Mixed H_2/l_1 Optimization Problem via Lagrange Multiplier Theory

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Abstract— **The paper considers the mixed H_2/l_1 optimization problem, which minimizes the H_2-norm of the closed loop map while maintaining the l_1-norm of the other closed loop map at a prescribed level. Using the duality theory of Lagrange multipliers and an approximation analysis, an optimization problem in the dual space is constructed which has the same optimal value as the primal mixed H_2/l_1 optimization problem.**

Keywords— **l_1 control, H_2 control, approximate analysis, Lagrange multiplier theory.**

I. Introduction

Most controller synthesis problems can be formulated as follows: Given a plant $\hat{P}$, design a controller $\hat{C}$ such that the closed loop system is stable and satisfies some given (optimal) performance criteria. When the optimal performance criterion is the H_∞-norm (H_2-norm or l_1-norm respectively) of the closed loop transfer function based on the Youla parameterization [1] — a parameterization of the class of controllers which stabilize the plant, the controller synthesis problem can be changed into the H_∞-norm (H_2-norm or l_1-norm respectively) model matching problem — the problem of finding the optimal stable free parameter which minimizes the H_∞-norm (H_2-norm or l_1-norm respectively) of a map of the free parameter. Consequently, H_∞, H_2 and l_1 control designs have been introduced respectively in [2]–[4].

Mixed performance controls, such as mixed H_2/H_∞ control [5], mixed l_1/H_∞ control [6], mixed l_1/H_2 control [7] and mixed H_2/l_1 control [8], have been the pole of attraction for many researchers lately. Mixed performance control can directly accommodate realistic situations where a system must satisfy several different performance constraints. Based on the Youla parameterization, a mixed performance control problem can be changed into a special optimization problem with two kinds of norms.

The topic of this paper is mixed H_2/l_1 control. A scalar discrete time mixed H_2/l_1 control problem was addressed by Voulgaris [8],[9] through minimizing the H_2-norm of the closed loop map while maintaining its l_1-norm at a prescribed level. Based on the duality theory, a finite step method was presented to solve exactly the mixed H_2/l_1 optimization problem. A more general class of discrete time mixed H_2/l_1 control problems, in which Voulgaris' problem is a special case, was addressed in [10],[11]. This class of problems is to minimize the H_2-norm of the closed loop map while maintaining the l_1-norm of the other closed loop map at a prescribed level. An approximation analysis on its solution in primal space was presented in [11],[12]. This paper is devoted to construct the dual problem of this general mixed H_2/l_1 problem.

II. Notation and mathematical preliminaries

Let R denote the field of real numbers, R^m denote the m-dimensional real vectors, and Z_+ denote the nonnegative integers. A causal SISO LTI transfer function $\hat{G}$ can be described as

$$\hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots,$$

where $G(k) \in R$, $\forall k \in Z_+$. As $\hat{G}$ can be represented uniquely by its impulse response sequence $[G(0) \quad G(1) \quad G(2) \quad \cdots]^T$, $\hat{G}$ and its impulse response sequence are not differentiated in notation throughout this paper. Define

$$\begin{aligned} l_e &= \left\{ \hat{G} \,\middle|\, \begin{array}{c} \hat{G} = G(0) + G(1)\lambda + G(2)\lambda^2 + \cdots \\ G(k) \in R, \forall k \in Z_+ \end{array} \right\}, \\ l_\infty &= \left\{ \hat{G} \in l_e \,\middle|\, \sup_k |G(k)| < \infty \right\}, \end{aligned}$$

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$$l_2 = \left\{ \hat{G} \in l_e \left| \sum_{k=0}^{\infty} (G(k))^2 < \infty \right. \right\},$$
$$l_1 = \left\{ \hat{G} \in l_e \left| \sum_{k=0}^{\infty} |G(k)| < \infty \right. \right\}.$$

For any $\hat{G} \in l_\infty$, the l_∞-norm of $\hat{G}$ is given by

$$\|\hat{G}\|_\infty = \sup_k |G(k)| .$$

For any $\hat{G} \in l_2$, the l_2-norm of $\hat{G}$ is

$$\|\hat{G}\|_2 = \sqrt{\sum_{k=0}^{\infty} (G(k))^2}.$$

$\|\hat{G}\|_2$ is also the H_2-norm of $\hat{G}$. For any $\hat{G} \in l_1$, the l_1-norm of $\hat{G}$ is

$$\|\hat{G}\|_1 = \sum_{k=0}^{\infty} |G(k)| .$$

It is easily seen that $l_1 \subset l_2 \subset l_\infty$, and that $\forall \hat{G}_1, \hat{G}_2 \in l_1$, $\hat{G}_1 \hat{G}_2 \in l_1$.

