Mixed μ Robust Finite Word Length Controller Design

Jun Wu[1], Gang Li[2], Sheng Chen[3], Jian Chu[1]

[1]State Key Laboratory of Industrial Control Technology
Institute of Cyber-Systems and Control, Zhejiang University
Hangzhou 310027, P. R. China

[2]College of Information Engineering
Zhejiang University of Technology
Hangzhou 310014, P. R. China

[3]School of Electronics and Computer Science
University of Southampton
Southampton SO17 1BJ, UK

Outline

Outline

Fragility Problem

- Control system designed by maximising its robustness to plant uncertainty alone may exhibit poor stability margin with respect to controller coefficient perturbation
- Two types of finite word length errors in controller implementation are:
  - Rounding errors that occur in arithmetic operations, and
  - Controller parameter representation errors

  These two types of errors are typically investigated separately for mathematical tractability
- We consider second type of FWL errors, which has critically influence on close-loop stability

Existing Approaches

- Two strategies for considering FWL controller parameter representation errors
  - Indirect approach: search for an "optimal" realisation of the given controller that is most robust to FWL errors
  - Direct approach: design controller realisation by considering both robust control criteria and FWL errors
- In literature, direct approach is also referred to as non-fragile, defragile or resilient control
  - Some works assume controller parameter perturbation block is 2-norm bounded
  - More realistic ones assume parameter perturbation is independent and magnitude bounded
- Yang *et al.* [20] design robust FWL H_2 controller by considering all vertices of FWL perturbation hypercube

Outline

Our Contributions

With similar hypothesis to Yang *et al.* [20], we study robust FWL H_∞ output feedback controller, and our contributions are

- FWL robust control performance measure is proposed, which takes into account robust control requirements and FWL effects on controller implementation
- Robust FWL controller design problem is naturally formulated as a mixed μ problem which can be solved effectively with the aid of mixed μ theory
- Our proposed method is computationally more attractive than Yang *et al.* [20]

Outline

Plant Model

- Plant is described by known nominal model $\hat{\mathbf{P}}_g(w)$ and unknown but bounded structured uncertainty $\hat{\mathbf{U}}(w)$, where $w \in \mathbb{C}$
- $\hat{\mathbf{P}}_g(w)$ is given as

$$\begin{aligned}
\mathbf{x}_P(k+1) &= \mathbf{A}_P\mathbf{x}_P(k) + \mathbf{B}_v\mathbf{v}(k) + \mathbf{B}_w\mathbf{w}(k) + \mathbf{B}_P\mathbf{u}_P(k) \\
\mathbf{h}(k) &= \mathbf{C}_h\mathbf{x}_P(k)\mathbf{D}_{1,1}\mathbf{v}(k) + \mathbf{D}_{1,2}\mathbf{w}(k) \\
\mathbf{z}(k) &= \mathbf{C}_z\mathbf{x}_P(k) + \mathbf{D}_{2,1}\mathbf{v}(k) + \mathbf{D}_{2,2}\mathbf{w}(k) + \mathbf{D}_{2,3}\mathbf{u}_P(k) \\
\mathbf{y}_P(k) &= \mathbf{C}_P\mathbf{x}_P(k) + \mathbf{D}_{3,2}\mathbf{w}(k)
\end{aligned}$$

  $\mathbf{x}_P(k) \in \mathbb{R}^n$: state, $\mathbf{v}(k) \in \mathbb{R}^{n_1}$: uncertainty-linked input, $\mathbf{w}(k) \in \mathbb{R}^{n_2}$: external disturbance input, $\mathbf{u}_P(k) \in \mathbb{R}^s$: control input, $\mathbf{h}(k) \in \mathbb{R}^{n_1}$: uncertainty-linked output, $\mathbf{z}(k) \in \mathbb{R}^{n_2}$: controlled output, $\mathbf{y}_P(k) \in \mathbb{R}^t$: measured output

