inequality constraints imposed on inputs and outputs:

$$\mathsf{E}\boldsymbol{g}\big(\boldsymbol{u}(k), \boldsymbol{y}(k)\big) \leqslant \boldsymbol{0}, \tag{2.2}$$

where E denotes the mathematical expectation operator and $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^r$ is a measurable mapping.

The stochastic steady-state optimization problem can be described by:

$$\text{'Find } \arg\min_{\boldsymbol{u}(k)} j_k \text{ s.t. (2.1) and (2.2) hold'}, \tag{2.3}$$

where $j_k = \mathsf{E}\big[v\big(\boldsymbol{y}(k), \boldsymbol{u}(k)\big) \mid \mathcal{L}_{k-1}, \boldsymbol{u}(k)\big]$ is the performance index at k, $\mathsf{E}(\cdot \mid \cdot)$ is the conditional expectation, and $v : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is the objective function. $\mathcal{L}_{k-1}$ is the real-time information at time $k-1$, which is defined by

$$\mathcal{L}_{k-1} \triangleq \{\boldsymbol{y}^{k-1}, \boldsymbol{u}^{k-1}\},$$

where $\boldsymbol{y}^{k-1}$ and $\boldsymbol{u}^{k-1}$ are the output and input data up to $k-1$: that is,

$$\boldsymbol{y}^{k-1} \triangleq \{\boldsymbol{y}(k-1), \boldsymbol{y}(k-2), \ldots\}, \qquad \boldsymbol{u}^{k-1} \triangleq \{\boldsymbol{u}(k-1), \boldsymbol{u}(k-2), \ldots\}.$$

Here, as well as in the sequel, $\boldsymbol{y}(k-i)$ should be interpreted as a realization of the stochastic process $\boldsymbol{y}(k)$ at $k-i$, $i = 1, 2, \ldots$.

Solving the optimization problem (2.3) is, in general, very difficult because the real parameters $\boldsymbol{\alpha}(k)$ are unknown and the process is contaminated by noise. A common strategy is to employ a two-step approach. According to point-estimation theory (Jazwinski, 1970), the optimal parameter estimation $\hat{\boldsymbol{\alpha}}(k)$ is the conditional expectation:

$$\hat{\boldsymbol{\alpha}}(k) = \mathsf{E}[\boldsymbol{\alpha}(k) \mid \mathcal{L}_{k-1}].$$

If $\boldsymbol{\xi}(k)$ is an independent process, the optimal one-step-ahead prediction of $\boldsymbol{y}(k)$ is:

$$\hat{\boldsymbol{y}}(k \mid k-1) = \mathsf{E}\big[\boldsymbol{h}\big(\boldsymbol{\alpha}(k), \boldsymbol{u}(k)\big) \mid \mathcal{L}_{k-1}, \boldsymbol{u}(k)\big],$$

which depends on $\boldsymbol{u}(k)$, $\hat{\boldsymbol{\alpha}}(k)$, and the higher-order conditional moments of $\boldsymbol{\alpha}(k)$ based on $\mathcal{Z}_{k-1}$. The prediction can be written approximately as:

$$\hat{\boldsymbol{y}}(k \mid k-1) \approx \boldsymbol{h}\big(\hat{\boldsymbol{\alpha}}(k), \boldsymbol{u}(k)\big). \tag{2.4}$$

The model-based optimization problem then can be simplified as:

$$\text{'Find } \arg\min_{\boldsymbol{u}(k)} \hat{J}_k \text{ s.t. (2.4) holds and } \boldsymbol{g}\big(\boldsymbol{u}(k), \hat{\boldsymbol{y}}(k \mid k-1)\big) \leqslant \boldsymbol{0}\text{'},$$

where $\hat{J}_k = v\big(\hat{\boldsymbol{y}}(k \mid k-1), \boldsymbol{u}(k)\big)$.

Note that the formulation of $\boldsymbol{h}(\cdot)$ is assumed to be known. However, $\boldsymbol{\alpha}(k)$ is allowed to be time-dependent to accommodate some of the structural uncertainties. Moreover, some structural errors can also be included into the noise $\boldsymbol{\xi}(k)$. It is recognized that if $\boldsymbol{h}(\cdot)$ is unknown and only an estimate $\hat{\boldsymbol{h}}(\cdot)$ is available, the problem becomes more complicated. In the next section the LQG problem is discussed and some quantitative results are produced.

3. Steady-state optimization of the LQG problem with unknown parameters

Consider the linear steady-state system

$$\boldsymbol{y}(k) = A(k)\boldsymbol{u}(k) + \boldsymbol{b}(k) + \boldsymbol{\xi}(k), \tag{3.1}$$

where $A(k) = [a_{ij}(k)]_{m \times n} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b}(k) \in \mathbb{R}^m$, and $\boldsymbol{\xi}(k)$ is an m-dimensional Gaussian white noise: that is,

$$\mathsf{E}[\boldsymbol{\xi}(k)] = \boldsymbol{0} \quad \text{and} \quad \operatorname{cov}[\boldsymbol{\xi}(k), \boldsymbol{\xi}(s)] = \begin{cases} R & (s = k) \\ 0 & (s \neq k) \end{cases} \quad \text{for all } k,$$

where R is the covariance matrix of $\boldsymbol{\xi}(k)$. In order to rewrite (3.1) in a more convenient form, the system matrix $A(k)$ is first written in terms of column vectors:

$$A(k) = \big[\boldsymbol{a}_1(k), \ldots, \boldsymbol{a}_n(k)\big], \qquad \boldsymbol{a}_j(k) \in \mathbb{R}^m \quad (j = 1, \ldots, n).$$

