Rotating fermions

Victor Eugen Ambrus

Supervisor: Prof. Elizabeth Winstanley

University of Sheffield

04/04/2012

Outline

Quantum field theory in curved spaces

- Semi-classical approach - quantized theory of fields in the presence of classical gravity
- Starting point: the semi-classical Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi \left\langle T_{\mu\nu} \right\rangle_{\text{ren}}$$

- $g_{\mu\nu}$ - classical background space-time
- $T_{\mu\nu}$ - renormalized expectation value of the stress-tensor describing the quantum fields
- The energy density, T_{tt}, contains information about the distribution of energy and the production of particles

Steps for calculating the energy density

- Construct the Lagrangian $\mathcal{L}$ for the quantum field in a background $g_{\mu\nu}$
- Derive the field equations using the Euler-Lagrange formalism
- Solve the field equations and construct a complete set of modes
- Split set into particle/anti-particle modes and perform second quantization

$$\Psi = \sum_k (f_k^+ b_k + f_k^- d_k^\dagger) \qquad \left\{b_k, b_{k'}^\dagger\right\} = \left\{d_k, d_{k'}^\dagger\right\} = \delta_{kk'}$$

- Construct the energy density operator

$$T_{tt} = \frac{i}{2}\left(\overline{\Psi}\gamma_0 D_0 \Psi - \overline{D_0 \Psi}\gamma_0 \Psi\right) \qquad D_\mu = \partial_\mu - \frac{1}{2}\Gamma_{\hat{\beta}\hat{\gamma}\hat{\alpha}} D[\Sigma^{\hat{\beta}\hat{\gamma}}]\omega_\mu^{\hat{\alpha}}$$

- Quantum operators are typically divergent $\Rightarrow$ renormalize to remove infinities

Toy model: rigidly rotating space-time

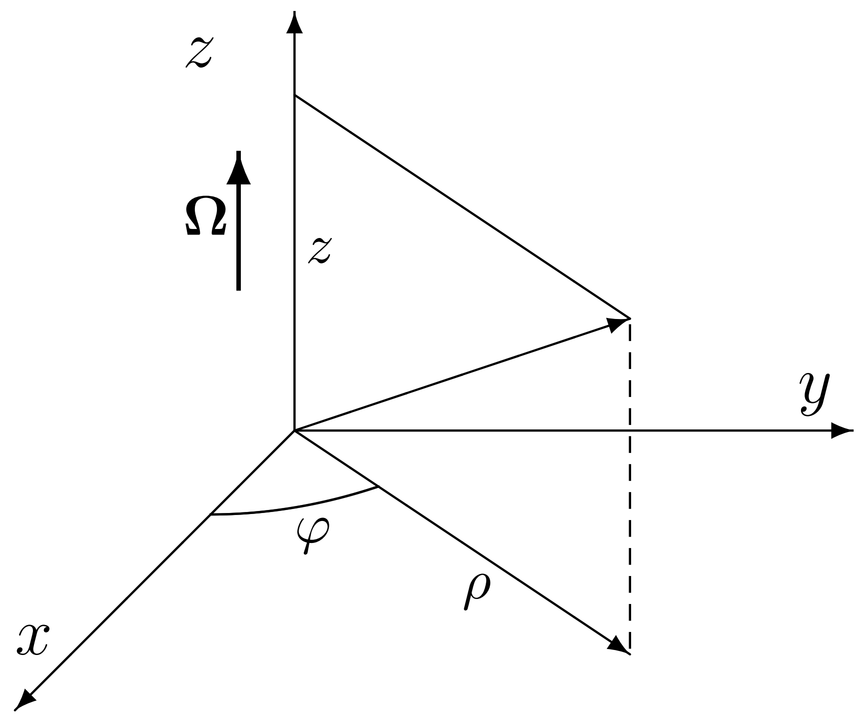
- Obtained from the Minkowski space-time using the coordinate transformation:

$$\varphi \to \varphi - \Omega t$$

- Metric in co-rotating coordinates:

$$g_{\mu\nu} = \begin{pmatrix} -(1-\rho^2\Omega^2) & 0 & \Omega\rho^2 & 0 \\ 0 & 1 & 0 & 0 \\ \Omega\rho^2 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Coordinate singularity on the speed of light surface $\rho\Omega = 1$