- $\hat{\mathbf{P}}_g(w)$ connects with $\hat{\mathbf{U}}(w)$ through $\mathbf{h}$ and $\mathbf{v}$

$$\mathbf{v} = \hat{\mathbf{U}}(w)\mathbf{h}$$

Structured Uncertainty

- Unknown structured uncertainty $\hat{\mathbf{U}}(w)$ takes the form

$$\hat{\mathbf{U}}(w) = \mathrm{diag}\left(\hat{\mathbf{U}}_1(w), \cdots, \hat{\mathbf{U}}_{b+d}(w)\right)$$

where $\hat{\mathbf{U}}_i(w) = \varphi_i(w)\mathbf{I}_{p_i}$ with $\varphi_i(w) \in \mathbb{C}$, $\forall w \in \mathbb{C}$, $\forall i \in \{1, \cdots, b\}$; and $\hat{\mathbf{U}}_i(w) \in \mathbb{C}^{p_i \times p_i}$, $\forall w \in \mathbb{C}$, $\forall i \in \{b+1, \cdots, b+d\}$, while

$$\sum_{i=1}^{b+d} p_i = n_1, p_i \geq 1$$

- Given a constant $\tau > 0$, $\hat{\mathbf{U}}(w)$ is included in the set

$$\mathcal{H}_\tau \triangleq \left\{ \hat{\mathbf{U}}(w) \;\middle|\; \begin{array}{l} \hat{\mathbf{U}}(w) = \mathrm{diag}\left(\hat{\mathbf{U}}_1(w), \cdots, \hat{\mathbf{U}}_{b+d}(w)\right) \\ \hat{\mathbf{U}}(w) \in \mathcal{F},\ \hat{\mathbf{U}}(w) \text{ is stable},\ \|\hat{\mathbf{U}}(w)\|_\infty < \tau \end{array} \right\}$$

with $\mathcal{F}$: the set of all causal finite linear time-invariant systems

Controller

- Controller $\hat{\mathbf{C}}(w)$ of mth-order is described by

$$\begin{aligned} \mathbf{x}_C(k+1) &= \mathbf{A}_C\mathbf{x}_C(k) + \mathbf{B}_C\mathbf{y}_P(k) \\ \mathbf{u}_P(k) &= \mathbf{C}_C\mathbf{x}_C(k) + \mathbf{D}_C\mathbf{y}_P(k) \end{aligned}$$

and the controller is also denoted by its parameters as

$$\mathbf{X} \triangleq \begin{bmatrix} \mathbf{D}_C & \mathbf{C}_C \\ \mathbf{B}_C & \mathbf{A}_C \end{bmatrix} \in \mathbb{R}^{(s+m)\times(t+m)}$$

- $\mathbf{X}$ is perturbed to $\mathbf{X} + \mathbf{\Delta}$ due to FWL fixed-point implementation, with $\mathbf{\Delta}$ belonging to the hypercube

$$\mathcal{D}_\beta \triangleq \{\mathbf{\Delta} \mid \mathbf{\Delta} \in \mathbb{R}^{(s+m)\times(t+m)}, \|\mathbf{\Delta}\|_\mathrm{m} \le \beta\}$$

where $0 \le \beta \in \mathbb{R}$ is the maximum representation error, $\mathbf{\Delta} = [\delta_{i,j}]$ and $\|\mathbf{\Delta}\|_\mathrm{m} = \max_{i,j} |\delta_{i,j}|$

Closed-Loop System

- Closed-loop system, which consists of $\hat{\mathbf{P}}_g(w)$, $\hat{\mathbf{U}}(w)$, $\mathbf{X}$ and $\boldsymbol{\Lambda}$, is denoted as $\hat{\boldsymbol{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda})$, where $\boldsymbol{\Lambda}$ is equivalent to $\boldsymbol{\Delta}$ as

$$\boldsymbol{\Lambda} \triangleq \mathrm{diag}\left(\delta_{1,1}, \delta_{2,1}, \cdots, \delta_{s+m,1}, \delta_{1,2}, \cdots, \delta_{1,t+m}, \cdots, \delta_{s+m,t+m}\right)$$

$$\boldsymbol{\Lambda} \in \mathcal{O}_\beta \triangleq \{\mathbf{Q} \mid \mathbf{Q} \in \mathbb{R}^{N\times N}, \mathbf{Q} \text{ is diagonal}, \bar{\sigma}(\mathbf{Q}) \le \beta\}$$

  with $\bar{\sigma}(\mathbf{Q})$ denoting the maximum singular value of $\mathbf{Q}$

- Further denote the closed-loop transfer function from $\mathbf{w}(k)$ to $\mathbf{z}(k)$ by $\hat{\boldsymbol{\Phi}}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda})$