The parameter vector is given by

$$\boldsymbol{\alpha}(k) \triangleq \big[\boldsymbol{a}_1^{\mathsf{T}}(k), \ldots, \boldsymbol{a}_n^{\mathsf{T}}(k), \boldsymbol{b}^{\mathsf{T}}(k)\big]^{\mathsf{T}} \in \mathbb{R}^{(n+1)m},$$

where $^{\mathsf{T}}$ denotes the transpose. Next, an $m(n+1) \times m$ matrix $\Phi(k)$ is introduced:

$$\Phi(k) \triangleq \big[u_1(k)\mathrm{I}_m, \ldots, u_n(k)\mathrm{I}_m, \mathrm{I}_m\big]^{\mathsf{T}},$$

where I_m is the $m \times m$ identity matrix and $u_j(k)$ $(j = 1, \ldots, n)$ are the components of $\boldsymbol{u}(k)$. Then (3.1) can be written in the form:

$$\boldsymbol{y}(k) = \Phi^{\mathsf{T}}(k)\boldsymbol{\alpha}(k) + \boldsymbol{\xi}(k).$$

The objective function is quadratic:

$$v\big(\boldsymbol{y}(k), \boldsymbol{u}(k)\big) = [\boldsymbol{y}(k) - \boldsymbol{y}_{\mathrm{d}}]^{\mathsf{T}} C[\boldsymbol{y}(k) - \boldsymbol{y}_{\mathrm{d}}] + \boldsymbol{u}^{\mathsf{T}}(k) D \boldsymbol{u}(k),$$

where $\boldsymbol{y}_{\mathrm{d}} \in \mathbb{R}^m$ is a constant vector, and C and D are symmetric positive definite

$m \times m$ and $n \times n$ matrices, respectively. Let us first consider using the two-step strategy to solve this LQG problem.

Assuming that the LS estimation is employed, the estimate of $\boldsymbol{\alpha}(k)$ at k is

$$\hat{\boldsymbol{\alpha}}(k) = \mathsf{E}[\boldsymbol{\alpha}(k) \mid \mathcal{L}_{k-1}] = (\Phi_{k-1}^{\mathsf{T}} W_{k-1} \Phi_{k-1})^{-1} \Phi_{k-1}^{\mathsf{T}} W_{k-1} y_{k-1},$$

where

$$\Phi_{k-1} \triangleq \begin{bmatrix} \Phi^{\mathsf{T}}(1) \\ \vdots \\ \Phi^{\mathsf{T}}(k-1) \end{bmatrix}, \qquad y_{k-1} \triangleq \begin{bmatrix} y(1) \\ \vdots \\ y(k-1) \end{bmatrix},$$
$$W_{k-1} \triangleq \begin{bmatrix} \beta^{k-2} \mathrm{I}_m & & 0 \\ & \ddots & \\ & & \beta \mathrm{I}_m \\ 0 & & & \mathrm{I}_m \end{bmatrix} = \begin{bmatrix} \beta W_{k-2} & 0 \\ 0 & \mathrm{I}_m \end{bmatrix}, \qquad W_1 = \mathrm{I}_m \tag{3.2}$$

are the matrix of input data, the vector of output data, and the weighting matrix, respectively; β is the forgetting factor. If the definition

$$P(k-1) \triangleq (\Phi_{k-1}^{\mathsf{T}} W_{k-1} \Phi_{k-1})^{-1} = \left(\sum_{s=1}^{k-1} \beta^{k-1-s} \Phi(s) \Phi^{\mathsf{T}}(s) \right)^{-1}$$

is made, then the recursive least-squares (RLS) estimation formula can be obtained (e.g. Van den Boom, 1982) as follows.

Initial conditions are $\hat{\boldsymbol{\alpha}}(1) = \mathbf{0}$, $P(0) = \theta^2 \mathrm{I}_{m(n+1)}$, $\theta^2 \geqslant 10^6$, and $\beta = 0{\cdot}95 \sim 0{\cdot}997$.

(i) $\hat{\boldsymbol{\alpha}}(k+1) = \hat{\boldsymbol{\alpha}}(k) + K(k)[y(k) - \hat{y}(k \mid k-1)]$ where

$$K(k) = P(k-1)\Phi(k)[\beta \mathrm{I}_m + \Phi^{\mathsf{T}}(k) P(k-1) \Phi(k)]^{-1},$$

is the Kalman gain matrix and $\hat{y}(k \mid k-1)$ is the output prediction which is given by

$$\begin{aligned} \hat{y}(k \mid k-1) &= \mathsf{E}[y(k) \mid \mathcal{L}_{k-1}, \boldsymbol{u}(k)] = \Phi^{\mathsf{T}}(k) \hat{\boldsymbol{\alpha}}(k) \\ &= \hat{A}(k) \boldsymbol{u}(k) + \hat{\boldsymbol{b}}(k). \end{aligned} \tag{3.3}$$

Here $\hat{A}(k)$ and $\hat{\boldsymbol{b}}(k)$ are the estimates of $A(k)$ and $\boldsymbol{b}(k)$ at k, respectively.

