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Optimizing Stability Bounds of Finite-Precision PID Controllers Using Adaptive Simulated Annealing

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Abstract

Based on a lower bound stability measure for sampled-data controller structures subject to finite-word-length (FWL) constraints, the optimal realization of the digital PID controller with FWL considerations is formulated as a nonlinear optimization problem. An efficient strategy based on adaptive simulated annealing (ASA) is adopted to solve this complex optimization problem. A numerical example of optimizing the finite-precision PID controller for a steel rolling mill system is used to demonstrate the effectiveness of the proposed method.

1 Introduction

Controller implementations with fixed-point arithmetic offer the advantages of speed, memory space, cost and simplicity over floating-point arithmetic. However, a designed stable closed-loop system may become unstable when the infinite-precision controller is implemented using a fixed-point processor due to FWL effects. In recent years, many results have been reported in the literature, addressing FWL issues [1]-[9]. It remains an unsolved problem to compute the exact stability robustness measure for obtaining realizable digital controller design with FWL effects [2]. To overcome this problem, a tractable lower bound stability measure has been derived [6],[7].

Based on this lower bound stability measure, the optimal realization of the digital PID controller under FWL constraints can be solved as a nonlinear optimization problem with four variables [8],[9]. The optimization criteria are however nonsmooth and nonconvex functions, and conventional optimization methods may fail to obtain an optimal solution. To overcome this difficulty, we propose to use an efficient global optimization method, called the ASA [10]–[13], to search for a true optimal PID controller realization. The effectiveness of the proposed optimization strategy is illustrated by the example of a steel rolling mill control problem.

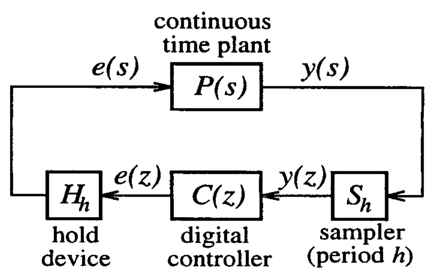


Figure 1: Sampled-data control system.

2 Stability robustness measures

Consider the sampled-data system depicted in Fig. 1, where $P(s)$ is strictly proper. The plant $P(z) = S_h P(s) H_h$ has a state-space description $(A_z, B_z, C_z, 0)$ with $A_z \in \mathbf{R}^{m\times m}$, $B_z \in \mathbf{R}^{m\times l}$ and $C_z \in \mathbf{R}^{q\times m}$. The controller $C(z)$ has a state-space description (A_c, B_c, C_c, D_c) with $A_c \in \mathbf{R}^{n\times n}$, $B_c \in \mathbf{R}^{n\times q}$, $C_c \in \mathbf{R}^{l\times n}$ and $D_c \in \mathbf{R}^{l\times q}$. We will refer to (A_c, B_c, C_c, D_c) as a realization of $C(z)$. The realizations of $C(z)$ are not unique. If (A_c, B_c, C_c, D_c) is a realization of $C(z)$, so is $(\mathcal{T}^{-1}A_c\mathcal{T}, \mathcal{T}^{-1}B_c, C_c\mathcal{T}, D_c)$ for any similarity transformation $\mathcal{T} \in \mathbf{R}^{n\times n}$. The transition matrix of the closed-loop system is given by:

$$\bar{A}(X) = \begin{bmatrix} A_z + B_z D_c C_z & B_z C_c \\ B_c C_z & A_c \end{bmatrix}$$

$$= \begin{bmatrix} A_z & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_z & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \begin{bmatrix} C_z & 0 \\ 0 & I_n \end{bmatrix}$$

$$= M_0 + M_1 X M_2, \quad (1)$$

where the controller matrix

$$X = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$$

$$= \begin{bmatrix} p_1 & \cdots & p_{q+n} \\ p_{q+n+1} & \cdots & p_{2(q+n)} \\ \vdots & \cdots & \vdots \\ p_{(l+n-1)(q+n)+1} & \cdots & p_{(l+n)(q+n)} \end{bmatrix}. \quad (2)$$

When (A_c, B_c, C_c, D_c) is implemented using a finite-precision processor, X is perturbed to $X + \Delta X$ with

$$\Delta X = \begin{bmatrix} \Delta p_1 & \cdots & \Delta p_{q+n} \\ \Delta p_{q+n+1} & \cdots & \Delta p_{2(q+n)} \\ \vdots & \cdots & \vdots \\ \Delta p_{(l+n-1)(q+n)+1} & \cdots & \Delta p_{(l+n)(q+n)} \end{bmatrix}. \quad (3)$$

