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Symmetric Radial Basis Function Network Equaliser

S. Chen[†], A. Wolfgang[†], S. Benedetto[‡], P. Duhanel[§] and L. Hanzo[†]

[†] **School of ECS, University of Southampton, UK**

[‡] **Dipartimento di Elettronica, Politecnico di Torino, Italy**

[§] **CNRS/LSS, Supelec, France**

Outline

- ❑ Channel equalisation revisit

  Symmetric property of Bayesian solution

- ❑ Symmetric RBF equaliser

  Exploit prior knowledge (symmetry)

- ❑ Adaptive equalisation algorithms
  - ☆ Kernel-type algorithm (Off-line, block data)
  - ☆ Nonlinear LBER algorithm (On-line, recursive)
  - ☆ Clustering algorithm (On-line, recursive)

  Better performance due to exploiting prior knowledge

Equaliser Model

- Channel model: received signal

$$x(k) = \sum_{i=0}^{n_c-1} c_i b(k-i) + n(k)$$

- Signal model: equaliser input $\mathbf{x}(k) = [x(k)\ x(k-1) \cdots x(k-n_e+1)]^T$

$$\mathbf{x}(k) = \mathbf{C}\mathbf{b}(k) + \mathbf{n}(k) = \bar{\mathbf{x}}(k) + \mathbf{n}(k)$$

- Equaliser: uses information contained in $\mathbf{x}(k)$ to detect symbol $b(k-\tau)$

  c_i, $0 \le i \le n_c - 1$, are channel taps, with channel length n_c

  $\mathbf{C}$ is $n_e \times L$ channel convolution matrix, with $L = n_c + n_e - 1$

  n_e is equaliser order and τ decision delay

  $n(k)$ is AWGN, with $E[|n(k)|^2] = \sigma_n^2$

Bayesian Equaliser

- Assume BPSK: $b(k) \in \{-1, +1\}$, denote $N_b = 2^L$ legitimate combinations of $\mathbf{b}(k)$ as $\mathbf{b}_q$, $1 \le q \le N_b$, and $(\tau + 1)$-th element of $\mathbf{b}_q$ as $b_{q,\tau}$
- Noiseless channel output $\bar{\mathbf{x}}(k)$ takes values from signal state set

$$\mathcal{X} = \{\bar{\mathbf{x}}_q = \mathbf{C}\mathbf{b}_q, \ 1 \le q \le N_b\}$$

- Optimal Bayesian equaliser

$$y_{Bay}(k) = \sum_{q=1}^{N_b} \mathrm{sgn}(b_{q,\tau}) \beta_q e^{-\frac{\|\mathbf{x}(k) - \bar{\mathbf{x}}_q\|^2}{2\sigma_n^2}}$$

where $\beta_q = \frac{1}{N_b (2\pi\sigma_n^2)^{L/2}}$, with decision

$$\hat{b}(k - \tau) = \mathrm{sgn}(y_{Bay}(k)) = \begin{cases} +1, & y_{Bay}(k) \ge 0 \\ -1, & y_{Bay}(k) < 0 \end{cases}$$

Symmetry Property

- Signal state set $\mathcal{X}$ can be divided into two subsets conditioned on $b(k-\tau)$

$$\mathcal{X}^{(\pm)} = \{\bar{\mathbf{x}}_i \in \mathcal{X}, 1 \le i \le N_{sb} : \; b(k-\tau) = \pm 1\}$$

  Sizes of $\mathcal{X}^{(+)}$ and $\mathcal{X}^{(-)}$ are both $N_{sb} = N_b/2$

- Symmetry property: for any $\bar{\mathbf{x}}_i^{(+)} \in \mathcal{X}^{(+)}$ there exists a $\bar{\mathbf{x}}_i^{(-)} \in \mathcal{X}^{(-)}$ such that $\bar{\mathbf{x}}_i^{(-)} = -\bar{\mathbf{x}}_i^{(+)}$

- With this **symmetry** property, Bayesian equaliser can be re-arranged as

$$y_{Bay}(k) = \sum_{q=1}^{N_{sb}} \beta_q \left(e^{-\frac{\|\mathbf{x}(k)-\bar{\mathbf{x}}_q^{(+)}\|^2}{2\sigma_n^2}} - e^{-\frac{\|\mathbf{x}(k)+\bar{\mathbf{x}}_q^{(+)}\|^2}{2\sigma_n^2}} \right)$$

  where $\bar{\mathbf{x}}_q^{(+)} \in \mathcal{X}^{(+)}$, Bayesian solution has an inherent odd symmetry

