

Resonances in black hole spacetimes and neutron stars

Béatrice Bonga - Virtual Seminar University of Southampton - 28 May 2020

*Based on BB, Yang, Hughes [PRL, 1905.00030], Yang, BB, Peng, Li [PR, 1910.07337]
and Pan, Lyu, BB, Ortiz and Yang [arXiv: 2003.03330]*

Tidal forces & resonances

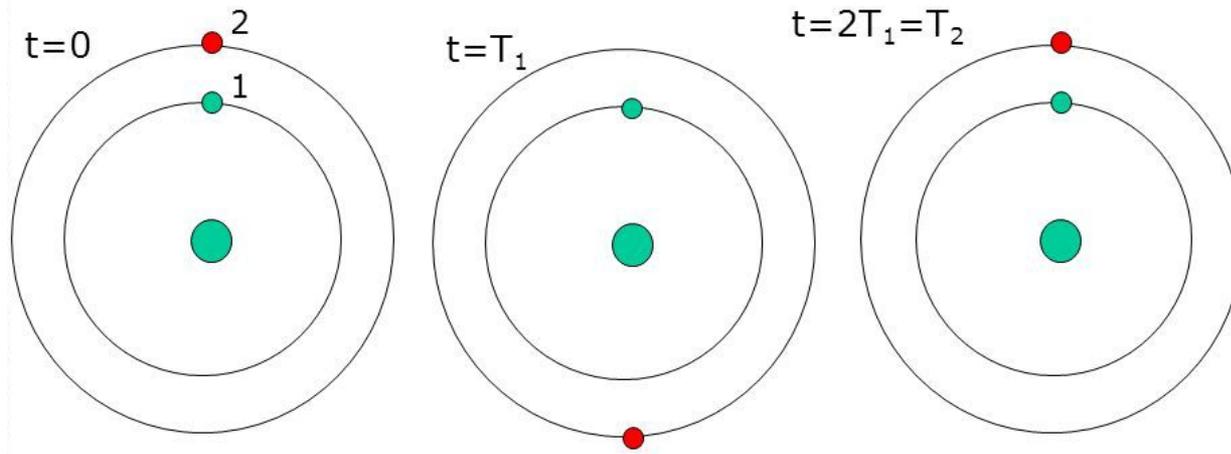
Multiple black holes

- Mean motion resonances
- Tidal resonances

Neutron stars

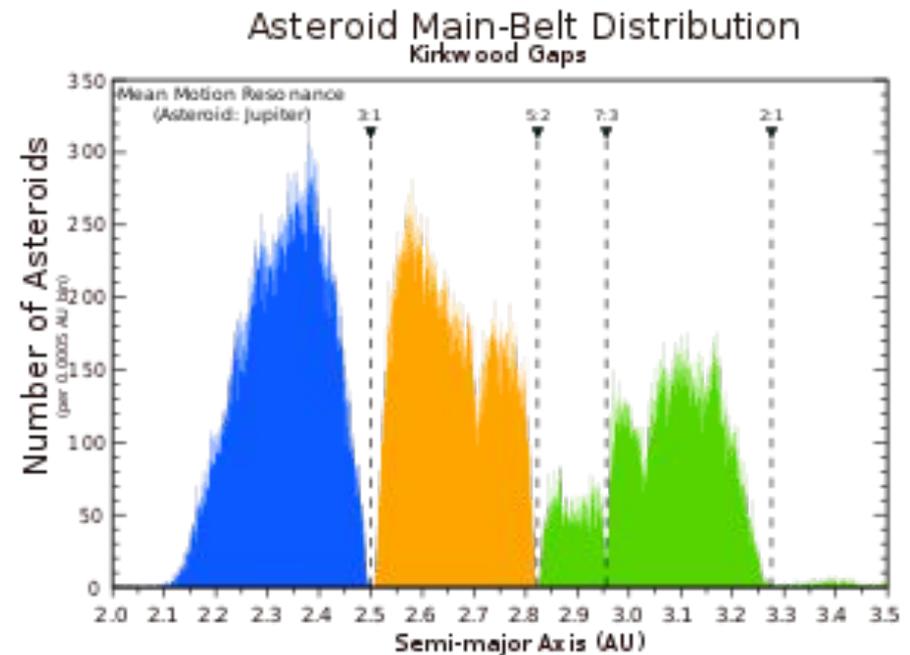
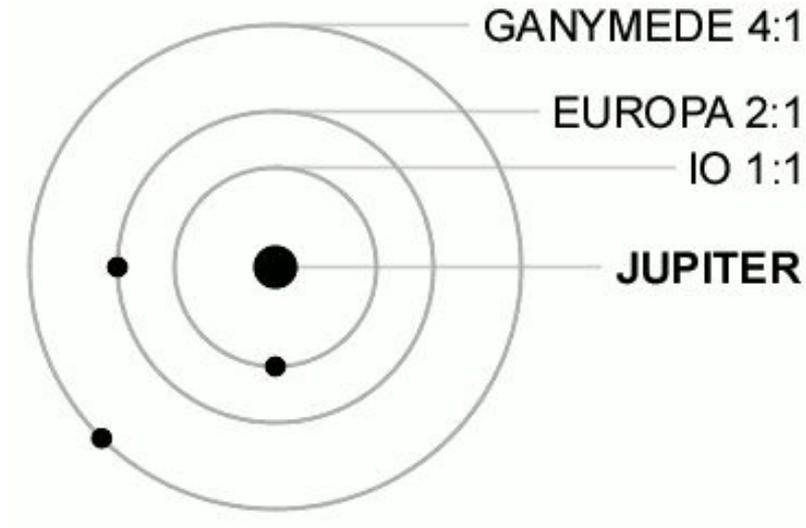
Interface-mode
resonance