To see when the round off error will cause the closed-loop system to become unstable, define

$$\mu_0(X) \triangleq \inf\{\mu(\Delta X) : \bar{A}(X + \Delta X) \text{ is unstable}\}, \quad (4)$$

where

$$\mu(\Delta X) \triangleq \max_{i \in \{1, \cdots, N\}} |\Delta p_i| \quad (5)$$

and $N = (l+n)(q+n)$. $\mu_0(X)$ is a stability robustness measure of the controller realization X with FWL considerations [6],[7]. However, how to compute $\mu_0(X)$ is still an unsolved problem. To overcome this difficulty, a lower bound of $\mu_0(X)$ has been introduced as [7]

$$\mu_1(X) \triangleq \min_{i \in \{1, \cdots, m+n\}} \frac{1 - |\lambda_i|}{\sum_{j=1}^{N} \left| \frac{\partial \lambda_i}{\partial p_j} \right|}, \quad (6)$$

where $\{\lambda_i, i = 1, \cdots, m+n\}$ are the eigenvalues of $\bar{A}(X)$. $\mu_1(X)$ can be computed easily as [7]

$$\frac{\partial \lambda_i}{\partial X} = \begin{bmatrix} \frac{\partial \lambda_i}{\partial p_1} & \cdots & \frac{\partial \lambda_i}{\partial p_{q+n}} \\ \frac{\partial \lambda_i}{\partial p_{q+n+1}} & \cdots & \frac{\partial \lambda_i}{\partial p_{2(q+n)}} \\ \vdots & \cdots & \vdots \\ \frac{\partial \lambda_i}{\partial p_{(l+n-1)(q+n)+1}} & \cdots & \frac{\partial \lambda_i}{\partial p_{(l+n)(q+n)}} \end{bmatrix}$$

$$= M_1^T \mathbf{y}_i^* \mathbf{x}_i^T M_2^T, \quad (7)$$

where $\mathbf{x}_i$ is a right eigenvector related to λ_i and $\mathbf{y}_i$ is the corresponding reciprocal left eigenvector.

Let $B_s^{\min}$ be the minimal word length that can guarantee the closed-loop stability. We can calculate a $\hat{B}_s^{\min}$ based on $\mu_1(X)$ as a super estimate of $B_s^{\min}$. Furthermore, there are different realizations for a given $C(z)$ and $\mu_1(X)$ is a function of the realization. We can search for a realization that maximizes $\mu_1(X)$. Such a realization will be referred to as an optimal realization in the sense that it has maximum stability robustness to FWL effects. The controller implemented with an optimal realization means that the stability of the closed-loop system is guaranteed with a minimum hardware requirement in terms of word length.

3 Optimal realization of PID controller structures

We assume that $C(z)$ is a physically realizable non-interacting PID controller. Let an initial realization of $C(z)$ be $(A_c^0 \in \mathbf{R}^{2\times 2}, B_c^0 \in \mathbf{R}^{2\times 1}, C_c^0 \in \mathbf{R}^{1\times 2}, D_c^0 \in \mathbf{R})$. Then the initial controller matrix is

$$X_0 = \begin{bmatrix} D_c^0 & C_c^0 \\ B_c^0 & A_c^0 \end{bmatrix} \in \mathbf{R}^{3\times 3}. \quad (8)$$

Any realization of $C(z)$ can be represented as

$$X_{\mathcal{T}} = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-1} \end{bmatrix} X_0 \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T} \end{bmatrix}, \quad (9)$$

where $\mathcal{T} \in \mathbf{R}^{2\times 2}$ and $\det(\mathcal{T}) \neq 0$. For the complex-valued matrix

$$M = \begin{bmatrix} M_{1,1} & \cdots & M_{1,n} \\ \vdots & \cdots & \vdots \\ M_{m,1} & \cdots & M_{m,n} \end{bmatrix} \quad (10)$$

define the norm

$$\|M\|_s \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} |M_{i,j}|. \quad (11)$$

The optimal realization of the digital PID controller is the solution of the following optimization problem:

$$\nu = \max_{X_{\mathcal{T}}} \mu_1(X_{\mathcal{T}}) =$$

$$\min_{\det(\mathcal{T})\neq 0} \max_{i \in \{1, \cdots, m+2\}} \left\| \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^T \end{bmatrix} \Phi_i \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-T} \end{bmatrix} \right\|_s, \quad (12)$$

where

$$\Phi_i = \frac{1}{1 - |\lambda_i^0|} \left. \frac{\partial \lambda_i}{\partial X} \right|_{X = X_0} \quad (13)$$

and $\{\lambda_i^0, i = 1, \cdots, m+2\}$ are the eigenvalues of $\bar{A}(X_0)$.