(ii) $P(k)$ is obtained according to formula

$$\begin{aligned} P(k) &= \beta^{-1}[\mathrm{I}_{m(n+1)} - K(k)\Phi^{\mathsf{T}}(k)]P(k-1) \\ &= \beta^{-1}\{P(k-1) - P(k-1)\Phi(k)[\beta \mathrm{I}_m + \Phi^{\mathsf{T}}(k)P(k-1)\Phi(k)]^{-1}\Phi^{\mathsf{T}}(k)P(k-1)\}. \end{aligned} \tag{3.4}$$

Using the prediction model (3.3) the optimization problem is simplified as a deterministic one:

$$\text{'Find } \arg\min_{\boldsymbol{u}(k)} \hat{J}_k \text{ s.t. (3.3) holds'}, \tag{3.5}$$

where $\hat{J}_k = [\hat{y}(k \mid k-1) - y_{\mathrm{d}}]^{\mathsf{T}} C [\hat{y}(k \mid k-1) - y_{\mathrm{d}}] + \boldsymbol{u}^{\mathsf{T}}(k) D \boldsymbol{u}(k)$. The solution of

(3.5) or the certainty-equivalent input $\hat{\boldsymbol{u}}^*(k)$ can readily be written as,

$$\hat{\boldsymbol{u}}^*(k) = [D + \hat{A}^{\mathsf{T}}(k)C\hat{A}(k)]^{-1}\hat{A}^{\mathsf{T}}(k)C[\boldsymbol{y}_{\mathrm{d}} - \hat{\boldsymbol{b}}(k)]. \tag{3.6}$$

Having obtained the certainty-equivalent input, consider the original optimization problem:

$$\text{'Find } \arg\min_{\boldsymbol{u}(k)} j_k \text{ s.t. (3.1) holds'}, \tag{3.7}$$

where

$$\begin{aligned} j_k &= \mathsf{E}\{[\boldsymbol{y}(k) - \boldsymbol{y}_{\mathrm{d}}]^{\mathsf{T}}C[\boldsymbol{y}(k) - \boldsymbol{y}_{\mathrm{d}}] + \boldsymbol{u}^{\mathsf{T}}(k)D\boldsymbol{u}(k) \mid \mathcal{L}_{k-1}, \boldsymbol{u}(k)\} \\ &= [\hat{\boldsymbol{y}}(k \mid -1) - \boldsymbol{y}_{\mathrm{d}}]^{\mathsf{T}}C[\hat{\boldsymbol{y}}(k \mid k-1) - \boldsymbol{y}_{\mathrm{d}}] + \boldsymbol{u}^{\mathsf{T}}(k)D\boldsymbol{u}(k) + \mathrm{tr}\,[CR_{\boldsymbol{y}}(k)]. \end{aligned} \tag{3.8}$$

Here tr denotes the trace of a matrix and $R_{\boldsymbol{y}}(k)$ is the covariance matrix of the prediction error which is defined as

$$R_{\boldsymbol{y}}(k) \triangleq \mathsf{E}[\tilde{\boldsymbol{y}}(k)\tilde{\boldsymbol{y}}^{\mathsf{T}}(k) \mid \mathcal{L}_{k-1}, \boldsymbol{u}(k)].$$

The prediction error $\tilde{\boldsymbol{y}}(k)$ is given by

$$\tilde{\boldsymbol{y}}(k) \triangleq \boldsymbol{y}(k) - \hat{\boldsymbol{y}}(k \mid k-1) = \Phi^{\mathsf{T}}(k)[\boldsymbol{\alpha}(k) - \hat{\boldsymbol{\alpha}}(k)] + \boldsymbol{\xi}(k).$$

Because $\boldsymbol{\xi}(k)$ is an independent process, $R_{\boldsymbol{y}}(k)$ can be expressed as

$$R_{\boldsymbol{y}}(k) = \Phi^{\mathsf{T}}(k)R_{\alpha}(k)\Phi(k) + R,$$

where $R_{\alpha}(k)$ is the covariance matrix of the parameter estimation error, given by

$$R_{\alpha}(k) \triangleq \mathsf{E}[\tilde{\boldsymbol{\alpha}}(k)\tilde{\boldsymbol{\alpha}}^{\mathsf{T}}(k) \mid \mathcal{L}_{k-1}]. \tag{3.9}$$

The parameter estimation error $\tilde{\boldsymbol{\alpha}}(k)$ of the LS estimation is

$$\tilde{\boldsymbol{\alpha}}(k) \triangleq \boldsymbol{\alpha}(k) - \hat{\boldsymbol{\alpha}}(k) = -(\Phi_{k-1}^{\mathsf{T}}W_{k-1}\Phi_{k-1})^{-1}\Phi_{k-1}^{\mathsf{T}}W_{k-1}\boldsymbol{\xi}_{k-1}, \tag{3.10}$$

where

$$\boldsymbol{\xi}_{k-1} = [\boldsymbol{\xi}^{\mathsf{T}}(1), \ldots, \boldsymbol{\xi}^{\mathsf{T}}(k-1)]^{\mathsf{T}}.$$

Because W_{k-1} is diagonal, substituting (3.10) into (3.9) gives

$$\begin{aligned} R_{\alpha}(k) &= (\Phi_{k-1}^{\mathsf{T}}W_{k-1}\Phi_{k-1})^{-1}(W_{k-1}\Phi_{k-1})^{\mathsf{T}}R_{k-1}(W_{k-1}\Phi_{k-1})(\Phi_{k-1}^{\mathsf{T}}W_{k-1}\Phi_{k-1})^{-1} \\ &= P(k-1)(W_{k-1}\Phi_{k-1})^{\mathsf{T}}R_{k-1}(W_{k-1}\Phi_{k-1})P(k-1). \end{aligned} \tag{3.11}$$

Here, R_{k-1} is given by $R_{k-1} = \bigoplus_{i=1}^{k-1} R$. Obviously, if R is positive definite, so is $R_{\alpha}(k)$; and $R_{\boldsymbol{y}}(k)$ depends on $\Phi(k)$, which in turn is dependent on $\boldsymbol{u}(k)$. Therefore, the separation theorem (Åström, 1970) does not hold and the certainty-equivalent input (3.6) is not optimal.