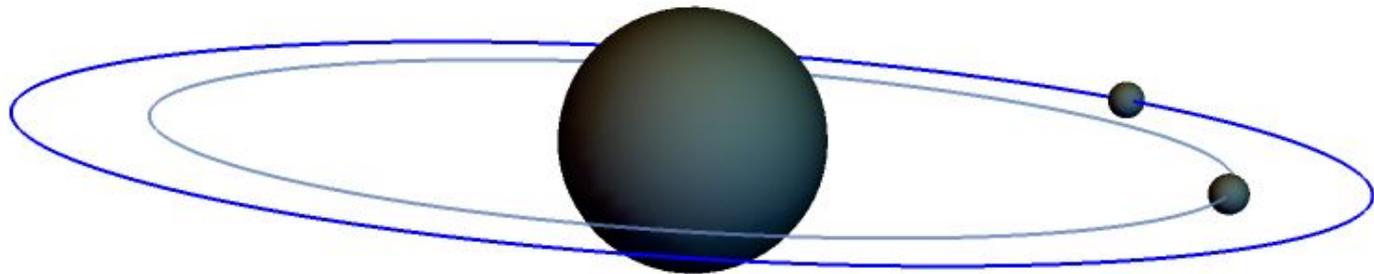
Mean motion resonance



Orbital resonance due to the gravitational interaction between two small bodies orbiting a central massive object

Stable and unstable configurations





Focus on mean motion resonance around massive black holes
in the *post-Newtonian regime*

Post-Newtonian Hamiltonian

Oscillatory terms,
secular terms and
resonant terms → only
include first order
resonance

$$H = H_1 + H_2 + H_{\text{int}}$$

$$H_1 = \frac{1}{2m_1} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{m_1 M}{r} + \frac{1}{c^2} \left[-\frac{1}{8m_1^3} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right)^2 + \frac{m_1 M^2}{2r} - \frac{3M}{2m_1 r} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) \right] + \mathcal{O}\left(\frac{1}{c^4}\right)$$

Canonical coordinates &
momenta

Action-angle
variables

Poincaré variables

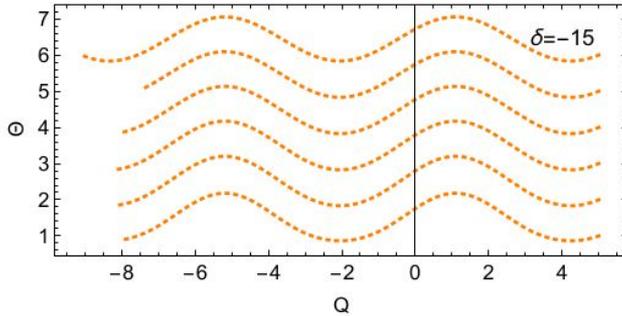
Small eccentricity
expansion

$$H = \alpha \underline{\Gamma} + \beta \underline{\Gamma}^2 + \kappa \sqrt{2\underline{\Gamma}} \cos \theta_2 + \mathcal{O}\left(\frac{1}{c^4}, f_d^2, e^3\right)$$