Let X be a normed linear space. The space of all bounded linear functionals on X is denoted as X^*, equipped with the naturally induced norm. For any $x \in X$ and $x^* \in X^*$, $< x, x^* >$ denotes the value of the bounded linear functional x^* at the point x. From standard functional analysis results [13], we have $(R^m)^* = R^m$, $(l_1)^* = l_\infty$ and $(l_2)^* = l_2$. For any $x \in R^{N+1}$ and $x^* \in R^{N+1}$,

$$< x, x^* >= \sum_{k=0}^{N} (x(k) x^*(k)) .$$

For any $x \in l_1$ and $x^* \in l_\infty$ (or for any $x \in l_2$ and $x^* \in l_2$),

$$< x, x^* >= \sum_{k=0}^{\infty} (x(k) x^*(k)) .$$

Given a convex cone P in X, it is possible to define an ordering relation on X as follows: $x_1 \geq x_2$ if and only if $x_1 - x_2 \in P$. The cone P defining this relation on X is called positive. Then it is natural to define a positive cone $P^\oplus$ inside X^* in the following way: $P^\oplus = \{x^* \in X^* \,| < x, x^* > \geq 0, \forall x \in P \}$, This in turn defines an ordering relation on X^*. For any vector space in this paper, the positive cone which defines an ordering relation is the set consisting of elements with nonnegative pointwise components. Let Z be vector spaces with positive cone. A mapping $G : X \to Z$ is convex if $G(tx_1 + (1-t)x_2) \leq tG(x_1) + (1-t)G(x_2)$ for all x_1, x_2 in X and $t \in R$ with $0 \leq t \leq 1$.

Lemma 1 [13]: Let f be a real valued convex functional defined on a convex subset Ω of a vector space X, G be a convex mapping of X into a normed space Z, and $H(x) = Ax - b$ is a map of X into the finite dimensional normed space Y. Suppose that A is linear, that $0 \in Y$ is an interior point of $\{y \in Y \,| H(x) = y \text{ for some } x \in \Omega \}$ and that there exists an $x_1 \in \Omega$ such that $G(x_1) < 0$ and $H(x_1) = 0$. Define the minimization problem:

$$\mu = \inf_{\substack{x \in \Omega \\ G(x) \leq 0 \\ H(x) = 0}} f(x). \tag{1}$$

Assume that μ is finite. Then the dual problem is

$$\mu = \max_{\substack{z^* \in Z^* \\ z^* \geq 0 \\ y^* \in Y^*}} \inf_{x \in \Omega} \left[f(x) + < G(x), z^* > + < H(x), y^* > \right].$$

III. MIXED H_2/l_1 OPTIMIZATION PROBLEM

The mixed H_2/l_1 optimization problem [11],[12] can be stated as: Given $\hat{T}_1 \in l_1$, $\hat{T}_2 \in l_2$,

$$\hat{V}_1 = \begin{bmatrix} V_1(0) & \cdots & V_1(m-1) & 1 \end{bmatrix}^T \in R^{m+1},$$
$$\hat{V}_2 = \begin{bmatrix} V_2(0) & \cdots & V_2(n-1) & 1 \end{bmatrix}^T \in R^{n+1},$$

and a constant γ, find $\hat{Q} \in l_1$ such that $\|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2$ is minimized and $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leq \gamma$.

For the above mixed H_2/l_1 optimization problem, in order to make its feasible region nonempty, it is assumed that $\gamma > \inf_{\hat{Q} \in l_1} \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1$. In addition, we also assume that all the poles of $\hat{V}_1$ are inside the open unit disk in complex plane, i.e., for

$$\prod_{i=1}^{m} (\lambda - \lambda_i) = \lambda^m + V_1(m-1)\lambda^{m-1} + \cdots + V_1(0),$$

$|\lambda_i| < 1, \forall i \in \{1, \cdots, m\}$. Under these assumptions, we have the following lemma.

