Optimum Delay order selection for Linear Equalization Problems

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Abstract: This paper shows that the BER performance using linear equalizer for channel equalization problem is significantly dependent on delay order. To obtain optimum performance, the equalizer output should be derived from the equalizer with delay order having the best BER performance. An efficient method to evaluate the upper bound BER performance of a linear equalizer to find the optimum delay is proposed. The method is novel as the evaluation is performed using only the channel statistics and the equalizer's weights.

Key words: Linear equalizers, delay order, MMSE, MBER, BER

1 Introduction

The transmission of digital signal across a communication channel is subjected to noise and Inter Symbol Interference (ISI). At the receiver, these effects must be compensated by an equalizer to achieve reliable data communications [1, 2]. We consider a linear discrete-time communication channel depicted in Fig. 1, whose output is given by

$$x(k) = \sum_{i=0}^{M} h_i s(k-i) + n(k) \tag{1}$$

where k denotes sample index, $n(k)$ is a white Gaussian noise with variance σ^2, h_i are the taps of the Channel Impulse Response (CIR) which has a memory M, and $s(k)$ is a binary input drawn from the symbol set $\{\pm 1\}$. Although the analysis presented assumes the case of binary transmit symbols, the results can be generalized to more complicated transmit signal sets.

The purpose of the equalizer is to use a vector of noisy observation $\mathbf{x}(k) = [x(k)\ x(k-1) \cdots x(k-N+1)]^T \in R^{N\times 1}$ to estimate $s(k-d)$, where N denotes the equalizer input length and d the decision delay order. The vector $\mathbf{x}(k)$ is given by

$$\mathbf{x}(k) = \mathcal{H}\mathbf{s}(k) + \mathbf{n}(k) = \hat{\mathbf{x}}(k) + \mathbf{n}(k) \tag{2}$$

where $\hat{\mathbf{x}}(k) \in R^{N\times 1}$ is the vector of noise-free input signal known as the channel state, $\mathbf{n}(k) \in R^{N\times 1}$ is the noise vector, $\mathbf{s}(k) = [s(k)\ s(k-1) \cdots s(k-L+1)]^T \in R^{L\times 1}$ is the vector of $L = M+N$ transmitted digital symbols, and $\mathcal{H} \in R^{N\times L}$ is the channel convolution matrix given by

$$\mathcal{H} = \begin{bmatrix} h_0 & h_1 & \cdots & h_m & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_m & 0 & \cdots \vdots \\ \vdots & \ddots & \ddots & \ddots & & & 0 \\ 0 & 0 & & h_0 & h_1 & \cdots & h_m \end{bmatrix} \tag{3}$$

It is obvious from Eqn.(2) that $\mathbf{x}(k)$ depends only on L symbols in $\mathbf{s}(k)$ and hence the valid range of delay order is $d \in D = \{0, 1, \cdots, L-1\}$. It is well known that the choice of d can affect BER performance significantly [3, 4].

This paper focuses on the effect of delay order on BER performance and presents an efficient scheme to evaluate the upper bound BER performance for a given set of equalizers to select the optimal delay.

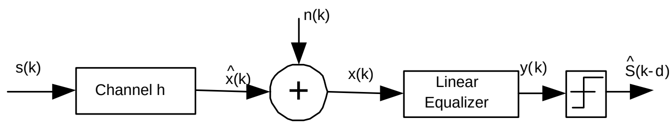

Figure 1. Model of the channel equalization problem

2 The set of Linear Equalizers

The output of a linear equalizer with weights $\mathbf{w} \in R^{N\times 1}$ is

$$y(k) = \mathbf{w}^T \mathbf{x}(k) \tag{4}$$

For each delay order, the weight vector $\mathbf{w}$ can be evaluated with respect to some criteria, e.g. the MMSE solution or the MBER solution [7, 8]. We define the set of weights for equalizers as

$$\mathcal{W} = [\mathbf{w}_1 \mathbf{w}_2 \ldots \mathbf{w}_L] \in R^{N\times L} \tag{5}$$

where the weights for delay $d \in D$ equalizer is $\mathbf{w}_{d+1} \in R^{N\times 1}$.