The following theorem shows that (12) can be solved by solving the two "simpler" optimization problems [8],[9].

Theorem. Let

$$\nu_1 = \min_{\substack{x\in(0,+\infty)\\ y\in(-\infty,+\infty)\\ w\in(0,+\infty)}} f_1(x, y, w), \tag{14}$$

where

$$f_1(x, y, w) = \max_{i\in\{1,\cdots,m+2\}} \|\Gamma_i\|_s \tag{15}$$

and

$$\Gamma_i = \begin{bmatrix} w & 0 & 0 \\ 0 & x & 0 \\ 0 & y & \frac{1}{x} \end{bmatrix} \Phi_i \begin{bmatrix} \frac{1}{w} & 0 & 0 \\ 0 & \frac{1}{x} & 0 \\ 0 & -y & x \end{bmatrix}. \tag{16}$$

Let

$$\nu_2 = \min_{\substack{x\in(-\infty,+\infty)\\ y\in(-\infty,+\infty)\\ u\in(0,+\infty)\\ w\in(0,+\infty)}} f_2(x, y, u, w), \tag{17}$$

where

$$f_2(x, y, u, w) = \max_{i\in\{1,\cdots,m+2\}} \|\Lambda_i\|_s \tag{18}$$

and

$$\Lambda_i = \begin{bmatrix} w & 0 & 0 \\ 0 & x & u \\ 0 & \frac{xy-1}{u} & y \end{bmatrix} \Phi_i \begin{bmatrix} \frac{1}{w} & 0 & 0 \\ 0 & y & -u \\ 0 & \frac{1-xy}{u} & x \end{bmatrix}. \tag{19}$$

Then

$$\nu = \min\{\nu_1, \nu_2\}. \tag{20}$$

Specifically, if $\nu = \nu_1$ and $(x_{\text{opt1}}, y_{\text{opt1}}, w_{\text{opt1}})$ is the solution of (14), the solution of (12) is given as:

$$\mathcal{T}_{\text{opt}} = \frac{1}{w_{\text{opt1}}} \begin{bmatrix} x_{\text{opt1}} & y_{\text{opt1}} \\ 0 & \frac{1}{x_{\text{opt1}}} \end{bmatrix}. \tag{21}$$

If $\nu = \nu_2$ and $(x_{\text{opt2}}, y_{\text{opt2}}, u_{\text{opt2}}, w_{\text{opt2}})$ is the solution of (17), the solution of (12) is given as:

$$\mathcal{T}_{\text{opt}} = \frac{1}{w_{\text{opt2}}} \begin{bmatrix} x_{\text{opt2}} & \frac{x_{\text{opt2}}y_{\text{opt2}}-1}{u_{\text{opt2}}} \\ u_{\text{opt2}} & y_{\text{opt2}} \end{bmatrix}. \tag{22}$$

The optimization problems (14) and (17) are highly complex and difficult. A conventional optimization method can only search for local optimal solutions. In order to obtain a global optimal solution, global optimization methods are required.

4 Adaptive simulated annealing

The ASA is an efficient algorithm for solving the following general constrained optimization problem:

$$\min_{\mathbf{w}} J(\mathbf{w}) \tag{23}$$

subject to

$$L_i \le w_i \le U_i, \; i = 1, \cdots, n_J, \tag{24}$$

and

$$a_j \le g_j(\mathbf{w}) \le b_j, \; j = 1, \cdots, m_J, \tag{25}$$

where $\mathbf{w} = [w_1 \cdots w_{n_J}]^T$ is the parameter vector. Fig. 2 shows the flow chart of ASA. Detailed description of the algorithm can be found in [10]–[13].

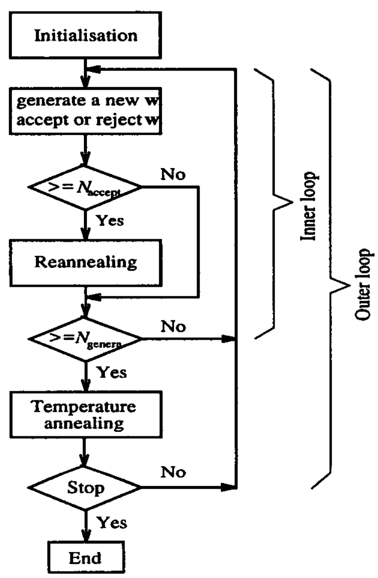


Figure 2: Flow chart of ASA.