Lemma 2 [12]: Suppose that $\|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leq \gamma$. Then $\|\hat{Q}\|_1 \leq L$, where $L = \frac{\|\hat{T}_1\|_1 + \gamma}{\prod_{i=1}^{m} (1 - |\lambda_i|)}$.

Define $\xi = \left\{ \hat{Q} \in l_1 \left| \|\hat{T}_1 - \hat{Q}\hat{V}_1\|_1 \leq \gamma, \|\hat{Q}\|_1 \leq L \right. \right\}$. From lemma 2, it is easy to see another description of the mixed H_2/l_1 optimization problem:

$$\mu = \inf_{\hat{Q} \in \xi} \|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2^2 \tag{2}$$

To obtain the dual problem of (2), we first rewrite (2) in the form of (1). Let

$$X = l_2 \times l_1 \times l_1 \times l_1 \times l_1,$$

$$
\begin{aligned}
\Omega &= \left\{ x = \begin{bmatrix} \hat{\Phi} \\ \hat{\Psi}_+ \\ \hat{\Psi}_- \\ \hat{Q}_+ \\ \hat{Q}_- \end{bmatrix} \middle| \begin{array}{c} \hat{\Phi} \in l_2 \\ 0 \le \hat{\Psi}_+ \in l_1 \\ 0 \le \hat{\Psi}_- \in l_1 \\ 0 \le \hat{Q}_+ \in l_1 \\ 0 \le \hat{Q}_- \in l_1 \end{array} \right\}, \\
Y &= l_1 \times l_2, Z = R^2, \\
f(x) &= x^T \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x, \\
G(x) &= \begin{bmatrix} 0 & \mathbf{E}^T & \mathbf{E}^T & 0 & 0 \\ 0 & 0 & 0 & \mathbf{E}^T & \mathbf{E}^T \end{bmatrix} x - \begin{bmatrix} \gamma \\ L \end{bmatrix}, \\
H(x) &= \begin{bmatrix} 0 & \mathbf{I} & -\mathbf{I} & \mathbf{V}_1 & -\mathbf{V}_1 \\ \mathbf{I} & 0 & 0 & \mathbf{V}_2 & -\mathbf{V}_2 \end{bmatrix} x - \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \end{bmatrix} \\
&= Ax - b,
\end{aligned}
$$

where

$$
\hat{\Phi} = \hat{T}_2 - \hat{Q}\hat{V}_2,
$$
$$
\hat{\Psi} = \hat{\Psi}_+ - \hat{\Psi}_- = \hat{T}_1 - \hat{Q}\hat{V}_1,\ \hat{\Psi}_+ \ge 0,\ \hat{\Psi}_- \ge 0,
$$
$$
\hat{Q} = \hat{Q}_+ - \hat{Q}_-,\ \hat{Q}_+ \ge 0,\ \hat{Q}_- \ge 0,\ \hat{Q}_+ + \hat{Q}_- \le L,
$$

$$
\mathbf{I} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix},
$$

$$
\mathbf{V}_1 = \begin{bmatrix} V_1(0) & & & 0 \\ \vdots & V_1(0) & & \\ V_1(m-1) & \vdots & \ddots & \\ 1 & V_1(m-1) & \ddots & \ddots \\ & 1 & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix},
$$

$$
\mathbf{V}_2 = \begin{bmatrix} V_2(0) & & & 0 \\ \vdots & V_2(0) & & \\ V_2(n-1) & \vdots & \ddots & \\ 1 & V_2(n-1) & \ddots & \ddots \\ & 1 & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix},
$$

$$
\mathbf{E} = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^T.
$$

With these definitions, (2) becomes

$$
\mu = \inf_{\substack{x \in \Omega \\ G(x) \le 0 \\ H(x) = 0}} f(x)
$$

which has the same form as (1). However, lemma 1 cannot be applied to (2). This is because $Y = l_1 \times l_2$ is infinite dimensional which does not satisfy the conditions of lemma 1. In this paper, this difficulty in setting up the dual problem of (2) is overcome by considering an approximation of (2).