Now consider an alternative method for solving (3.7). If R is known, and hence $R_{\alpha}(k)$ is known, then

$$\begin{aligned} \mathrm{tr}\,[CR_{\boldsymbol{y}}(k)] &= \mathrm{tr}\,\{C[\Phi^{\mathsf{T}}(k)R_{\alpha}(k)\Phi(k) + R]\} \\ &= \mathrm{tr}\,[R_{\alpha}(k)\Phi(k)C\Phi^{\mathsf{T}}(k)] + \mathrm{tr}\,(CR). \end{aligned}$$

Because

$$\Phi(k)C\Phi^{\mathrm{T}}(k)=\begin{bmatrix} u_1^2(k)C & \cdots u_1(k)u_n(k)C & u_1(k)C \\ \vdots & \vdots & \vdots \\ u_n(k)u_1(k)C\cdots & u_n^2(k)C & u_n(k)C \\ u_1(k)C & \cdots & u_n(k)C \quad C \end{bmatrix},$$

and $R_\alpha(k)=[R_{\alpha,ij}(k)]_{i,j=1,\ldots,n+1}$, where $R_{\alpha,ij}(k)\in\mathbb{R}^{m\times m}$, it follows that

$$\operatorname{tr}[CR_y(k)]=\sum_{i=1}^{n+1}\Big(u_i(k)\sum_{j=1}^{n+1}\operatorname{tr}[R_{\alpha,ij}(k)C]u_j(k)\Big)+\operatorname{tr}(CR), \tag{3.12}$$

where $u_{n+1}(k)=1$ and $R_{\alpha,ij}(k)=R_{\alpha,ji}^{\mathrm{T}}(k)$.

Using (3.12) in (3.8) and differentiating it with respect to $\boldsymbol{u}(k)$:

$$\frac{\partial j_k}{\partial \boldsymbol{u}(k)}=2\boldsymbol{u}^{\mathrm{T}}(k)[D+\hat{A}^{\mathrm{T}}(k)C\hat{A}(k)+Q(k)]+2[\hat{\boldsymbol{b}}(k)-y_{\mathrm{d}}]^{\mathrm{T}}C\hat{A}(k)+2\lambda^{\mathrm{T}}(k)=\boldsymbol{0}^{\mathrm{T}}. \tag{3.13}$$

$Q(k)$ and $\boldsymbol{\lambda}(k)$ are defined as follows:

$$\boldsymbol{\lambda}(k)\triangleq[\operatorname{tr}[R_{\alpha,1(n+1)}(k)C],\ldots,\operatorname{tr}[R_{\alpha,n(n+1)}(k)C]]^{\mathrm{T}}, \tag{3.14}$$

$$Q(k)\triangleq\begin{bmatrix} \operatorname{tr}[R_{\alpha,11}(k)C] & \operatorname{tr}[R_{\alpha,12}(k)C]\cdots\operatorname{tr}[R_{\alpha,1n}(k)C] \\ \operatorname{tr}[R_{\alpha,21}(k)C] & \operatorname{tr}[R_{\alpha,22}(k)C]\cdots\operatorname{tr}[R_{\alpha,2n}(k)C] \\ \vdots & \vdots \quad \vdots \\ \operatorname{tr}[R_{\alpha,n1}(k)C] & \operatorname{tr}[R_{\alpha,n2}(k)C]\cdots\operatorname{tr}[R_{\alpha,nn}(k)C] \end{bmatrix}. \tag{3.15}$$

The optimal input $\hat{\boldsymbol{u}}^*(k)$ can be obtained directly from (3.13):

$$\hat{\boldsymbol{u}}^*(k)=[D+\hat{A}^{\mathrm{T}}(k)C\hat{A}(k)+Q(k)]^{-1}\{\hat{A}^{\mathrm{T}}(k)C[y_{\mathrm{d}}-\hat{\boldsymbol{b}}(k)]-\boldsymbol{\lambda}(k)\}.$$

$R_\alpha(k)$ can be calculated in a recursive manner. According to (3.11) and (3.2),

$$R_\alpha(k)=$$
$$P(k-1)[\beta^2(W_{k-2}\Phi_{k-2})^{\mathrm{T}}R_{k-2}(W_{k-2}\Phi_{k-2})+\Phi(k-1)R\Phi^{\mathrm{T}}(k-1)]P(k-1).$$

Then using (3.4) gives,

$$R_\alpha(k)=$$
$$[\mathrm{I}_{m(n+1)}-K(k-1)\Phi^{\mathrm{T}}(k-1)]P(k-2)(W_{k-2}\Phi_{k-2})^{\mathrm{T}}R_{k-2}(W_{k-2}\Phi_{k-2})P(k-2)\cdot$$
$$[\mathrm{I}_{m(n+1)}-K(k-1)\Phi^{\mathrm{T}}(k-1)]^{\mathrm{T}}+P(k-1)\Phi(k-1)R\Phi^{\mathrm{T}}(k-1)P(k-1). \tag{3.16}$$