- For $0 < \xi \in \mathbb{R}$, the set of all mth-order robust H_∞ controllers, which do not consider FWL effect, is defined by

$$\mathcal{X}_m \triangleq \left\{ \mathbf{X} \,\middle|\, \begin{array}{l} \mathbf{X} \in \mathbb{R}^{(s+m)\times(t+m)}, \hat{\boldsymbol{\Phi}}(w, \hat{\mathbf{U}}(w), \ \mathbf{X}, \mathbf{0}) \text{ is stable,} \\ \forall \hat{\mathbf{U}}(w) \in \mathcal{H}_\tau, \ \|\hat{\boldsymbol{\Phi}}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{0})\|_\infty \le \xi \end{array} \right\}$$

Outline

Theoretical Measure

- For a controller $\mathbf{X} \in \mathcal{X}_m$, the FWL robust measure

$$v(\mathbf{X}) \triangleq \sup_{0 \le \beta \in \mathbb{R}} \left\{ \beta \;\middle|\; \begin{array}{l} \forall \hat{\mathbf{U}}(w) \in \mathcal{H}_\tau,\ \hat{\mathbf{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda}) \text{ is stable,} \\ \forall \mathbf{\Lambda} \in \mathcal{O}_\beta,\ \|\hat{\mathbf{\Phi}}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda})\|_\infty \le \xi \end{array} \right\}$$

  charaterises “robustness” of $\mathbf{X}$ to controller perturbation $\mathbf{\Lambda}$

  - $\mathcal{H}_\tau$ is the set of structured uncertainty
  - $\mathcal{O}_\beta$ defines FWL perturbation hypercube
  - $\hat{\mathbf{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda})$ is the whole closed-loop system
  - $\hat{\mathbf{\Phi}}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda})$ is the closed-loop transfer function from external perturbation input $\mathbf{w}(k)$ to controlled output $\mathbf{z}(k)$

- However, how to compute the value of $v(\mathbf{X})$ is unknown
- With aid of mixed μ theorem, we derive a tractable lower bound for $v(\mathbf{X})$

Mixed μ

- "Substitute out" $\hat{\mathbf{U}}(w)$ from $\hat{\mathbf{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{\Lambda}) \Rightarrow$ composite system of $\hat{\mathbf{P}}_g(w)$, $\mathbf{X}$ and $\mathbf{\Lambda}$, described by:

$$
\begin{aligned}
\mathbf{x}_{PC}(k+1) &= \left(\overline{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \mathbf{\Lambda} \mathbf{C}_u\right) \mathbf{x}_{PC}(k) + \mathbf{B}_{\overline{v}} \mathbf{v}(k) + \overline{\mathbf{B}}(\mathbf{X}) \mathbf{w}(k) \\
\mathbf{h}(k) &= \mathbf{C}_{\overline{h}} \mathbf{x}_{PC}(k) + \mathbf{D}_{1,1} \mathbf{v}(k) + \mathbf{D}_{1,2} \mathbf{w}(k) \\
\mathbf{z}(k) &= \overline{\mathbf{C}}(\mathbf{X}) \mathbf{x}_{PC}(k) + \mathbf{D}_{2,1} \mathbf{v}(k) + \overline{\mathbf{D}}(\mathbf{X}) \mathbf{w}(k)
\end{aligned}
$$

- Define the matrix

$$
\mathbf{\Theta}(\mathbf{X}, \beta) \triangleq \begin{bmatrix}
\overline{\mathbf{A}}(\mathbf{X}) & \mathbf{B}_u & \mathbf{B}_{\overline{v}} & \overline{\mathbf{B}}(\mathbf{X}) \\
\beta \mathbf{C}_u & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\tau \mathbf{C}_{\overline{h}} & \mathbf{0} & \tau \mathbf{D}_{1,1} & \tau \mathbf{D}_{1,2} \\
\frac{1}{\xi} \overline{\mathbf{C}}(\mathbf{X}) & \mathbf{0} & \frac{1}{\xi} \mathbf{D}_{2,1} & \frac{1}{\xi} \overline{\mathbf{D}}(\mathbf{X})
\end{bmatrix}
$$

  and the related set of allowable perturbations $\mathcal{K}_\theta$

- We can obtain a computable mixed μ: $\alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X}, \beta))$