IV. Two approximation problems and their duals

$\forall N \in Z_+$, define

$$
\xi_{+N} = \left\{ \hat{Q} \in R^{N+1} \middle| \hat{Q} \in \xi \right\}.
$$

The variable N-truncation problem of (2) is constructed as

$$
\mu_{+N} = \inf_{\hat{Q} \in \xi_{+N}} \|\hat{T}_2 - \hat{Q}\hat{V}_2\|_2^2. \tag{3}
$$

The mixed H_2/l_1 problem (2) can be approximated from upper side by the variable N-truncation problem (3), as stated in the following lemma.

Lemma 3 [12]: $\mu_{+0} \ge \mu_{+1} \ge \mu_{+2} \ge \cdots$ and $\lim_{N \to \infty} \mu_{+N} = \mu$.

$\forall N \in Z_+$, define the N-th truncation operator

$$
\Gamma_N : l_e \to R^{N+1}
$$

as

$$
\Gamma_N \hat{G} = G(0) + G(1)\lambda + \cdots + G(N)\lambda^N.
$$

Let

$$
\begin{aligned}
X &= R^{5N+2m+n+5}, \\
\Omega &= \left\{ x = \begin{bmatrix} \hat{\Phi} \\ \hat{\Psi}_+ \\ \hat{\Psi}_- \\ \hat{Q}_+ \\ \hat{Q}_- \end{bmatrix} \middle| \begin{array}{c} \hat{\Phi} \in R^{N+n+1} \\ 0 \le \hat{\Psi}_+ \in R^{N+m+1} \\ 0 \le \hat{\Psi}_- \in R^{N+m+1} \\ 0 \le \hat{Q}_+ \in R^{N+1} \\ 0 \le \hat{Q}_- \in R^{N+1} \end{array} \right\}, \\
Y &= R^{N+m+1} \times R^{N+n+1}, Z = R^2, \\
f(x) &= x^T \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \alpha(N),
\end{aligned}
$$

$$
\begin{aligned}
&G(x) = \\
&\begin{bmatrix} 0 & \mathbf{E}_{N+m}^T & \mathbf{E}_{N+m}^T & 0 & 0 \\ 0 & 0 & 0 & \mathbf{E}_N^T & \mathbf{E}_N^T \end{bmatrix} x - \begin{bmatrix} \gamma - \beta(N) \\ L \end{bmatrix}, \\
&H(x) = \\
&\begin{bmatrix} 0 & I & -I & \mathbf{U}_{1,N+m} & -\mathbf{U}_{1,N+m} \\ I & 0 & 0 & \mathbf{U}_{2,N+n} & -\mathbf{U}_{2,N+n} \end{bmatrix} x - \begin{bmatrix} \Gamma_{N+m}\hat{T}_1 \\ \Gamma_{N+n}\hat{T}_2 \end{bmatrix} \\
&= Ax - b
\end{aligned}
$$

where I denotes the identity matrix with a proper dimension,

$$
\mathbf{U}_{1,N+m} = \begin{bmatrix} V_1(0) & & 0 \\ \vdots & \ddots & \\ V_1(m-1) & \ddots & V_1(0) \\ 1 & \ddots & \vdots \\ & \ddots & V_1(m-1) \\ 0 & & 1 \end{bmatrix}
$$

which is a matrix in $R^{(N+m+1)\times(N+1)}$,

$$\mathbf{U}_{2,N+n} = \begin{bmatrix} V_2(0) & & 0 \\ \vdots & \ddots & \\ V_2(n-1) & \ddots & V_2(0) \\ 1 & \ddots & \vdots \\ & \ddots & V_2(n-1) \\ 0 & & 1 \end{bmatrix}$$

which is a matrix in $R^{(N+n+1)\times(N+1)}$,

$$\begin{aligned} \mathbf{E}_{N+m} &= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in R^{N+m+1}, \\ \mathbf{E}_{N} &= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in R^{N+1}, \\ \alpha(N) &= < \hat{T}_2 - \Gamma_{N+n}\hat{T}_2, \hat{T}_2 - \Gamma_{N+n}\hat{T}_2 >, \\ \beta(N) &= \left\| \hat{T}_1 - \Gamma_{N+m}\hat{T}_1 \right\|_1 . \end{aligned}$$