Scalar field theory in co-rotating coordinates at finite temperature

- Mode solutions:

$$f_{kqm} = \frac{1}{2\pi\sqrt{2|\omega|}} e^{-i\widetilde{\omega}t + im\varphi + ikz} J_m(q\rho), \qquad \widetilde{\omega} = \omega - \Omega m$$

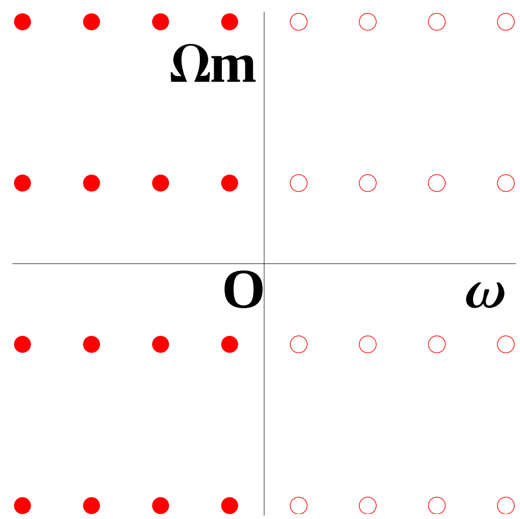
- Quantization procedure dictated by norm:

$$\langle f_k, f_{k'} \rangle = \text{sgn}(\omega_k)\delta_{kk'} \qquad (\omega_k = \pm \text{ Minkowski energy})$$

- For particles, $\langle f_k, f_{k'} \rangle \geq 0$[1]:

$$\Phi(x) = \sum_k \theta(\omega_k)(f_k a_k + f_k^* a_k^\dagger)$$

- Result: $T_{\hat{t}\hat{t}} = \sum_k \theta(\omega_k) \frac{1}{e^{\beta\widetilde{\omega}_k} - 1}(\dots)$
- Infinite contributions from $\omega = m\Omega$
- Thermal state unattainable



[1] Letaw & Pfautsch, Phys.Rev.D, 22(6):1345-1351, 1980

Dirac field theory at finite temperature

- Dirac equation:

$$\left[\gamma^0(H + \Omega M_z) - \boldsymbol{\gamma} \cdot \boldsymbol{P} - \mu\right] \Psi(x) = 0$$

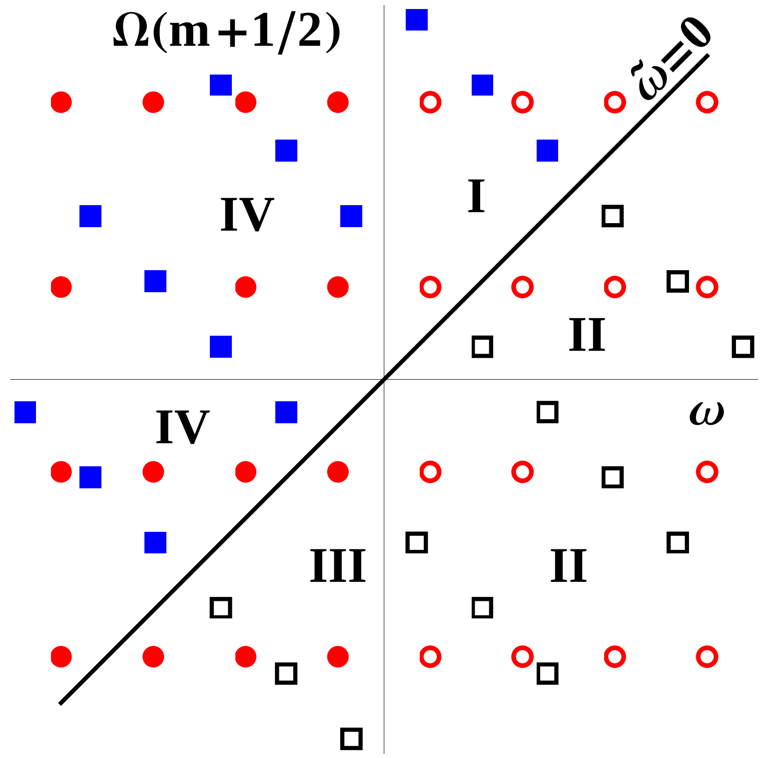
- Norm always positive:

$$\langle U_k, U_{k'} \rangle = \delta_{kk'}$$

- For particles, choose eigenvectors of H with positive eigenvalues $\widetilde{\omega}_k \geq 0$[2]:

$$\Psi(x) = \sum_k \theta(\widetilde{\omega}_k)(U_k b_k + V_k d_k^\dagger)$$

$$\widetilde{\omega} = \omega - \Omega(m + \tfrac{1}{2})$$



[2] Iyer, Phys.Rev.D, 26(8):1900-1905, 1982

Results: Thermal expectation value of the stress tensor

- Energy density for massless fermions ($\omega^2 = q^2 + k^2$):

$$\langle : T_{\hat{t}\hat{t}} : \rangle_\beta = \frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} \int_0^\infty \omega^2 \, d\omega \int_0^\omega dk \, \frac{1}{e^{\beta\widetilde{\omega}} + 1} \left[J_m^2(q\rho) + J_{m+1}^2(q\rho) \right]$$

$$= \underbrace{\frac{7\pi^2}{60\beta^4}}_{\text{Minkowski value}} \left\{ \frac{4}{3(1-\rho^2\Omega^2)^3} - \frac{1}{3(1-\rho^2\Omega^2)^2} \right\}$$

$$+ \frac{\Omega^2}{8\beta^2} \left\{ \frac{8}{3(1-\rho^2\Omega^2)^4} - \frac{16}{9(1-\rho^2\Omega^2)^3} + \frac{1}{9(1-\rho^2\Omega^2)^2} \right\}$$

- Expression agrees with Vilenkin's[3] result
- Parameters:

Ω - angular speed of the rotating observer

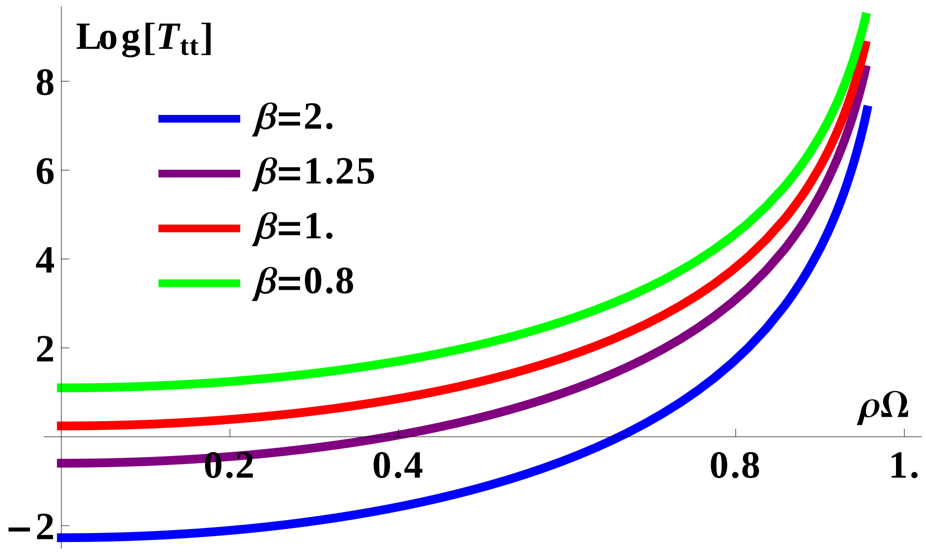
$\beta = 1/T$ - inverse temperature

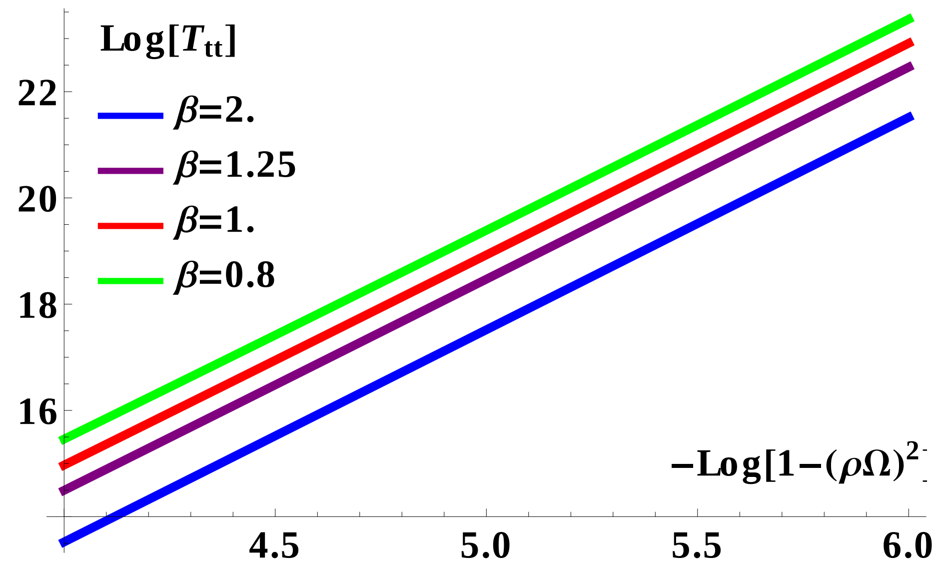
ρ - distance from the rotation axis

---

[3] Vilenkin, Phys.Rev.D, 21(8):2260-2269, 1980

Results: Thermal expectation values - plots





a) Logarithm of $\langle T_{\hat{t}\hat{t}} \rangle_\beta$ for inverse temperature $\beta = 0.8, 1.0, 1.2, 2.0$

b) Log-Log plot of $\langle T_{\hat{t}\hat{t}} \rangle_\beta$ against distance from speed of light surface

Cylindrical boundary

- Problems with the unbounded theory:
  - Scalar field thermal states are ill-defined, due to modes making infinite contributions
  - Dirac field observables diverge as inverse powers of the distance to the speed of light surface
- Solution: Enclose system inside a cylinder of radius R, parallel to $\boldsymbol{\Omega}$
- Side effect: the presence of the boundary alters the vacuum state $\Rightarrow$ the Casimir effect

Scalar field inside a cylinder

- Dirichlet boundary conditions on cylinder of radius $R \Rightarrow f_{\rho=R} = 0$[4]
- Transverse momentum q gets quantised:

$$f \sim J_m(q\rho) \qquad \Rightarrow \qquad q \to \frac{\xi_{ml}}{R}$$

- $\xi_{ml} < \xi_{m,l+1}$ are roots of J_m $(l = 1, 2, 3 \dots)$
- The useful property of the roots of J_m: $\xi_{m1} > m + \frac{1}{2} \dots$
- $\dots$ ensures that $\widetilde{\omega} > 0$:

$$\widetilde{\omega} \geq \frac{\xi_{m1}}{R} - m\Omega > \frac{m(1 - \Omega R) + \frac{1}{2}}{R}$$

- $\dots$ as long as boundary is inside the speed of light surface $(R\Omega \leq 1)$

[4] Duffy & Ottewill, Phyis.Rev.D, 67(044002), 2003

Fermions inside a cylinder

- The Dirac equation is of first order $\Rightarrow \Psi_{\rho=R} = 0$ does not work
- The theory is consistent if

$$\int_{-\infty}^{\infty} dz \int_0^{2\pi} d\varphi \, \overline{\psi} \gamma^{\hat{\rho}} \psi' = 0 \qquad \text{for all } \psi, \psi' \text{ obeying the Dirac equation}$$