2.1 Linear or nonlinear separability

Not all the delay order will result in the equalization problem being linearly separable [3, 6]. For a nonlinearly separable problem, the corresponding solution in Eqn.(5) is ineffective and a nonlinear approach [6] should be used.

Let us examine Eqn.(2) as a matrix operation,

$$\hat{\mathcal{X}} = \mathcal{H}\mathcal{S} + \mathcal{N} \tag{6}$$

where $\hat{\mathcal{X}} \in R^{N \times N_s}$ has $N_s = 2^L$ columns of channel state vectors, specifically $\mathcal{X} = [\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \cdots, \hat{\mathbf{x}}_{N_s}]$, and $\mathcal{S} \in R^{L \times N_s}$ with corresponding N_s columns of distinct transmission sequence, specifically $\mathcal{S} = [\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_{N_s}]$ and $\hat{\mathcal{N}} \in R^{N \times N_s}$ is the noise matrix. For the noiseless case, i.e. $\mathcal{N} = 0$, the estimate of $\mathcal{S}$ given $\mathcal{W}$ is

$$\hat{\mathcal{S}} = \mathcal{W}^T \hat{\mathcal{X}} \tag{7}$$

To verify if the equalization problem for delay d is linearly separable, one can evaluate for $d \in D$

$$\psi(d) = \sum_{i=1}^{N_s} |\text{sgn}(\hat{s}_{d+1,i}) - \text{sgn}(s_{d+1,i})| \tag{8}$$

where $| \bullet |$ is the absolute value operator, and $\hat{s}_{h,i}$ and $s_{h,i}$ are the elements at h^{th} row and i^{th} column of $\hat{\mathcal{S}}$ and $\mathcal{S}$, respectively. If $\psi(d) = 0$, the equalization problem for delay d resulted in the correct classification of all the transmitted symbols in the noise free case, otherwise it did not.

Eqn.(8) shows a direct way to examine linear separability which involves evaluating $\hat{\mathcal{S}}$ and comparing all its elements to $\mathcal{S}$. This procedure is inefficient as it is computationally intensive while only yielding information regarding linear or nonlinear separability. To extract more quantitative information, the following subsection examines an efficient method to evaluate the upper bound BER performance of the equalizer using only the matrix $\mathcal{H}$ and $\mathcal{W}$.

3 Bit error rate and Delay order

Given the equalizer's weight vector $\mathbf{w}$, the BER of the equalizer for a fixed delay order d can be evaluated by [7, 8]

$$\text{BER}(d, \mathbf{w}) = \frac{1}{N_s} \sum_{i=1}^{N_s} p_e(\hat{\mathbf{x}}_i; d, \mathbf{w}) \tag{9}$$

where $p_e(\hat{\mathbf{x}}_i; d, \mathbf{w})$ denotes the probability of error due to the received channel state being $\hat{\mathbf{x}}_i$, and is evaluated by

$$p_e(\hat{\mathbf{x}}_i; d, \mathbf{w}) = \begin{cases} Q\left(\frac{|\zeta_{i,\mathbf{w}}|}{\sigma}\right), & \hat{\mathbf{x}}_i \text{ correctly classified} \\ 1 - Q\left(\frac{|\zeta_{i,\mathbf{w}}|}{\sigma}\right), & \text{otherwise} \end{cases} \tag{10}$$

where $Q(\bullet)$ is the Gaussian error function [2], $\zeta_{i,\mathbf{w}}$ is the distance of the channel state $\hat{\mathbf{x}}_i$ to the decision boundary and is given by

$$\zeta_{i,\mathbf{w}} = \frac{|\mathbf{w}^T \hat{\mathbf{x}}_i|}{\|\mathbf{w}\|} \tag{11}$$