With these definitions, (3) can be expressed in the form of (1). It is easy to see that $f(x)$ so constructed is a convex functional, Ω is a convex subset, $G(x)$ is a convex map, Y is finite dimensional, A is linear, μ_{+N} is finite, $0 \in Y$ is an interior point of $\{y \in Y \,|H(x) = y \text{ for some } x \in \Omega\}$, there exists $x_1 \in \Omega$ with $G(x_1) < 0$ and $H(x_1) = 0$. Hence, lemma 1 can be applied to derive the dual of the problem (3)

$$\begin{aligned} \mu_{+N} = & \max_{\substack{y_1^* \in R^{N+m+1} \\ y_2^* \in R^{N+n+1} \\ 0 \le z_1^* \in R \\ 0 \le z_2^* \in R}} \inf_{x \in \Omega} \Big[< \hat{\Phi}, \hat{\Phi} > + \alpha(N) + \\ & < \hat{\Psi}_+ - \hat{\Psi}_- - \Gamma_{N+m}\hat{T}_1 + \mathbf{U}_{1,N+m}(\hat{Q}_+ - \hat{Q}_-), y_1^* > \\ & + < \hat{\Phi} - \Gamma_{N+n}\hat{T}_2 + \mathbf{U}_{2,N+n}(\hat{Q}_+ - \hat{Q}_-), y_2^* > \\ & + < \mathbf{E}_{N+m}^T(\hat{\Psi}_+ + \hat{\Psi}_-) - \gamma + \beta(N), z_1^* > \\ & + < \mathbf{E}_N^T(\hat{Q}_+ + \hat{Q}_-) - L, z_2^* > \Big] \\ = & \max_{\substack{y_1^* \in R^{N+m+1} \\ y_2^* \in R^{N+n+1} \\ 0 \le z_1^* \in R \\ 0 \le z_2^* \in R}} \inf_{x \in \Omega} \Big[< \hat{\Phi}, \hat{\Phi} > + < \hat{\Phi}, y_2^* > \\ & + < \hat{\Psi}_+, \mathbf{E}_{N+m} z_1^* + y_1^* > + < \hat{\Psi}_-, \mathbf{E}_{N+m} z_1^* - y_1^* > \\ & + < \hat{Q}_+, \mathbf{E}_N z_2^* + \mathbf{U}_{1,N+m}^T y_1^* + \mathbf{U}_{2,N+n}^T y_2^* > \\ & + < \hat{Q}_-, \mathbf{E}_N z_2^* - \mathbf{U}_{1,N+m}^T y_1^* - \mathbf{U}_{2,N+n}^T y_2^* > \\ & + \alpha(N) - < \gamma - \beta(N), z_1^* > - < L, z_2^* > \\ & - < \Gamma_{N+m}\hat{T}_1, y_1^* > - < \Gamma_{N+n}\hat{T}_2, y_2^* > \Big] . \end{aligned}$$

As $\mu_{+N} \in [0, \infty)$, it is evident that

$$\mathbf{E}_{N+m} z_1^* + y_1^* \ge 0.$$

This is because if $\mathbf{E}_{N+m} z_1^* + y_1^* < 0$, $\hat{\Psi}_+$ can be chosen as such a large value that $\mu_{+N} < 0$, which contradicts the fact that $\mu_{+N} \ge 0$. Similarly, it can be obtained that

$$\begin{aligned} \mathbf{E}_{N+m} z_1^* - y_1^* &\ge 0, \\ \mathbf{E}_N z_2^* + \mathbf{U}_{1,N+m}^T y_1^* + \mathbf{U}_{2,N+n}^T y_2^* &\ge 0, \\ \mathbf{E}_N z_2^* - \mathbf{U}_{1,N+m}^T y_1^* - \mathbf{U}_{2,N+n}^T y_2^* &\ge 0, \end{aligned}$$

and the above infimization is achieved when $\hat{\Psi}_+ = 0$, $\hat{\Psi}_- = 0$, $\hat{Q}_+ = 0$ and $\hat{Q}_- = 0$. Moreover