Tractable Measure

- **Result**: $\exists\ 0 \le \beta \in \mathbb{R}$ such that $\alpha_{\mathcal{K}_\theta}(\boldsymbol{\Theta}(\mathbf{X}, \beta)) < 1$, then $\mathbf{X} \in \mathcal{X}_m$ and $\forall \hat{\mathbf{U}}(w) \in \mathcal{H}_\tau$, $\forall \boldsymbol{\Lambda} \in \mathcal{O}_\beta$
$$\hat{\boldsymbol{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda}) \text{ is stable, } \|\hat{\boldsymbol{\Phi}}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda})\|_\infty \le \xi$$
- Define a subset of $\mathcal{X}_m$ as
$$\widetilde{\mathcal{X}}_m \triangleq \{\mathbf{X} \mid \mathbf{X} \in \mathbb{R}^{(s+m)\times(t+m)}, \alpha_{\mathcal{K}_\theta}(\boldsymbol{\Theta}(\mathbf{X}, 0)) < 1\}$$
- For $\mathbf{X} \in \widetilde{\mathcal{X}}_m$, the FWL robust measure
$$\widetilde{\upsilon}(\mathbf{X}) \triangleq \sup_{0 \le \beta \in \mathbb{R}} \{\beta \mid \alpha_{\mathcal{K}_\theta}(\boldsymbol{\Theta}(\mathbf{X}, \beta)) < 1\}$$
is a lower bound of $\upsilon(\mathbf{X})$
- $\widetilde{\upsilon}(\mathbf{X})$ can be computed using combined linear matrix inequality technique and bisection search

Outline

Design Problem

- Robust FWL controller design: given $\hat{\mathbf{P}}_g(w)$, τ, ξ, m and nonempty $\widetilde{\mathcal{X}}_m$, find a controller $\mathbf{X}_{\text{opt}} \in \widetilde{\mathcal{X}}_m$ that achieves

$$\gamma = \sup_{\mathbf{X} \in \widetilde{\mathcal{X}}_m} \widetilde{\upsilon}(\mathbf{X})$$

  - This design makes the FWL tolerance as large as possible, while satisfying a suboptimal robust control requirement
- This robust FWL controller design can be solved with aid of bilinear matrix inequality
- Complexity comparison with Yang *et al.* [20]
  - Our FWL robust H_∞ controller design solves one BMI of size $2(n + m + N + n_1 + n_2)$
  - FWL robust H_2 controller design [20] requires to solve at least 2^N BMIs of size no less than $4n$

Outline

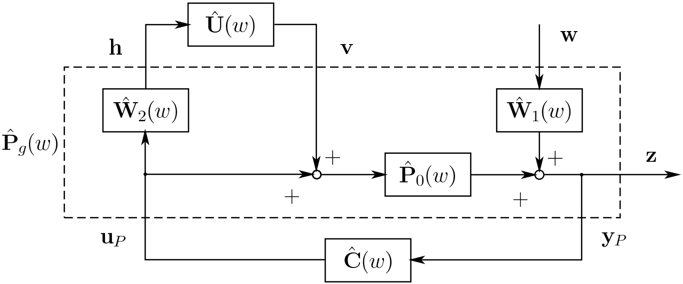
Design Problem

- Nominal plant model $\hat{\mathbf{P}}_g(w)$ is given by

$$\hat{\mathbf{P}}_0(w) = \frac{3.3750 \times 10^{-3} w + 1.3669 \times 10^{-2} w^2 + 3.4605 \times 10^{-3} w^3}{1 - 3.0488 w + 3.1001 w^2 - 1.0513 w^3},$$

$$\hat{\mathbf{W}}_1(w) = \frac{4.9875 \times 10^{-3} w}{1 - 9.9501 \times 10^{-1} w}, \quad \hat{\mathbf{W}}_2(w) = \frac{5.8512 \times 10^{-1} w - 5.5933 \times 10^{-1} w^2}{1 - 1.3390 w + 3.7908 \times 10^{-1} w^2}$$

- Plant model uncertainty $\hat{\mathbf{U}}(w) \in \mathcal{H}_\tau$ with $\tau = 0.4$



Design Solution

- Constant that bounds closed-loop H_∞ norm from **w** to **z** was set to $\xi = 0.3$, and controller order was chosen to be $m = 2$
- Solving optimal FWL robust design problem yields the controller

$$\mathbf{X}_{\mathrm{opt1}} = \left[\begin{array}{c|cc} -103.44 & -15.600 & -1.4984 \\ \hline -16.070 & -1.4261 & 0.25055 \\ -19.469 & -3.0400 & 0.37517 \end{array}\right]$$

with $\widetilde{\upsilon}(\mathbf{X}_{\mathrm{opt1}}) = 8.2842 \times 10^{-3}$

- For any FWL perturbation to $\mathbf{X}_{\mathrm{opt1}}$ smaller than 8.2842×10^{-3} and for any $\hat{\mathbf{U}}(w) \in \mathcal{H}_\tau$ with $\tau = 0.4$,
  - the closed-loop system maintains stability, and
  - closed-loop H_∞ norm from **w** to **z** is always less than 0.3