- Symmetric property of Bayesian equaliser is difficult to learn accurately from noisy data by a **traditional** RBF network

Symmetric RBF Equaliser

- **RBF network** equaliser

$$\hat{b}(k-\tau) = \mathrm{sgn}(y_{RBF}(k)) \quad \text{with} \quad y_{RBF}(k) = \sum_{i=1}^{M} \theta_i \phi_i(\mathbf{x}(k))$$

  θ_i are RBF weights and M number of RBF units

- Symmetric RBF node

$$\phi_i(\mathbf{x}) = \varphi(\mathbf{x}; \boldsymbol{\mu}_i, \rho_i^2) - \varphi(\mathbf{x}; -\boldsymbol{\mu}_i, \rho_i^2)$$

  $\boldsymbol{\mu}_i \in \mathcal{R}^{n_e}$ is RBF centre, ρ_i^2 RBF variance, and $\varphi(\bullet)$ usual RBF function

- Gaussian RBF function

$$\varphi(\mathbf{x}; \boldsymbol{\mu}_i, \rho^2) = e^{-\frac{\|\mathbf{x}-\boldsymbol{\mu}_i\|^2}{2\rho^2}}$$

- This symmetric RBF network has an inherent **odd symmetry**

Block-Based Kernel Algorithm

- Given training set $D_K = \{\mathbf{x}(k), d(k) = b(k-\tau)\}_{k=1}^{K}$, set $M = K$ and $\boldsymbol{\mu}_i = \mathbf{x}(i)$ for $1 \le i \le K$, and fix all RBF variances to ρ^2
- Define $\varepsilon(i) = d(i) - y_{RBF}(i)$. Then we arrive regression model

$$\mathbf{d} = \boldsymbol{\Phi}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$$

where $\mathbf{d} = [d(1)\ d(2) \cdots d(K)]^T$, $\boldsymbol{\varepsilon} = [\varepsilon(1)\ \varepsilon(2) \cdots \varepsilon(K)]^T$,

$$\boldsymbol{\Phi} = [\boldsymbol{\phi}_1\ \boldsymbol{\phi}_2 \cdots \boldsymbol{\phi}_M] \in \mathcal{R}^{K \times M}$$

with $\boldsymbol{\phi}_i = [\phi_i(\mathbf{x}(1))\ \phi_i(\mathbf{x}(2)) \cdots \phi_i(\mathbf{x}(K))]^T$, and $\boldsymbol{\theta} = [\theta_1\ \theta_2 \cdots \theta_M]^T$

- Sparse kernel modelling problem $\Rightarrow$ select sparse M_{spa}-term subset model
- **OFS-FRCSM**:

  Orthogonal Forward Selection based on incrementally maximising Fisher Ratio of Class Separability Measure

Bit Error Rate of RBF Equaliser

- Generic symmetric RBF equaliser

$$y_{RBF}(k;\mathbf{w}) = \sum_{i=1}^{M} \theta_i \left(\varphi(\mathbf{x}(k);\boldsymbol{\mu}_i,\rho_i^2) - \varphi(\mathbf{x}(k);-\boldsymbol{\mu}_i,\rho_i^2) \right)$$

  where $\mathbf{w}$ includes all RBF centres $\boldsymbol{\mu}_i$, variances ρ_i^2 and weights θ_i

- Define probability density function (PDF) of signed decision variable $y_s(k) = \mathrm{sgn}(b(k-\tau))y_{RBF}(k;\mathbf{w})$ as $p_y(y_s)$

- Error probability or BER of RBF equaliser

$$P_E(\mathbf{w}) = \mathrm{Prob}\{y_s(k) < 0\} = \int_{-\infty}^{0} p_y(y_s)\, dy_s$$