Finally defining

$$M(k)\triangleq\mathrm{I}_{m(n+1)}-K(k-1)\Phi^{\mathrm{T}}(k-1),\qquad L(k)\triangleq P(k-1)\Phi(k-1),$$

we may write (3.16) as

$$R_\alpha(k)=M(k)R_\alpha(k-1)M^{\mathrm{T}}(k)+L(k)RL^{\mathrm{T}}(k).$$

In reality, R is often unknown and must be estimated by using, for example, some off-line estimation method. If $u(k)$ is a constant input and $y(k)$ is the corresponding output observation, the following simple scheme can be applied to obtain $\hat{R}$, an estimate of R:

$$\bar{y} = \frac{1}{n}\sum_{k=1}^{N} y(k), \qquad \hat{R} = \frac{1}{N}\sum_{k=1}^{N} [y(k) - \bar{y}][y(k) - \bar{y}]^{\mathrm{T}}.$$

The estimate of $R_\alpha(k)$, denoted by $\hat{R}_\alpha(k)$, is given by the recursive formula

$$\hat{R}_\alpha(k) = M(k)\hat{R}_\alpha(k-1)M^{\mathrm{T}}(k) + L(k)\hat{R}L^{\mathrm{T}}(k).$$

$\hat{R}_\alpha(0)$ may be chosen as a zero matrix. The estimates of $\lambda(k)$ and $Q(k)$, $\hat{\lambda}(k)$ and $\hat{Q}(k)$, can be obtained by substituting $\hat{R}_\alpha(k)$ into (3.14) and (3.15), respectively. Based on the above discussion, a modified two-step algorithm (Fig. 1) can be formulated:

Given $u(1)$, $\hat{\alpha}(1)$, $P(0)$, $\hat{R}_\alpha(0)$, β, and $\hat{R}$, for $k = 2, 3, \ldots$:

(1) The task of the estimation unit is to estimate the parameters $\hat{\alpha}(k)$, and $\hat{R}_\alpha(k)$.

(2) A modification unit is introduced whose job is to compute $\hat{\lambda}(k)$ and $\hat{Q}(k)$, which are termed modifiers.

(3) The optimization problem is obtained by modifying (3.5) with $\hat{\lambda}(k)$ and

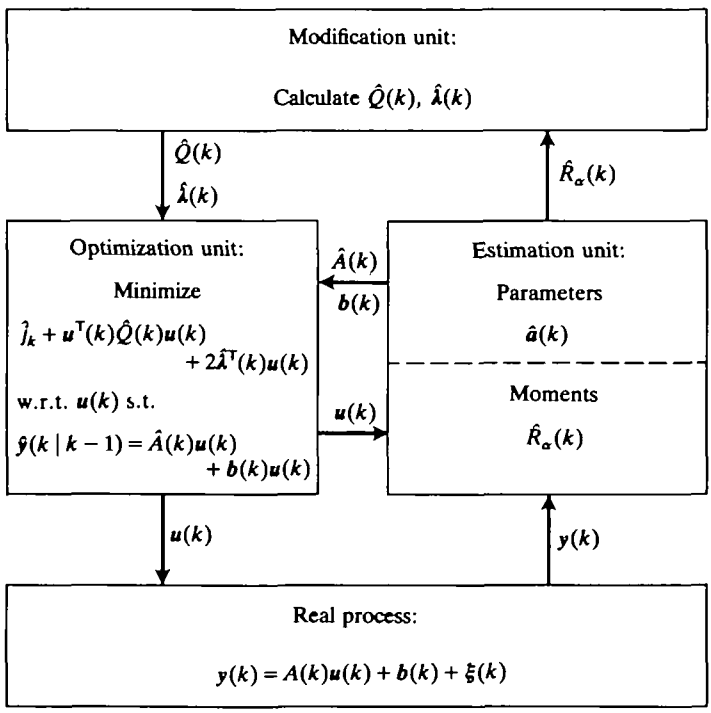

FIG. 1

$\hat{Q}(k)$ provided by the modification unit:

$$\text{'Find } \arg\min_{u(k)} [\hat{J}_k + u^T(k)\hat{Q}(k)u(k) + 2\hat{\lambda}^T(k)u(k)] \text{ s.t. (3.3) holds'.} \quad (3.17)$$

The solution of (3.17), for given modifiers, is given by:

$$\hat{u}^*(k) = [D + \hat{A}^T(k)C\hat{A}(k) + \hat{Q}(k)]^{-1}\{\hat{A}^T(k)C[y_d - \hat{b}(k)] - \hat{\lambda}(k)\} \quad (3.18)$$

This is applied to the process and the above steps are repeated.

It is observed that two terms are added in the optimization objective function, to take account of the inherent interaction between parameter estimation and system optimization because imperfect model parameters are employed. If we write

$$\hat{Q}(k) = [\hat{q}_{ij}(k)]_{i,j=1,\ldots,n}, \qquad \hat{\lambda}(k) = [\hat{\lambda}_1(k), \ldots, \hat{\lambda}_n(k)]^T,$$
$$\tilde{a}_i(k) = a_i(k) - \hat{a}_i(k) \quad (i = 1, \ldots, n), \qquad \tilde{b}(k) = b(k) - \hat{b}(k),$$

then it is clear that

$\hat{q}_{ij}(k)$ is an estimate of tr $\{E[\tilde{a}_i(k)\tilde{a}_j^T(k) \mid \mathcal{L}_{k-1}]C\}$ and
$\hat{\lambda}_i(k)$ is an estimate of tr $\{E[\tilde{a}_i(k)\tilde{b}^T(k) \mid \mathcal{L}_{k-1}]C\}$.

The standard two-step algorithm can readily be implemented by setting $\hat{\lambda}(k) = 0$ and $\hat{Q}(k) = 0$ in this algorithm.