The ASA has several important advantages. It can find a global minimum solution, uses only the value of the cost function in the optimization process, is very simple to program, and has very few algorithm parameters that require tuning. The algorithm is very efficient because it uses a very fast annealing schedule and employs a reannealing scheme to adapt itself. These features makes the ASA an ideal method for solving the optimization problems (14) and (17).

5 A numerical example

The approach is applied to design the optimal PID controller for a steel rolling mill system. The continuous-time plant model $P(s)$ is given in [14]. Discretizing $P(s)$

with the sampling period $h = 0.001$ yields $P(z)$:

$$A_z = \begin{bmatrix} 0.9951 & -9.7260 & 0.0049 \\ 0.0010 & 0.9884 & -0.0010 \\ 0.0067 & 13.3732 & 0.9933 \end{bmatrix},$$

$$B_z = \begin{bmatrix} 0.2486 \\ 0.0001 \\ 0.0006 \end{bmatrix}, \; C_z = [1\ 0\ 0]. \qquad (26)$$

A stabilized PID controller for vibration suppression and disturbance rejection is designed and the digital PID controller is given by

$$C(z) = -\frac{0.01426}{z-1} - \frac{1.1956}{z-0.3333} + 1.3512. \qquad (27)$$

The initial realization of $C(z)$ is set to

$$A_c^0 = \begin{bmatrix} 1 & 0 \\ 0 & 0.3333 \end{bmatrix}, \quad B_c^0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$
$$C_c^0 = [0.01426\ 1.1956], \quad D_c^0 = [1.3512]. \quad (28)$$

From $\bar{A}(X_0)$, the poles of the ideal closed-loop system can be computed and are given as:

$$\begin{bmatrix} \lambda_{1,2} \\ \lambda_{3,4} \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} 0.9089 \pm 0.2371i \\ 0.9431 \pm 0.0725i \\ 0.9422 \end{bmatrix}, \qquad (29)$$

where $i = \sqrt{-1}$. The corresponding eigenvalue sensitivity matrices are:

$$\Phi_{1,2} = \begin{bmatrix} 5.3222 \pm 2.4117i & 16.3783 \mp 16.1509i \\ -0.2336 \pm 0.2303i & 0.5165 \pm 1.1835i \\ 7.6870 \pm 8.1766i & 40.9014 \mp 16.6993i \end{bmatrix.$$
$$\begin{matrix} \end{matrix}\begin{bmatrix} -6.4294 \mp 6.8389i \\ 0.4878 \mp 0.1992i \\ -6.4145 \mp 16.8486i \end{bmatrix},$$

$$\Phi_{3,4} = \begin{bmatrix} 0.6130 \pm 6.0505i & 55.7394 \pm 35.2729i \\ -0.7948 \mp 0.5030i & -9.6135 \pm 3.4161i \\ -0.2065 \pm 11.8384i & 99.6482 \pm 81.0121i \end{bmatrix.$$
$$\begin{matrix} \end{matrix}\begin{bmatrix} 0.1727 \mp 9.9017i \\ 1.1885 \pm 0.9662i \\ 2.6112 \mp 19.1031i \end{bmatrix},$$

$$\Phi_5 = \begin{bmatrix} -8.0215 & -138.6951 & 13.1745 \\ 1.9778 & 34.1969 & -3.2483 \\ -15.7514 & -272.3494 & 25.8702 \end{bmatrix}. \quad (30)$$

For the optimization problem (14), starting from a variety of the initial points (x, y, w), the ASA algorithm always converged to the solution: $x_{\text{opt1}} = 2.3704$, $y_{\text{opt1}} = 3.3598$ and $w_{\text{opt1}} = 0.2004$ with $\nu_1 = 136.5897$. The corresponding realization is

$$X_{\text{opt1}} = \begin{bmatrix} 1.3512 & 0.1687 & 2.7560 \\ 0.5888 & 1 & 0.9450 \\ -0.4750 & 0 & 0.3333 \end{bmatrix}. \qquad (31)$$

The evolution of the cost function $f_1(x, y, w)$ in a typical run is shown in Fig.3 (a). It is worth pointing out that in the previous study [8] a conventional optimization method, the Rosenbrock algorithm, failed to find this global optimum. Instead, a local minimum with $\nu_1 = 148.1432$ was found.