- Nonlinear minimum bit error rate (MBER) solution

$$\mathbf{w}_{\mathrm{MBER}} = \arg\min_{\mathbf{w}} P_E(\mathbf{w})$$

Approximate Minimum Bit Error Rate

- Unknown $p_y(y_s) \Rightarrow$ Parzen window estimate

$$\tilde{p}_y(y_s) = \frac{1}{K\sqrt{2\pi}\sigma} \sum_{k=1}^{K} e^{-\frac{(y_s - \mathrm{sgn}(b(k-\tau))y_{RBF}(k;\mathbf{w}))^2}{2\sigma^2}}$$

where σ^2 is kernel variance

- With this estimated PDF, estimated or approximate BER is given by

$$\tilde{P}_E(\mathbf{w}) = \int_{-\infty}^{0} \tilde{p}_y(y_s)\, dy_s = \frac{1}{K} \sum_{k=1}^{K} Q\left(\tilde{g}_k(\mathbf{w})\right)$$

where $Q(\bullet)$ is the usual Gaussian error function and

$$\tilde{g}_k(\mathbf{w}) = \frac{\mathrm{sgn}(b(k-\tau))y_{RBF}(k;\mathbf{w})}{\sigma}$$

- Approximate MBER solution for $\mathbf{w}$ is obtained by minimising $\tilde{P}_E(\mathbf{w})$

Nonlinear Least Bit Error Rate

- To derive a sample-by-sample adaptive algorithm, consider single-sample PDF "estimate" of $p_y(y_s)$

$$\tilde{p}_y(y_s,k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y_s - \text{sgn}(b(k-\tau)) y_{RBF}(k;\mathbf{w}))^2}{2\sigma^2}}$$

- Conceptually, with this instantaneous PDF "estimate" we have one-sample BER "estimate" $\tilde{P}_E(\mathbf{w},k) = Q\left(\tilde{g}_k(\mathbf{w})\right)$
- Using instantaneous or stochastic gradient $\nabla \tilde{P}_E(\mathbf{w},k)$ gives rise to

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \frac{\xi}{\sqrt{2\pi}\sigma} e^{-\frac{y^2_{RBF}(k;\mathbf{w}(k-1))}{2\sigma^2}} \text{sgn}(b(k-\tau)) \frac{\partial y_{RBF}(k;\mathbf{w}(k-1))}{\partial \mathbf{w}}$$

which we refer to as Nonlinear Least Bit Error Rate (NLBER)

- Step size ξ and kernel width σ should be chosen carefully to ensure fast convergence and small steady-state BER misadjustment

Clustering Algorithm

- Set $M = N_{sb}$, $\theta_i = \beta > 0$ and $\rho_i^2 = \hat{\sigma}_n^2 \ \forall i$, where $\hat{\sigma}_n^2$ is estimated σ_n^2
- Then symmetric RBF equaliser realises exactly Bayesian equaliser by placing its centres $\boldsymbol{\mu}_i = \mathbf{x}_i^{(+)} \in \mathcal{X}^{(+)}$
- Denote legitimate combinations of $\mathbf{b}(k)$ corresponding to $b(k-\tau) = +1$ as $\mathbf{b}_i^{(+)}$, $1 \leq i \leq N_{sb}$
- During training, supervised clustering

$$\begin{aligned}
&\textbf{if } (\mathbf{b}(k) == \mathbf{b}_i^{(+)}) \\
&\qquad \check{\mathbf{x}}(k) = \mathbf{x}(k); \\
&\textbf{else if } (\mathbf{b}(k) == -\mathbf{b}_i^{(+)}) \\
&\qquad \check{\mathbf{x}}(k) = -\mathbf{x}(k); \\
&\boldsymbol{\mu}_i(k) = \boldsymbol{\mu}_i(k-1) + \mu \left(\check{\mathbf{x}}(k) - \boldsymbol{\mu}_i(k-1) \right);
\end{aligned}$$