4. Convergence analysis of the algorithm

Firstly, the convergence of the modified two-step algorithm for the LQG problem with unknown parameters given in the previous section is shown to depend on the convergence of the RLS parameter estimation. The result is also valid for the standard two-step algorithm by simply setting $\hat{Q}(k)$ and $\hat{\lambda}(k)$ to zero for all k.

Assume that the RLS estimation is convergent: that is, the sequence of parameter estimates $(\hat{\alpha}(k))_{k=1,2,\ldots}$ is a Cauchy sequence. For any $\delta > 0$, there then exists a k_0 such that

$$\|\hat{\alpha}(k) - \hat{\alpha}(s)\| < \delta \quad \forall (k, s) > (k_0, k_0)$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\|\hat{R}_\alpha(k) - \hat{R}_\alpha(s)\| \leq M\delta \quad \forall (k, s) > (k_0, k_0)$$

where M is a positive constant. From (3.18),

$$\|\hat{u}^*(k) - \hat{u}^*(s)\| =$$
$$\|[D + \hat{A}^T(k)C\hat{A}(k) + \hat{Q}(k)]^{-1}\{\hat{A}^T(k)C[y_d - \hat{b}(k)] - \hat{\lambda}(k)\} -$$
$$[D + \hat{A}^T(s)C\hat{A}(s) + \hat{Q}(s)]^{-1}\{\hat{A}^T(s)C[y_d - \hat{b}(s)] - \hat{\lambda}(s)\}\|$$
$$\leq u_1 + u_2$$

where

$$u_1 = \|[D + \hat{A}^T(k)C\hat{A}(k) + \hat{Q}(k)]^{-1} \cdot \{\hat{A}^T(k)C[y_d - \hat{b}(k)] - \hat{\lambda}(k) - \hat{A}^T(s)C[y_d - \hat{b}(s)] + \hat{\lambda}(s)\}\|,$$

$$u_2 = \|\{[D + \hat{A}^T(k)C\hat{A}(k) + \hat{Q}(k)]^{-1} - [D + \hat{A}^T(s)C\hat{A}(s) + \hat{Q}(s)]^{-1}\} \cdot \{\hat{A}^T(s)C[y_d - \hat{b}(s)] - \hat{\lambda}(s)\}\|.$$

Note that $\hat{A}^T(k)C\hat{A}(k)$ and $\hat{Q}(k)$ are symmetric non-negative definite matrices. Therefore $D + \hat{A}^T(k)C\hat{A}(k) + \hat{Q}(k) \geq D$ and

$$u_1 \leq \|D^{-1}\{\hat{A}^T(k)C[y_d - \hat{b}(k)] - \hat{A}^T(s)C[y_d - \hat{b}(s)] - [\hat{\lambda}(k) - \hat{\lambda}(s)]\}\| < l_1\delta,$$

where l_1 is a positive constant. Similarly, with the help of the matrix-inversion lemma, it can be shown that $u_2 \leq l_2\delta$, where l_2 is a positive constant. Let $l = l_1 + l_2$, then $\|\hat{u}^*(k) - \hat{u}^*(s)\| < l\delta$. This proves that $(\hat{u}^*(k))_{k=1,2,\ldots}$ is also a Cauchy sequence. Thus, it can be concluded that the algorithm is convergent if the RLS estimation is convergent.

It has been proved that LS estimation with $\beta = 1$ is convergent in probability if the parameters in the system equation (3.1) are constant, the noise is white Gaussian and the input sequence $(u(k))$ is persistently exciting (Van den Boom, 1982): that is, $\hat{\alpha}(k)$ converges in probability to α as $k \to \infty$, where α is the real parameter vector.

If $\beta = 1$, then $P(k)$ and $K(k)$ in RLS estimation tend to become zero when k is large enough: that is, $P(k) \approx 0$ and $K(k) \approx 0$ for $k \geq k_0$. This will result in $u(k) \approx u^*$ $(k \geq k_0)$, where u^* is a constant vector. The input sequence will not then be persistently exciting. However, Ljung has proved that if β is appropriately chosen, RLS estimation for Gaussian noise has the best convergence property (Ljung, 1977a, 1977b). Finally, it is well known that if the parameters in (3.1) change slowly, RLS estimation with $\beta < 1$ can follow the change.

5. Simulation study

Consider a time-invariant LQG problem with

$$A = \begin{bmatrix} 2 & -1 \\ -0{\cdot}5 & 2{\cdot}5 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad y_d = \begin{bmatrix} 5 \\ 8 \\ 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0{\cdot}001 & 0 \\ 0 & 0{\cdot}001 \end{bmatrix},$$

and assume that the noise $\xi(k)$ has zero mean and covariance

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This example was used in computer simulation to compare the modified two-step algorithm with the standard two-step algorithm. The recursive procedure was terminated after 200 recursive steps. The following statistical criteria were used to assess the performance of the two algorithms.

Two scalars $\bar{j}_N$ and σ_j^2:

$$\bar{j}_N = \frac{1}{N}\sum_{k=1}^{N} \hat{j}_k^* \qquad (N = 1, \ldots, 200), \qquad \sigma_j^2 = \frac{1}{200}\sum_{k=1}^{200} (\hat{j}_k^* - j^*)^2,$$

where j^* is the real optimal mean performance, and $\hat{j}_k^*$ is the value of the real performance function at k.