- MIT bag model:[5]:

$$i\gamma^{\hat{\rho}}\psi = \sigma\psi, \qquad \sigma^2 = 1$$

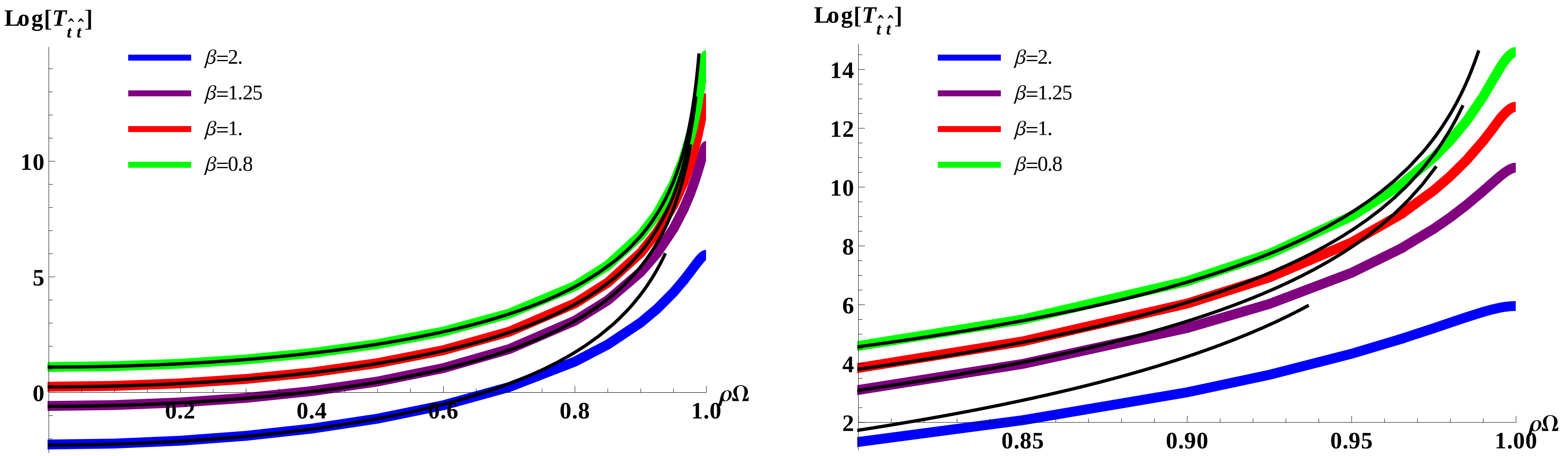
- Quantisation for massless fermions:

$$J_m(q_{ml}R) = \pm J_{m+1}(q_{ml}R).$$

[5] Chodos et al, Phys.Rev.D, 9(12):3471-3495, 1974

Numerical results: Energy density

- $\langle T_{\hat{t}\hat{t}} \rangle_\beta$ for massless fermions obeying MIT bag boundary conditions on the speed of light surface ($R\Omega = 1$):



- Stress-tensor finite on the speed of light surface
- $\beta = 1/T$ is the inverse temperature of the system

Casimir divergence

- $\langle T_{\hat{t}\hat{t}} \rangle$ obtained from the difference between the Euclidean Green's function satisfying Dirichlet boundary conditions and its unbounded analogue
- Asymptotic analysis shows:

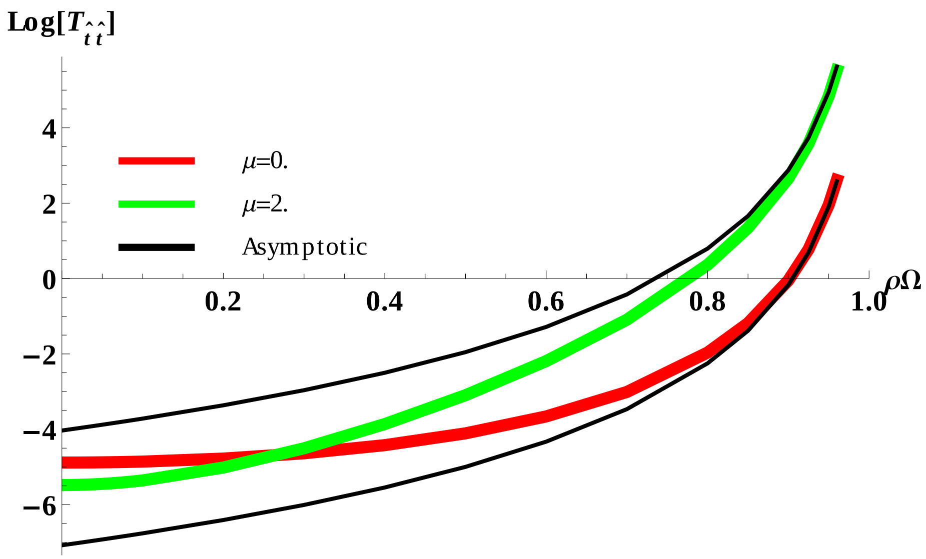
$$T^{\text{scalar}}_{\hat{t}\hat{t}} = -\frac{1}{720\pi^2 R^4} \times \frac{1}{(1-\rho/R)^3} + \mathcal{O}[(1-\rho/R)^{-2}]$$

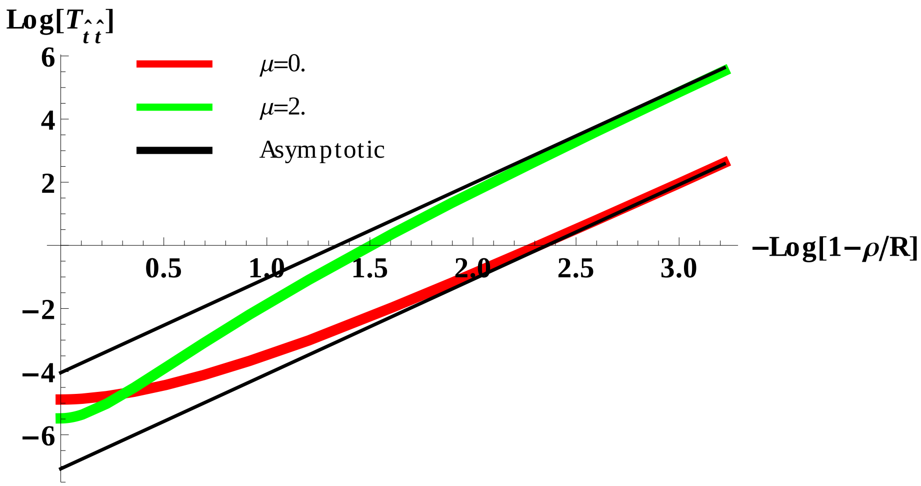
$$T^{\text{fermions}}_{\hat{t}\hat{t}} = -\frac{1}{120\pi^2 R^4} \times \frac{1+10\mu R}{(1-\rho/R)^3} + \mathcal{O}[(1-\rho/R)^{-2}]$$

- $T_{\hat{t}\hat{t}}$ diverges as the inverse cube of the distance from the bounding surface
- Contributions due to mass subleading for scalars but not for fermions
- Extends Saharian's[6] $2D$ results

[6] Bezerra de Mello, Moraes & Saharian, Phys.Rev.D, 85(045016), 2012

Casimir divergence - plots





a) The logarithm of the stress tensor for two values of the mass $\mu = 0, 2$

b) Log-Log plot of $T_{\hat{t}\hat{t}}$ against the distance from the boundary

Summary

- The scalar field theory at finite temperature as seen by a rotating observer is ill-defined
- The thermal field theory for rigidly rotating fermions in an unbounded spacetime is regular up to the speed of light surface
- Confinement of the system in a cylindrical boundary placed inside or on the speed of light surface eliminates divergences for both scalars and fermions
- Casimir divergence of the vacuum expectation value of the energy density due to presence of boundary