- In data transmission, use unsupervised or enhanced κ-means clustering

A Simulation Example

- Three-tap channel

$$x(k) = 0.3\ b(k) + 0.8\ b(k-1) + 0.3\ b(k-2) + n(k)$$

- Equaliser order $n_e = 4$

$$\mathbf{x}(k) = [x(k)\ x(k-1)\ x(k-2)\ x(k-3)]^T$$

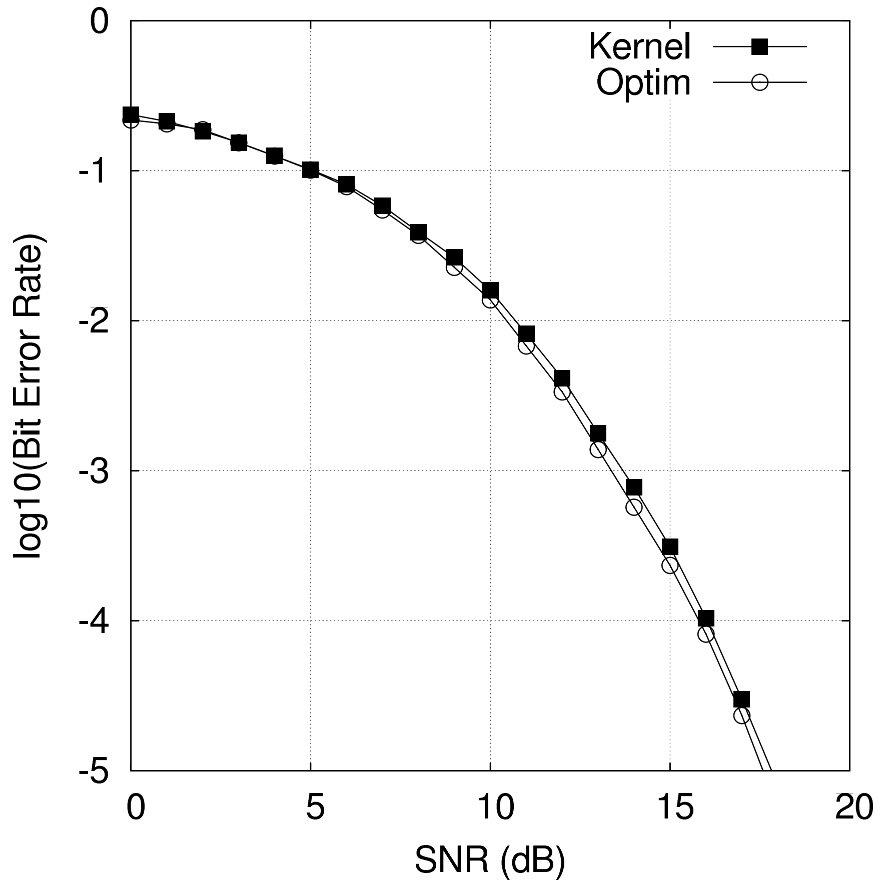
  and decision delay $\tau = 2$

- Size of $\mathcal{X}^{(+)}$ is $N_{sb} = 32$

OFS-FRCSM Results

Bit error rate comparison of optimal Bayesian equaliser and symmetric RBF equaliser based on OFS-FRCSM algorithm

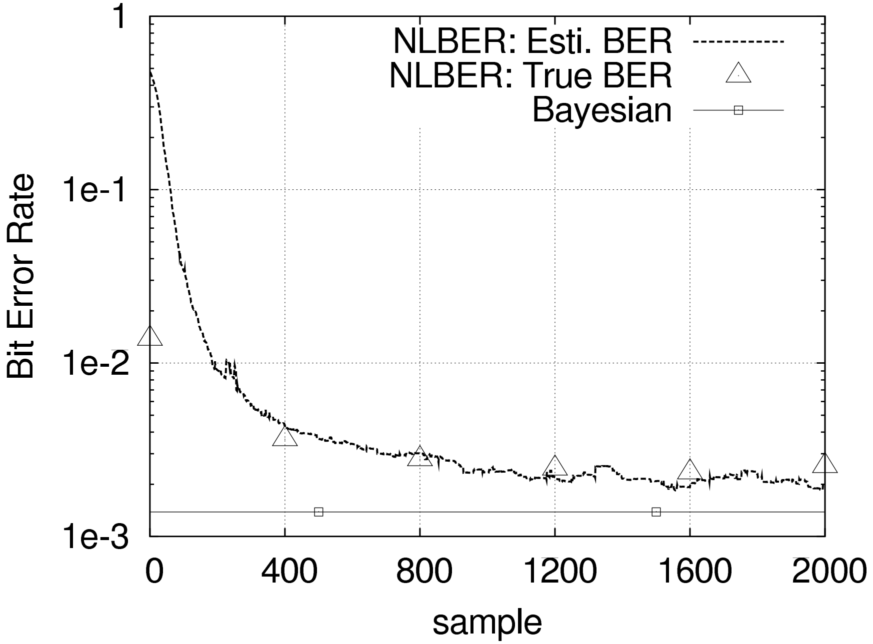
- Symmetric RBF size $M = 30$
- Data block length $K = 600$
- RBF variance $\rho^2 = \sigma_n^2$