Two vectors $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_\alpha$:

These are defined as the vectors whose elements are the diagonal elements of the matrices

$$\frac{1}{200}\sum_{k=1}^{200} [\hat{\boldsymbol{u}}^*(k) - \boldsymbol{u}^*][\hat{\boldsymbol{u}}^*(k) - \boldsymbol{u}^*]^T, \qquad \frac{1}{200}\sum_{k=1}^{200} [\hat{\boldsymbol{\alpha}}(k) - \boldsymbol{\alpha}^*][\hat{\boldsymbol{\alpha}}(k) - \boldsymbol{\alpha}^*]^T,$$

respectively, where $\boldsymbol{u}^*$ are the real optimal inputs, $\hat{\boldsymbol{u}}^*(k)$ are the inputs at k, and $\boldsymbol{\alpha}^*$ are the real parameters, with $\hat{\boldsymbol{\alpha}}(k)$ the parameter estimates at k. The three statistics $\bar{j}_N$, σ_j^2, and $\boldsymbol{\sigma}_u$ are directly connected with the control objective.

$\bar{j}_N$ can be observed in Fig. 2, and the other three statistics are given in Table 1, where case (a) represents the two-step algorithm and cases (b) to (e) represent the modified two-step algorithm with different estimated $\hat{R}$ and initial $\hat{R}_\alpha(0)$. Sequences of inputs generated in cases (a) and (b) are illustrated in Fig. 3 and Fig. 4.

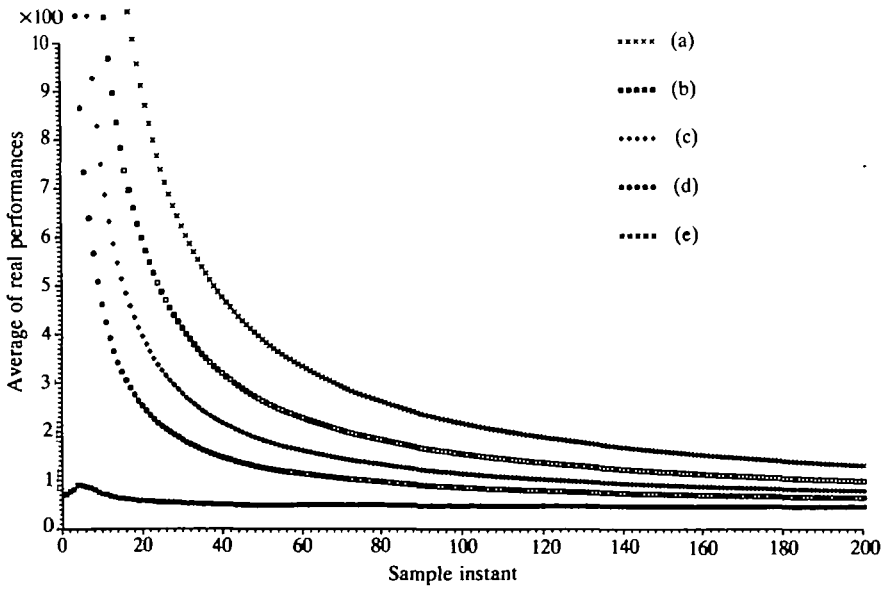

FIG. 2

TABLE 1
Statistics and initial conditions

Case	(a)	(b)	(c)	(d)	(e)
σ_j:	1215·5	774·8	492·2	275·5	15·6
α_u:	3·8382	2·3612	1·4215	0·9316	0·3410
	4·9732	3·1911	2·0875	1·2817	0·1460
α_a:	0·0811	0·1156	0·1058	0·1797	139·21
	0·0561	0·0720	0·0774	0·1398	88·410
	0·1400	0·1278	0·1564	0·6202	45·088
	0·0482	0·0627	0·0586	0·1237	177·49
	0·1098	0·1238	0·1281	0·1744	111·05
	0·1393	0·1422	0·1572	0·6431	51·741
	0·6717	0·8218	0·6837	0·3894	0·2706
	0·7594	0·9690	1·0225	0·6269	0·8448
	0·8711	0·8356	1·0947	0·4251	0·3784
$\hat{R}_a(0)$:		0	0	0	$2I_9$
$\hat{R}$:		$0·5I_3$	†	‡	†

$\hat{a}(1) = 0$, $u_1(1) = 0·1$, $u_2(1) = -0·1$, $P(0) = 10^6 I_9$ and $\beta = 0·97$ for all cases.
† diag (1·2, 0·8, 0·9), ‡ diag (1·5, 1·6, 1·4).