Bit Length Estimate

- Using fixed point processor of c-bit length to implement $\mathbf{X}$, c bits are assigned as:
  - 1 sign bit, c_{int} bits for integer part, c_{fra} bits for fraction part
- To guarantee dynamic range of $\mathbf{X}$, $c_{int} = \lceil \log_2 \|\mathbf{X}\|_{\mathrm{m}} \rceil$,
- Fraction bit length bounds the absolute values of FWL errors by $2^{-(c_{fra}+1)}$, and to maintain closed-loop performance, at least

$$c_{fra} = \lceil -\log_2 \widetilde{\upsilon}(\mathbf{X}) \rceil - 1$$

- Minimal bit length guaranteeing closed-loop performance, estimated based on $\widetilde{\upsilon}(\mathbf{X})$, is

$$\widetilde{c}(\mathbf{X}) \triangleq \lceil \log_2 \|\mathbf{X}\|_{\mathrm{m}} \rceil + \lceil -\log_2 \widetilde{\upsilon}(\mathbf{X}) \rceil$$

In this example, $\widetilde{c}(\mathbf{X}_{\mathrm{opt1}}) = 14$

Outline

Design Problem

- Example from Yang *et al.* [20] was for FWL H_2 control under plant parameter uncertainty
- Noting $\|\hat{\mathbf{\Phi}}_{wz}\|_\infty \geq \|\hat{\mathbf{\Phi}}_{wz}\|_2$ and structured uncertainty includes parameter uncertainty, we substituted $\|\hat{\mathbf{\Phi}}_{wz}\|_\infty$ for $\|\hat{\mathbf{\Phi}}_{wz}\|_2$ and plant structured uncertainty for plant parameter uncertainty
- Nominal plant model $\hat{\mathbf{P}}_g(w)$ is defined by

$$\mathbf{A}_P = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \mathbf{B}_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{B}_w = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{B}_P = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{C}_h = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{C}_z = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{D}_{2,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{D}_{2,3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{C}_P = \begin{bmatrix} 0 & -1 \end{bmatrix}, \mathbf{D}_{3,2} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \mathbf{D}_{1,1} = \mathbf{D}_{1,2} = \mathbf{D}_{2,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Plant structure uncertainty is defined by

$$\hat{\mathbf{U}}(w) = \varphi(w) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{H}_\tau \text{ with } \varphi(w) \in \mathbb{C} \text{ and } \tau = 0.13$$

Design Solution

- Set constant $\xi = 4.9676$. We designed 1st-order controller by solving the FWL robust design problem, leading to

$$\mathbf{X}_{\text{opt2}} = \left[\begin{array}{c|c} 1.0853 & -0.36600 \\ \hline 1.1031 & -0.34734 \end{array}\right]$$

  with $\widetilde{\upsilon}(\mathbf{X}_{\text{opt2}}) = 0.0275$, which can be implemented with a fixed point processor of $\widetilde{c}(\mathbf{X}_{\text{opt2}}) = 7$ bits

- As $\|\hat{\mathbf{\Phi}}_{wz}\|_\infty \geq \|\hat{\mathbf{\Phi}}_{wz}\|_2$, system was guaranteed to be closed-loop stable and $\|\hat{\mathbf{\Phi}}_{wz}\|_2 < 4.9676$ when $\tau = 0.13$ and the FWL bound was 0.0275
- Yang *et al.* [20] obtained a controller achieving $\|\hat{\mathbf{\Phi}}_{wz}\|_2 < 3.0822$ when $\tau = 0.13$ and the FWL bound 0.0275
- Our method required to solve one BMI of size 22, while Yang *et al.* [20] required to solve 32 BMIs of size 8

Conclusions

We have used mixed μ theory to directly design optimal robust FWL controllers, and our novel contributions include:

- A robust FWL control performance measure taking into account both robust control requirements and FWL implementation considerations
- This robust FWL control performance measure can be computed conveniently using LMI
- Optimal robust FWL controller design is formulated as a mixed μ problem, which can be solved by means of BMI

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