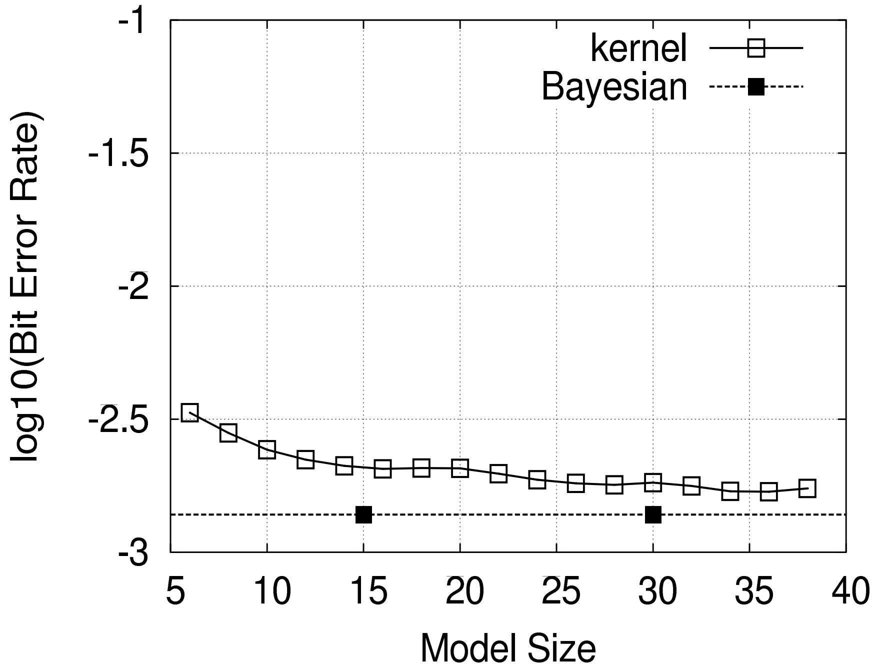
NLBER Results

- SNR= 13 dB, and average over 10 runs
- First M data points as initial centres $\boldsymbol{\mu}_i(0)$, $\theta_i(0) = 0.01$ and $\rho_i^2(0) = 4\sigma_n^2$
- Step size $\mu = 0.1$ and kernel width $\sigma = \sigma_n$ for NLBER

Learning Curve ($M = 30$)