$$\begin{aligned} & \inf_{x \in \Omega} \Big[< \hat{\Phi}, \hat{\Phi} > + < \hat{\Phi}, y_2^* > + \alpha(N) \\ & - < \gamma - \beta(N), z_1^* > - < L, z_2^* > \\ & - < \Gamma_{N+m}\hat{T}_1, y_1^* > - < \Gamma_{N+n}\hat{T}_2, y_2^* > \Big] \\ & = \inf_{x \in \Omega} \Big[< \hat{\Phi} + \tfrac{y_2^*}{2}, \hat{\Phi} + \tfrac{y_2^*}{2} > - < \tfrac{y_2^*}{2}, \tfrac{y_2^*}{2} > \\ & + \alpha(N) - < \gamma - \beta(N), z_1^* > - < L, z_2^* > \\ & - < \Gamma_{N+m}\hat{T}_1, y_1^* > - < \Gamma_{N+n}\hat{T}_2, y_2^* > \Big] \\ & = \inf_{x \in \Omega} \Big[-\tfrac{1}{4} < y_2^*, y_2^* > + \alpha(N) \\ & - < \gamma - \beta(N), z_1^* > - < L, z_2^* > \\ & - < \Gamma_{N+m}\hat{T}_1, y_1^* > - < \Gamma_{N+n}\hat{T}_2, y_2^* > \Big] . \end{aligned}$$

Consequently, the dual of the variable N-truncation problem is

$$\begin{aligned} \mu_{+N} &= \max \Big[-\frac{1}{4} < y_2^*, y_2^* > - < \Gamma_{N+m}\hat{T}_1, y_1^* > \\ &- < \gamma - \beta(N), z_1^* > - < L, z_2^* > \\ &+ \alpha(N) - < \Gamma_{N+n}\hat{T}_2, y_2^* > \Big] . \qquad (4) \\ \text{s.t.} \quad & -\mathbf{E}_{N+m} z_1^* \le y_1^* \le \mathbf{E}_{N+m} z_1^*, \\ & -\mathbf{E}_N z_2^* \le \mathbf{U}_{1,N+m}^T y_1^* + \mathbf{U}_{2,N+n}^T y_2^*, \\ & \mathbf{U}_{1,N+m}^T y_1^* + \mathbf{U}_{2,N+n}^T y_2^* \le \mathbf{E}_N z_2^*, \\ & y_1^* \in R^{N+m+1}, y_2^* \in R^{N+n+1}, \\ & 0 \le z_1^* \in R, 0 \le z_2^* \in R. \end{aligned}$$

$\forall N \in Z_+$, define

$$\xi_{-N} = \left\{ \hat{Q} \in l_1 \,\Big|\, \|\Gamma_N(\hat{T}_1 - \hat{Q}\hat{V}_1)\|_1 \le \gamma, \|\hat{Q}\|_1 \le L \right\} .$$

Obviously,

$$\xi_{-0} \supset \xi_{-1} \supset \xi_{-2} \supset \cdots \supset \xi.$$

The constraint N-truncation problem of (2) is constructed as

$$\mu_{-N} = \inf_{\hat{Q} \in \xi_{-N}} \left\| \Gamma_N(\hat{T}_2 - \hat{Q}\hat{V}_2) \right\|_2^2 . \qquad (5)$$

The mixed H_2/l_1 problem (2) can be approximated from lower side by this constraint N-truncation problem, as summarized in the following lemma.

Lemma 4 [12]: $\mu_{-0} \le \mu_{-1} \le \mu_{-2} \le \cdots$ and $\lim_{N\to\infty} \mu_{-N} = \mu$.

Using the same method for the dual of the variable N-truncation problem, the dual of the constraint N-truncation problem can be obtained as:

$$
\begin{aligned}
\mu_{-N} \quad &= \quad \max\left[-\frac{1}{4} < y_2^*, y_2^* > \right.\\
&- \quad < \gamma, z_1^* > - < L, z_2^* >\\
&- \quad \left. < \Gamma_N \hat{T}_1, y_1^* > - < \Gamma_N \hat{T}_2, y_2^* > \right]. \qquad (6)\\
\text{s.t.} \qquad & -\mathbf{E}_N z_1^* \leq y_1^* \leq \mathbf{E}_N z_1^*,\\
& -\mathbf{E}_N z_2^* \leq \mathbf{V}_{1N}^T y_1^* + \mathbf{V}_{2N}^T y_2^* \leq \mathbf{E}_N z_2^*,\\
& y_1^* \in R^{N+1}, y_2^* \in R^{N+1},\\
& 0 \leq z_1^* \in R, 0 \leq z_2^* \in R.
\end{aligned}
$$