The vector multiplication $\mathbf{w}^T \hat{\mathbf{x}}_i$ is the response of the linear equalizer and $\|\mathbf{w}\| = \sqrt{\mathbf{w}^T \mathbf{w}}$. For the channel state $\hat{\mathbf{x}}_i$ to be classified correctly, the output of the equalizer must satisfy

$$\text{sgn}(\mathbf{w}^T \hat{\mathbf{x}}_i) = \text{sgn}(s_{d+1,i}) \tag{12}$$

Eqn.9 shows that the BER performance is the average probability of error for all the channel states. However, since the function $Q(\bullet)$ decays exponentially, the BER performance is dominated by the largest $p_e(\hat{\mathbf{x}}_i; d, \mathbf{w})$ when $\sigma \to 0$. Hence, an upper bound for the BER performance is

$$\text{BER}_{UB}(d, \mathbf{w}) = \max_{1 \le i \le N_s} \{p_e(\hat{\mathbf{x}}_i; d, \mathbf{w})\} \tag{13}$$

Therefore the evaluation of $\text{BER}_{UB}(d, \mathbf{w}_{d+1})$ will indicate which delay will result in the best BER performance.

3.1 Distance of channel state to decision boundary

When the equalization problem is linearly separable, all the channel states can be classified correctly. In this case, the probability of error is most affected by the channel state nearest to the decision boundary, i.e. the channel state with the minimum $\zeta_{i,\mathbf{w}}$, and hence the largest $\{p_e(\hat{\mathbf{x}}_i; d, \mathbf{w})\}$.

To find this nearest distance, the direct approach is to evaluate all the distance and find its minimum, i.e.

$$\{\zeta_{i,\mathbf{w}}\}_{i=1}^{N_s} = \frac{1}{\|\mathbf{w}\|} \mathbf{w}^T \hat{\mathcal{X}} = \left\{ \frac{1}{\|\mathbf{w}\|} \mathbf{w}^T \hat{\mathbf{x}}_i \right\}_{i=1}^{N_s} \tag{14}$$

An alternative approach is this:
Substituting $\hat{\mathcal{X}} = \mathcal{H}\mathcal{S}$ and letting $\mathbf{p}^T = [p_1 \; p_2 \cdots p_L] = \mathbf{w}^T \mathcal{H}$ in Eqn.14 yields

$$\{\zeta_{i,\mathbf{w}}\}_{i=1}^{N_s} = \frac{1}{\|\mathbf{w}\|} \mathbf{p}^T \mathcal{S} = \left\{ \frac{1}{\|\mathbf{w}\|} \mathbf{p}^T \mathbf{s}_i \right\}_{i=1}^{N_s} \tag{15}$$

The correct classification criterion in Eqn.12 also implies

$$\text{sgn}(\zeta_{i,\mathbf{w}}) = \text{sgn}(s_{d+1,i}) \tag{16}$$

For the above condition to be satisfied, Eqn.15 shows that

$$|p_{d+1}| > \sum_{\substack{j \neq d+1 \\ j = 1, \cdots, L}} |p_j| \tag{17}$$

and the minimum distance of $\{\zeta_{i,\mathbf{w}}\}_{i=1}^{N_s}$ for delay order d is therefore

$$\begin{aligned} \lambda(d, \mathbf{w}) &= \min_{1 \le i \le N_s} \{\zeta_{i,\mathbf{w}}\} \\ &= \frac{1}{\|\mathbf{w}\|} \left(|p_{d+1}| - \sum_{\substack{j \neq d+1 \\ j = 1, \cdots, L}} |p_j| \right) \end{aligned} \tag{18}$$

A positive value of $\lambda(d, \mathbf{w})$ indicates that the equalization problem is linearly separable and its magnitude is the distance of the nearest channel state to the decision boundary.