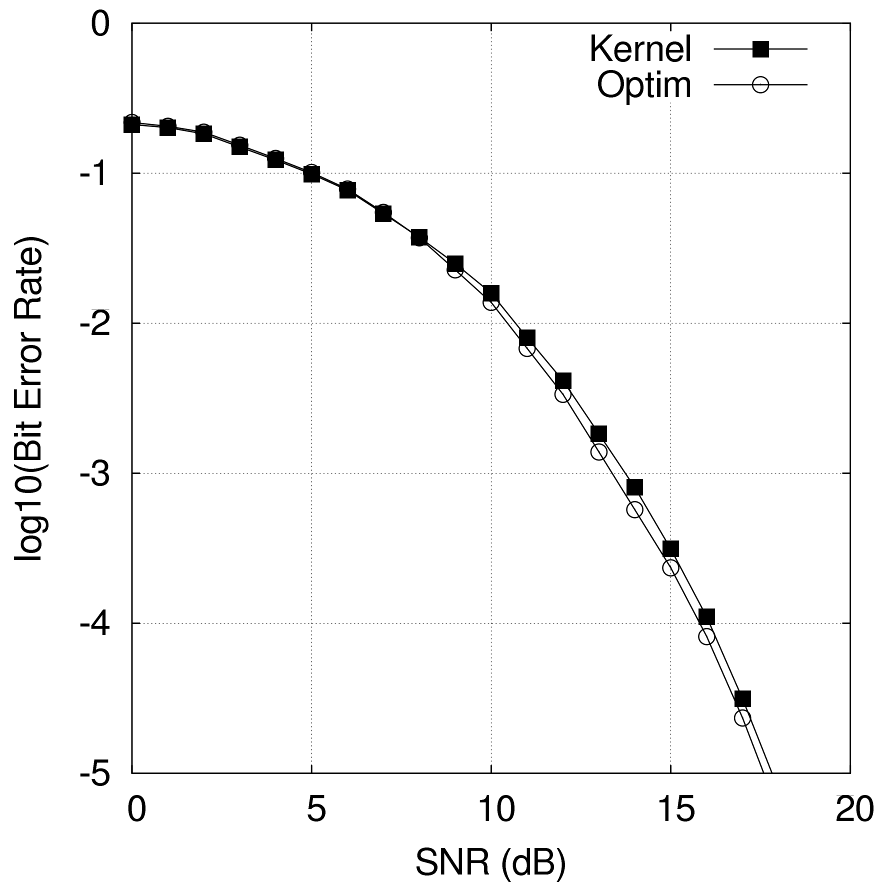
Influence of Model Size



NLBER Results (continue)

Bit error rate comparison of optimal Bayesian equaliser and symmetric RBF equaliser based on NLBER algorithm

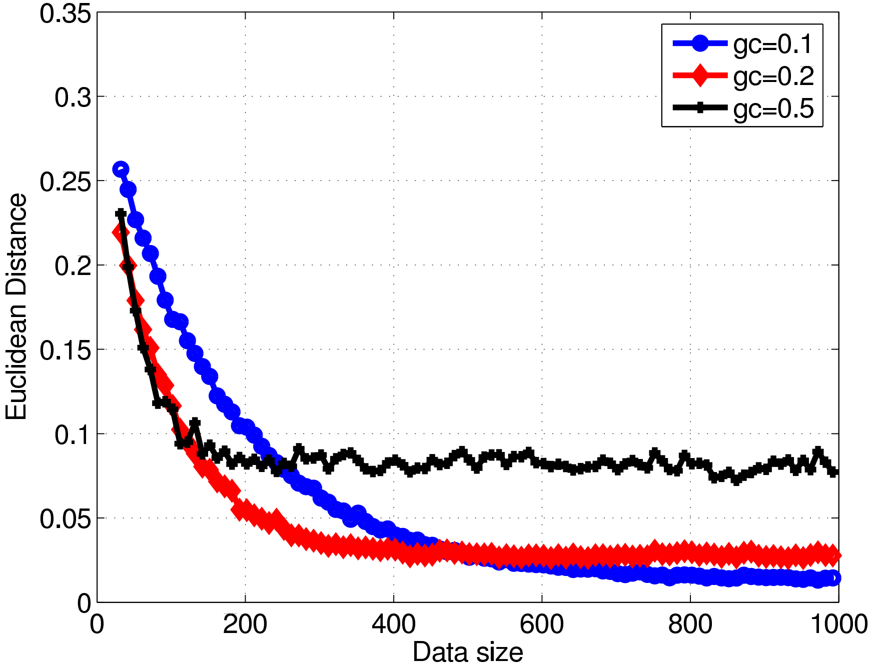
- Symmetric RBF size $M = 30$
- First 30 data as initial centres $\boldsymbol{\mu}_i(0)$, initial weights $\theta_i(0) = 0.01$ and initial variances $\rho_i^2(0) = 4\sigma_n^2$
- Step size $\mu = 0.1$ and kernel width $\sigma = \sigma_n$ for NLBER