Here

$$
\mathbf{V}_{1N} = \begin{bmatrix} V_1(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ V_1(N) & \cdots & V_1(0) \end{bmatrix} \in R^{(N+1)\times(N+1)},
$$

$$
\mathbf{V}_{2N} = \begin{bmatrix} V_2(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ V_2(N) & \cdots & V_2(0) \end{bmatrix} \in R^{(N+1)\times(N+1)}.
$$

V. The dual of mixed H_2/l_1 problem

Define

$$
W = \left\{ \omega = \begin{bmatrix} y_1^* \\ y_2^* \\ z_1^* \\ z_2^* \end{bmatrix} \middle| \begin{array}{c} y_1^* \in l_\infty, y_2^* \in l_2 \\ 0 \leq z_1^* \in R, 0 \leq z_2^* \in R \\ -\mathbf{E} z_1^* \leq y_1^* \leq \mathbf{E} z_1^* \\ -\mathbf{E} z_2^* \leq \mathbf{V}_1^T y_1^* + \mathbf{V}_2^T y_2^* \leq \mathbf{E} z_2^* \end{array} \right\}
$$

and

$$
\begin{aligned}
\varphi(\omega) = \qquad & -\frac{1}{4} < y_2^*, y_2^* > - < \gamma, z_1^* > - < L, z_2^* >\\
& - < \hat{T}_1, y_1^* > - < \hat{T}_2, y_2^* > .
\end{aligned}
$$

Construct an infinite dimensional optimization problem in the dual space as:

$$
\upsilon = \sup_{\omega \in W} \varphi(\omega). \qquad (7)
$$

$\forall N \in Z_+$, define W_{+N} as

$$
W_{+N} = \left\{ \omega \middle| \begin{array}{c} y_1^* \in l_\infty, y_2^* \in l_2 \\ 0 \leq z_1^* \in R, 0 \leq z_2^* \in R \\ -\mathbf{E} z_1^* \leq y_1^* \leq \mathbf{E} z_1^* \\ -\Gamma_N\left(\mathbf{E} z_2^*\right) \leq \Gamma_N\left(\mathbf{V}_1^T y_1^* + \mathbf{V}_2^T y_2^*\right) \\ \Gamma_N\left(\mathbf{V}_1^T y_1^* + \mathbf{V}_2^T y_2^*\right) \leq \Gamma_N\left(\mathbf{E} z_2^*\right) \end{array} \right\}.
$$

The constraint N-truncation problem of (7) can be constructed as

$$
\upsilon_{+N} = \sup_{\omega \in W_{+N}} \varphi(\omega). \qquad (8)
$$

The following proposition is a direct consequence of $W_{+N} \supset W$.

Proposition 1: For the problems (7) and (8), $\upsilon_{+N} \geq \upsilon$.

However,

$$
\begin{aligned}
&\sup_{\omega \in W_{+N}} \varphi(\omega) = \sup_{\omega \in W_{+N}} \left[-\tfrac{1}{4} < \Gamma_{N+n} y_2^*, \Gamma_{N+n} y_2^* > \right.\\
&- < \gamma, z_1^* > -\tfrac{1}{4} < y_2^* - \Gamma_{N+n} y_2^*, y_2^* - \Gamma_{N+n} y_2^* >\\
&- < L, z_2^* > - < \Gamma_{N+m} \hat{T}_1, \Gamma_{N+m} y_1^* >\\
&- < \hat{T}_1 - \Gamma_{N+m} \hat{T}_1, y_1^* - \Gamma_{N+m} y_1^* >\\
&- < \Gamma_{N+n} \hat{T}_2, \Gamma_{N+n} y_2^* >\\
&\left. - < \hat{T}_2 - \Gamma_{N+n} \hat{T}_2, y_2^* - \Gamma_{N+n} y_2^* > \right]\\
&= \sup_{\omega \in W_{+N}} \left[-\tfrac{1}{4} < \Gamma_{N+n} y_2^*, \Gamma_{N+n} y_2^* > \right.\\
&- < \gamma, z_1^* > - < L, z_2^* >\\
&- < (y_2^* - \Gamma_{N+n} y_2^*)/2 + (\hat{T}_2 - \Gamma_{N+n} \hat{T}_2),\\
&(y_2^* - \Gamma_{N+n} y_2^*)/2 + (\hat{T}_2 - \Gamma_{N+n} \hat{T}_2) >\\
&+ < \hat{T}_2 - \Gamma_{N+n} \hat{T}_2, \hat{T}_2 - \Gamma_{N+n} \hat{T}_2 >\\
&- < \Gamma_{N+m} \hat{T}_1, \Gamma_{N+m} y_1^* >\\
&\left. + z_1^* \|\hat{T}_1 - \Gamma_{N+m} \hat{T}_1\|_1 - < \Gamma_{N+n} \hat{T}_2, \Gamma_{N+n} y_2^* > \right]\\
&= \sup_{\omega \in W_{+N}} \left[-\tfrac{1}{4} < \Gamma_{N+n} y_2^*, \Gamma_{N+n} y_2^* > + \alpha(N) \right.\\
&- < \gamma - \beta(N), z_1^* > - < L, z_2^* >\\
&\left. - < \Gamma_{N+m} \hat{T}_1, \Gamma_{N+m} y_1^* > - < \Gamma_{N+n} \hat{T}_2, \Gamma_{N+n} y_2^* > \right]
\end{aligned}
$$