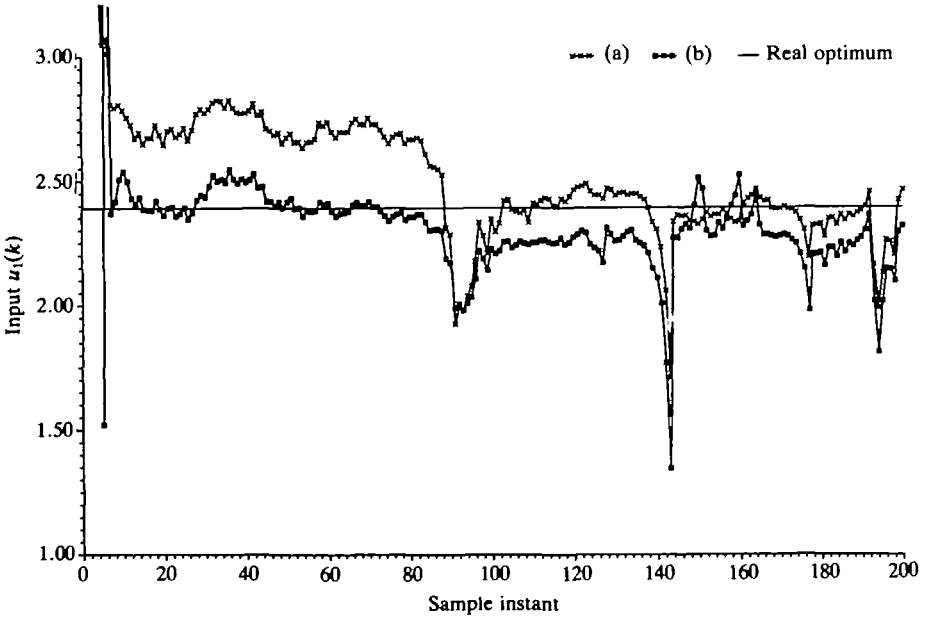

FIG. 3

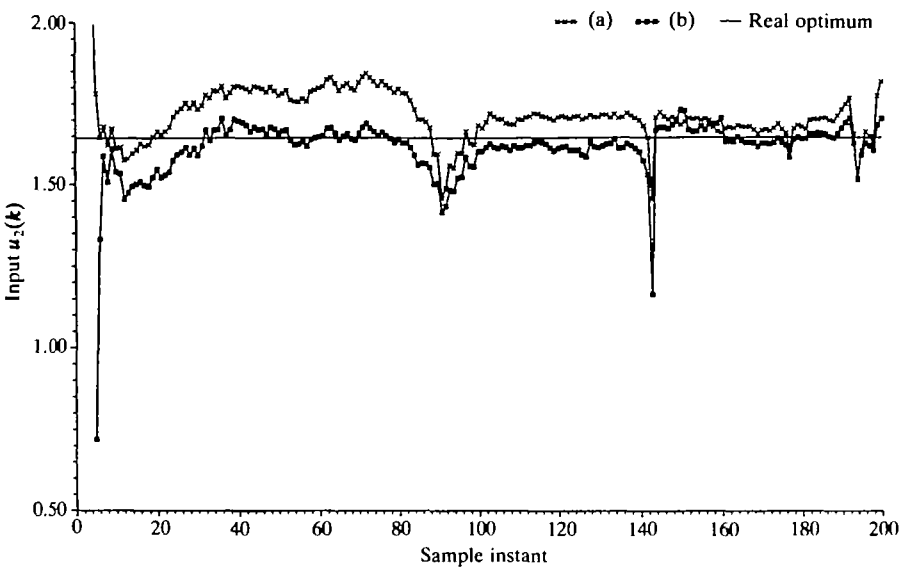

FIG. 4

The results show that, in the example tested, the modified two-step algorithm gave significant improvement in the real average performance and reduced the variances of the real performance, even with a poor estimate of $\hat{R}$ (case (b)). It is clear that the smaller mean performance and the smaller performance variances were the direct results of the smaller variances of the inputs around the real optimal values.

6. Conclusions

Theoretical analysis and simulation results indicate that the modified two-step algorithm derived in this paper is superior to the standard two-step algorithm for the LQG problem with unknown parameters.

The improvement achieved by the modified two-step algorithm depends on the off-line estimation of the noise covariance matrix. The simulation results, however, show that a rough estimate is enough to give substantial improvement in the control goal. If the estimation results indicate that the noise is a correlated process, the sampling interval must be increased appropriately so that the system displays steady-state characteristics.

There is scope for further research to investigate how to estimate the noise covariance matrix on-line and to study the possibility of extending the algorithm to more-general, nonlinear systems with constraints.

REFERENCES

ÅSTRÖM, K. J. 1970 *Introduction to Stochastic Control Theory*. New York: Academic Press.

BAR-SHALOM, Y., & TSE, E. 1976 Concepts and methods in stochastic control. In: *Control and Dynamic Systems: Advances in Theory and Application,* Vol. 12 (Ed C. T. Leondes). New York: Academic Press.

BRDYŚ, M., CHEN, S., & ROBERTS, P. D. 1986 An extension to the modified two step algorithm for steady state system optimisation and parameter estimation. *Int. J. Systems Sci.* (to be published).

ELLIS, J. E., & ROBERTS, P. D. 1985 On the practical viability of integrated system optimisation and parameter estimation. *IEE International Conference 'Control 85',* Publication No. 252, University of Cambridge, England, pp. 281–285.

JAZWINSKI, A. H. 1970 *Stochastic Processes and Filtering Theory.* New York: Academic Press.

LJUNG, L. 1977a Analysis of recursive stochastic algorithms. *IEEE Trans. autom. Control* AC-**22,** 551–575.

LJUNG, L. 1977b On positive real functions and the convergence of some recursive schemes. *IEEE Trans. on autom. Control* AC-**22,** 539–551.

ROBERTS, P. D. 1979 An algorithm for steady state system optimization and parameter estimation. *Int. J. Systems Sci.* **10,** 719–734.

ROBERTS, P. D., & ELLIS, J. E. 1981 Refinements to an algorithm for combined system optimisation and parameter estimation. *Proc. of the 1981 UKSC Conference on Computer Simulation,* Harrogate, England.

STEVENSON, I. A., BRDYŚ, M., & ROBERTS, P. D. 1985 Integrated system optimisation and parameter estimation for a travelling load furnace. *Preprints of the 7th IFAC/IFORS Symposium on Identification and System Parameter Estimation,* University of York, England, pp. 1641–1646.

VAN DEN BOOM, A. J. W. 1982 *System Identification.* Ph.D. Thesis, Proefschrift Eindhoven, Netherland. ISBN 90-9000352-5.