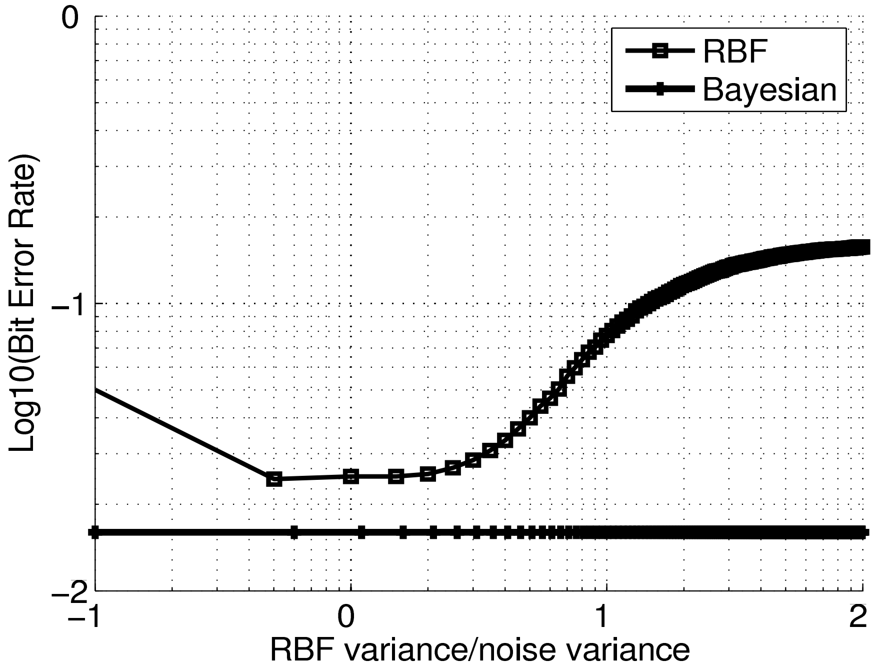
Clustering Results

- SNR$=$ 10 dB, symmetric RBF size $M = 32$ and RBF variance $\rho^2 = \sigma_n^2$
- Learning curve shows Euclidean distance between set of RBF centres and set of channel states

Learning Curve



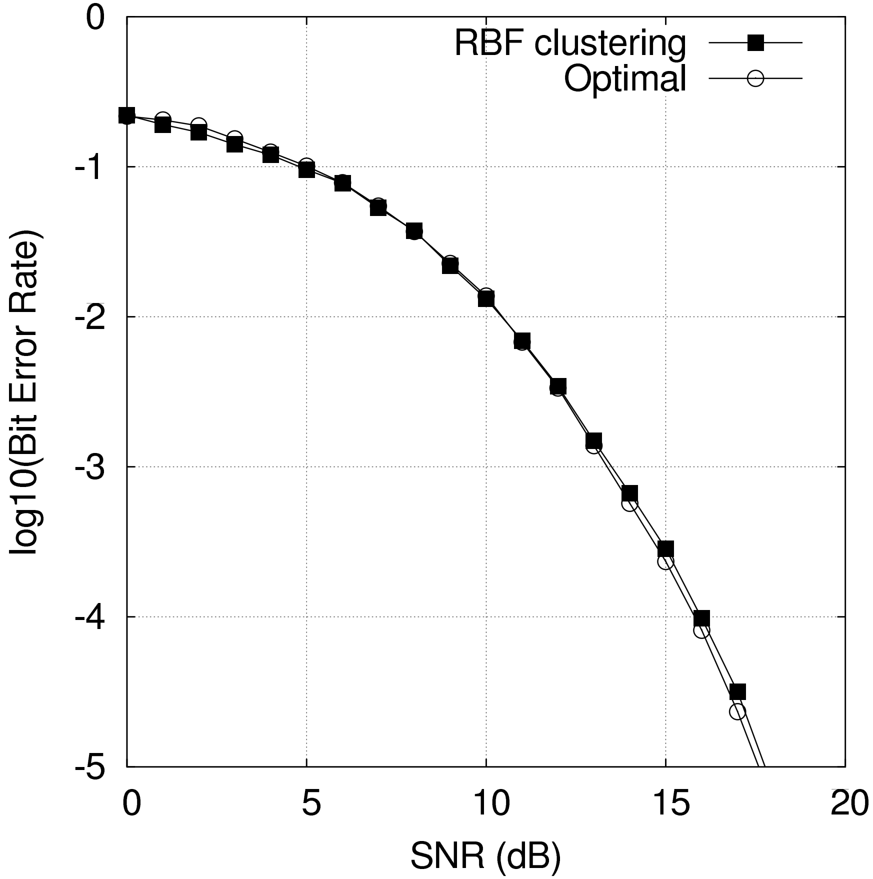
Influence of RBF variance



Clustering Results (continue)

Bit error rate comparison of optimal Bayesian equaliser and symmetric RBF equaliser based on clustering algorithm

- Symmetric RBF size $M = 32$
- RBF variance $\rho^2 = \sigma_n^2$



Conclusions

- Classical channel equalisation has been revisited
  - ☆ Inherently odd symmetry property of optimal Bayesian equaliser has been highlighted
- A novel symmetric radial basis function network has been proposed for channel equalisation
  - ☆ Which exploits prior knowledge on symmetry property of optimal solution, leading to enhanced performance
- Adaptive algorithms for training this symmetric RBF equaliser have been reviewed

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