A negative value indicates that the equalization problem is not linearly separable and its magnitude is the distance of the nearest wrongly classified channel state to the decision boundary. Hence, $\lambda(d, \mathbf{w})$ measures the degree of linear separability quantitatively. A negative $\lambda(d, \mathbf{w})$ with a larger magnitude means that nonlinear separability is more severe, and a larger positive value of $\lambda(d, \mathbf{w})$ indicates that the channel states are located further away from the linear decision boundary which implies better BER performance.

3.2 Selecting optimal delay

The following steps list the operations required to find optimum delay,

1. Evaluate $\mathcal{W}$ using MMSE or MBER criteria.
2. Evaluate $\lambda(d, \mathbf{w}_d)$.
3. The optimum delay with corresponding weights is

$$d^* = \arg \max_{d \in D} \{\lambda(d, \mathbf{w}_d)\} \tag{19}$$

The output of the equalizer is the decision from the equalizer with the optimum delay. Since the optimum delay order could be any value of D, and variable delay in decision output is undesirable, each equalizer's output is delayed by z^{L-1-d^*} to produce estimates of $s(k-L+1)$ simultaneously. Hence, regardless the value of d^*, the estimated output is always $s(k-L+1)$.
To illustrate the proposal, consider the equalization problem with the transfer function of the CIR given by $H_0(z) = 0.5 + 1.0z^{-1}$ and an equalizer length $N = 2$. Since $L = 3$, the valid delay orders are $d \in D = \{0, 1, 2\}$. Fig 2 illustrates the implementation. The normalized soft output of each equalizer with delay d for input $\mathbf{x}(k)$ is

$$f_d(\mathbf{x}(k)) = \frac{\mathbf{w}_d^T \mathbf{x}(k)}{\|\mathbf{w}_d\|} \tag{20}$$

and the output of the optimal delay equalizer (Fig 2)is

$$\hat{s}(k-L+1) = \text{sgn}(z^{L-1-d^*} f_{d^*}(\mathbf{x}(k))) \tag{21}$$

4 Simulation results

The following two sets of simulation results are presented.

Channel $H_0(z) = 0.5 + 1.0z^{-1}$, $N = 2$ and $d \in D = \{0, 1, 2\}$. The channel convolution matrix is

$$\mathcal{H} = \begin{bmatrix} 0.5 & 1.0 & 0.0 \\ 0.0 & 0.5 & 1.0 \end{bmatrix} \tag{22}$$

The weights were evaluated using the MMSE criterion. Using Eqn.(18), the evaluated $\lambda(d, \mathbf{w}_d)$ are $\lambda(0, \mathbf{w}_0) = -0.65$, $\lambda(1, \mathbf{w}_1) = 0.43$ and $\lambda(2, \mathbf{w}_2) = 0.65$. The results indicate that the delay $d = 0$ results in a nonlinearly separable equalization problem and $d^* = 2$ will produce the best BER performance. To confirm these predictions, simulation was

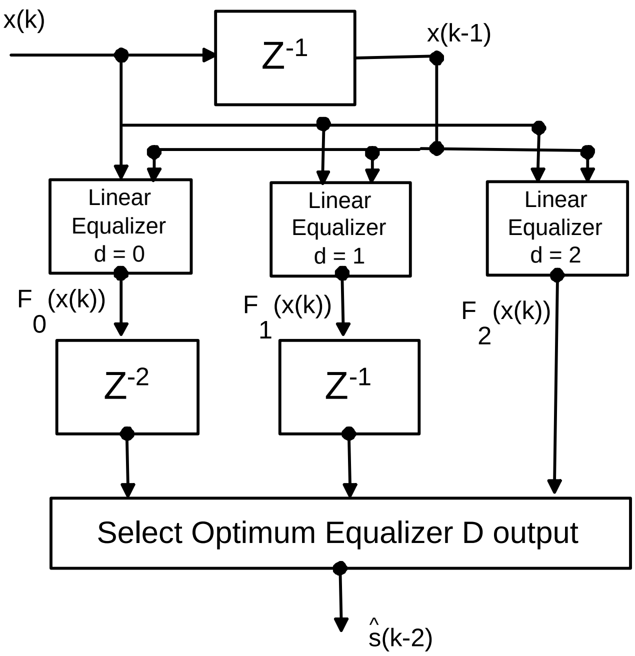

Figure 2. Schematic of combined linear equalizer for $H_0(z) = 0.5 + 1.0z^{-1}$, $N = 2$ and $L = 3$

conducted to evaluate BER performance using Eqn.(9), and the results are illustrated in Fig. 3. The results show that prediction by the value of $\lambda(d)$, namely $d = 0$ results in a nonlinearly separable equalization problem and $d = 2$ is the optimum delay order.