For the optimization problem (17), two solutions were found by the ASA, and they are:

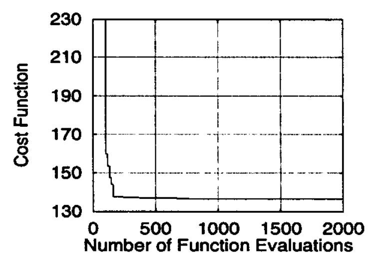
a) $x_{\text{opt2}}^{(1)} = 2.7967$, $y_{\text{opt2}}^{(1)} = 0.1540$, $u_{\text{opt2}}^{(1)} = 0.3512$, $w_{\text{opt2}}^{(1)} = 0.2565$, $\nu_2 = 111.9901$ with

$$X_{\text{opt2}}^{(1)} = \begin{bmatrix} 1.3512 & 1.7925 & 0.6277 \\ -0.4553 & 0.6204 & -0.1664 \\ -0.6273 & -0.6548 & 0.7129 \end{bmatrix}; \qquad (32)$$

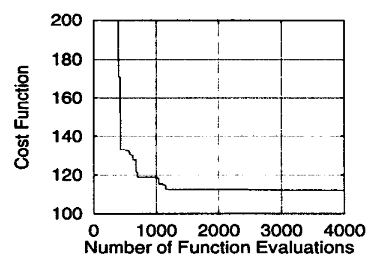
b) $x_{\text{opt2}}^{(2)} = -3.0481$, $y_{\text{opt2}}^{(2)} = -0.1868$, $u_{\text{opt2}}^{(2)} = 0.2895$, $u_{\text{opt2}}^{(2)} = 0.4824$, $\nu_2 = 111.9899$ with

$$X_{\text{opt2}}^{(2)} = \begin{bmatrix} 1.3512 & 0.6274 & -0.5069 \\ -0.6274 & 0.7129 & 0.1852 \\ 1.6101 & 0.5883 & 0.6204 \end{bmatrix}. \qquad (33)$$

The evolution of the cost function $f_2(x, y, u, w)$ in a typical run is shown in Fig.3 (b).



(a)



(b)

Figure 3: Typical convergence performance of the ASA in optimizing: (a) cost function $f_1(x, y, w)$ with initial $(x, y, w) = (1.0, 0.0, 1.0)$ and (b) cost function $f_2(x, y, u, w)$ with initial $(x, y, u, w) = (1.0, 1.0, 1.0, 1.0)$.

Since $\nu = \min\{\nu_1, \nu_2\} = \nu_2$, the optimal PID controller realization is either $X^{(1)}_{\text{opt2}}$ or $X^{(2)}_{\text{opt2}}$. Table 1 summarizes the stability lower bound measures, estimated minimal bit lengths and true minimal bit lengths that can ensure closed-loop stability for different controller realizations, where X_1 and X_2 are the two non-optimal realizations corresponding to $(x, y, w) = (1.0, 0.0, 1.0)$ and $(x, y, u, w) = (1.0, 1.0, 1.0, 1.0)$, respectively. The largest absolute parameter value is 1.6101 for $X^{(2)}_{\text{opt2}}$, and 1.7925 for $X^{(1)}_{\text{opt2}}$. For practical implementation, therefore, $X^{(2)}_{\text{opt2}}$ is preferred over $X^{(1)}_{\text{opt2}}$.

Table 1: Lower stability bounds, estimated minimal bit lengths and true minimal bit lengths for different controller realizations.

Realization	μ_1	$\hat{B}_s^{\min}$	$B_s^{\min}$
X_1	0.001900	10	7
X_{opt1}	0.007321	9	4
X_2	0.000716	11	8
$X^{(1)}_{\text{opt2}}$	0.008929	7	4
$X^{(2)}_{\text{opt2}}$	0.008929	7	4

6 Conclusions

Based on a lower bound measuring stability robustness of sampled-data systems with FWL considerations, the optimal realization of finite-precision PID controller can be interpreted as a nonlinear optimization problem. An efficient global optimization strategy based on the ASA has been developed to solve this FWL optimal realization problem. The theoretical results have been verified using a numerical example of the digital PID controller realization for a steel rolling mill system.

The results presented at this paper can be extended to high-order controllers, Ongoing work will also explore the integration of the proposed optimization procedure with the closed-loop controller performance and the sparseness consideration of optimal controller realizations. The ASA algorithm will offer an effective means for solving such a multi-objective constrained optimization problem.

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