and the constraint $\omega \in W_{+N}$ can be changed into

$$
\begin{aligned}
& -\mathbf{E}_{N+m} z_1^* \leq \Gamma_{N+m} y_1^* \leq \mathbf{E}_{N+m} z_1^*,\\
& -\mathbf{E}_N z_2^* \leq \mathbf{U}_{1,N+m}^T(\Gamma_{N+m} y_1^*) + \mathbf{U}_{2,N+n}^T(\Gamma_{N+n} y_2^*)\\
& \mathbf{U}_{1,N+m}^T(\Gamma_{N+m} y_1^*) + \mathbf{U}_{2,N+n}^T(\Gamma_{N+n} y_2^*) \leq \mathbf{E}_N z_2^*\\
& \Gamma_{N+m} y_1^* \in R^{N+m+1}, \Gamma_{N+n} y_2^* \in R^{N+n+1}\\
& 0 \leq z_1^* \in R, 0 \leq z_2^* \in R.
\end{aligned}
$$

Thus, it is verified that the problem (8) is exactly the problem (4), which leads to the following proposition.

Proposition 2: For the problems (4) and (8), $\mu_{+N} = \upsilon_{+N}$.

$\forall N \in Z_+$, define

$$
W_{-N} = \left\{ \omega \middle| \begin{array}{c} y_1^* \in R^{N+1}, y_2^* \in R^{N+1}, z_1^* \in R, \\ z_2^* \in R, z_1^* \geq 0, z_2^* \geq 0, \\ -\mathbf{E} z_1^* \leq y_1^* \leq \mathbf{E} z_1^*, \\ -\mathbf{E} z_2^* \leq \mathbf{V}_1^T y_1^* + \mathbf{V}_2^T y_2^* \leq \mathbf{E} z_2^* \end{array} \right\}.
$$

The variable N-truncation problem of (7) can be constructed as

$$
\upsilon_{-N} = \sup_{\omega \in W_{-N}} \varphi(\omega). \qquad (9)
$$

The following proposition is the direct result of $W_{-N} \subset W$.

Proposition 3: For the problems (7) and (9), $\upsilon_{-N} \leq \upsilon$.

In the same manner, the optimization problem (9) can be transformed into problem (6), and we have the following proposition.

Proposition 4: For the problems (6) and problem (9), $\mu_{-N} = \upsilon_{-N}$.

With lemmas 3–4 and propositions 1–4, we now can summarize the main result of this paper with the following proposition.

Proposition 5: For the problems (2) and (7), $\mu = \upsilon$.

VI. Conclusions

The optimization problem in the dual space sheds new lights on the mixed H_2/l_1 optimization problem. It can be seen that the well-known lemma 1 cannot be applied to the primal problem (2) directly. The idea in verifying the relation between the primal problem (2) and its dual problem (7) is to utilize their corresponding approximation problems (3) and (5) for which lemma 1 can be applied directly. It is interesting to notice that the variable truncation in the primal space becomes the constraint truncation in the dual space, and the constraint truncation in the primal space becomes the variable truncation in the dual space. The dual space approach is useful in research on the mixed H_2/l_1 optimization problem, as it is often that the dual problem can be solved more easily than the primal problem.

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