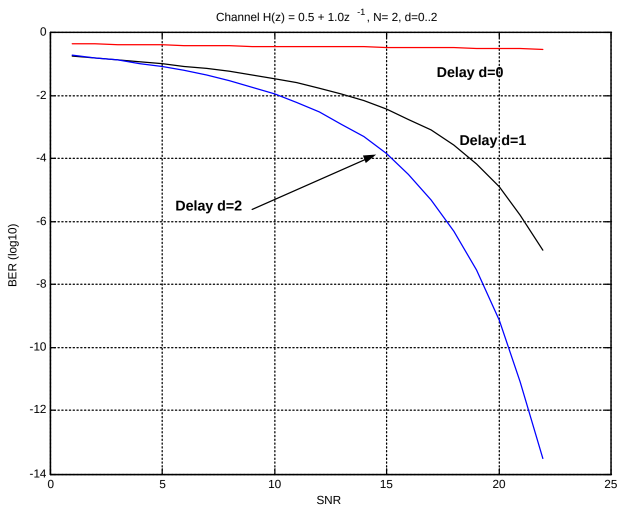

Figure 3. Simulation results for Channel $H_0(z) = 0.5 + 1.0z^{-1}$ with $N = 2$

Channel $H_2(z) = 0.6996 + 0.6646z^{-1} - 0.2623z^{-2}$, $N = 6$ and $d \in D = \{0, 1, \cdots, 7\}$. Using Eqn.(18), the evaluated $\lambda(d, \mathbf{w}_d)$ are $[\lambda(0, , \mathbf{w}_0)\ \lambda(1, \mathbf{w}_1) \cdots \lambda(7, \mathbf{w}_7)] = [-0.46, -0.24, 0.04, 0.22, 0.32, 0.37, 0.14, -0.87]$. The results indicate that $d = 0, 1, 7$ result in nonlinearly separable equalization problems and the optimal delay order is $d^* = 5$.

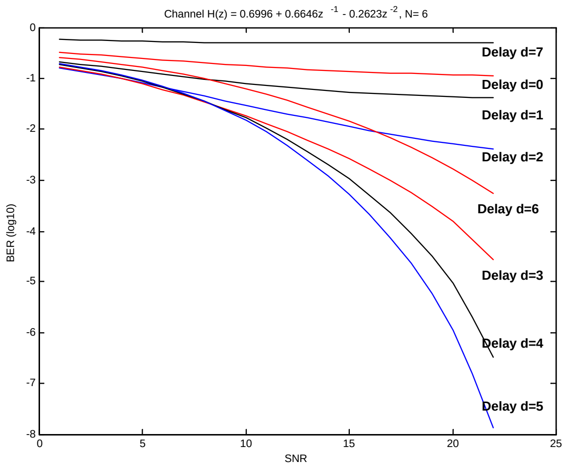

Figure 4. Simulation results for channel $H_2(z) = 0.6996 + 0.6646z^{-1} - 0.2623z^{-2}$ with $N = 6$

To confirm these predictions, simulation was conducted and the results illustrated in Fig. 4 agree with the predicted BER performance. As in the previous example, the MMSE cost functions was used to evaluate $\mathcal{W}$.

5 Conclusions

A simple and efficient method has been presented to evaluate quantitatively if a selected delay order will result in a linearly separable equalization problem. This provides a technique to determine the optimal delay order for the linear equalizer.

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