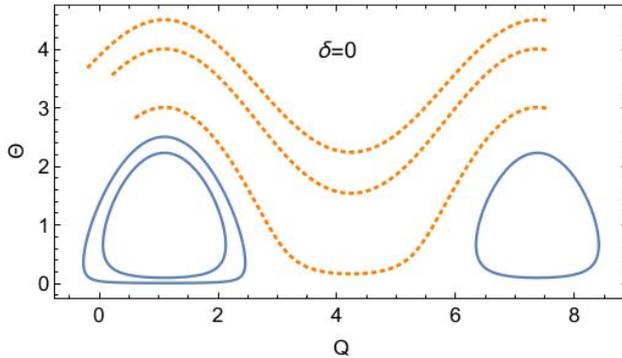
Measure of
proximity of
resonance

Resonant/libration
angle

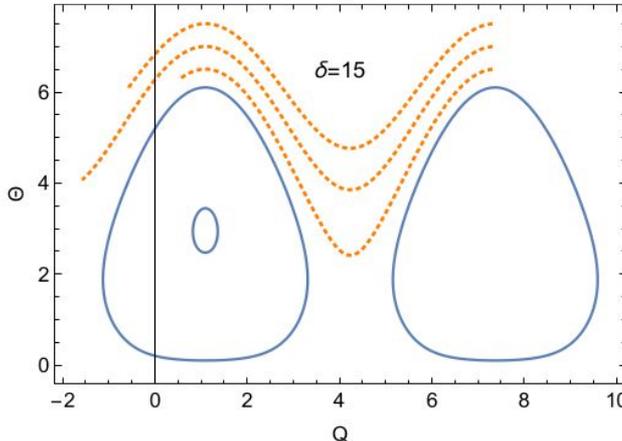
Phase space: guiding trajectories



No resonance possible



Resonance possible



Parametric evolution of α

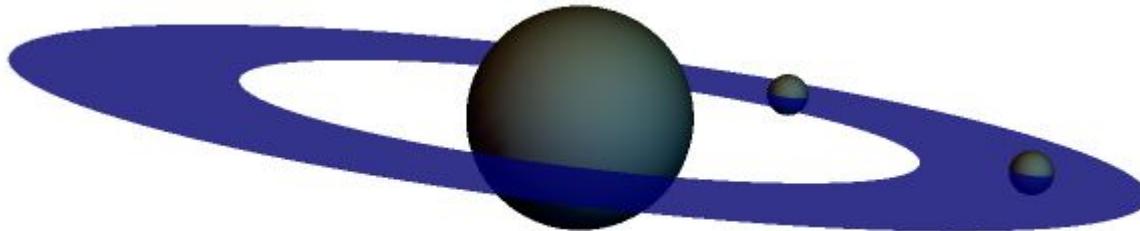
$$H = \alpha \Gamma + \beta \Gamma^2 + \kappa \sqrt{2\Gamma} \cos \theta_2 + \mathcal{O}\left(\frac{1}{c^4}, f_d^2, e^3\right)$$

Capture and escape

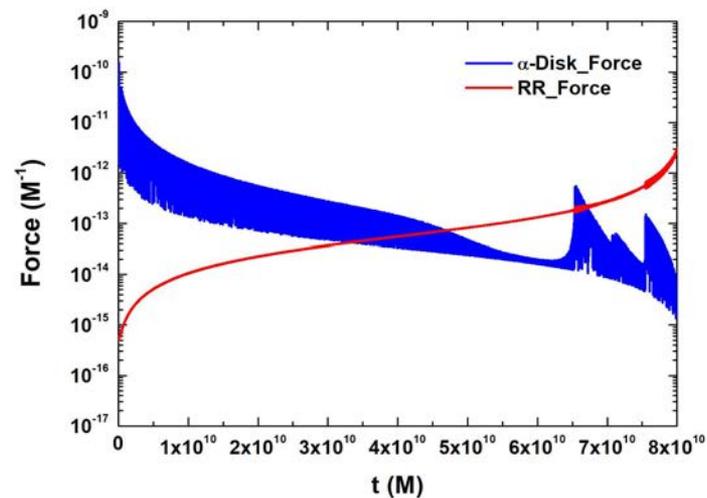
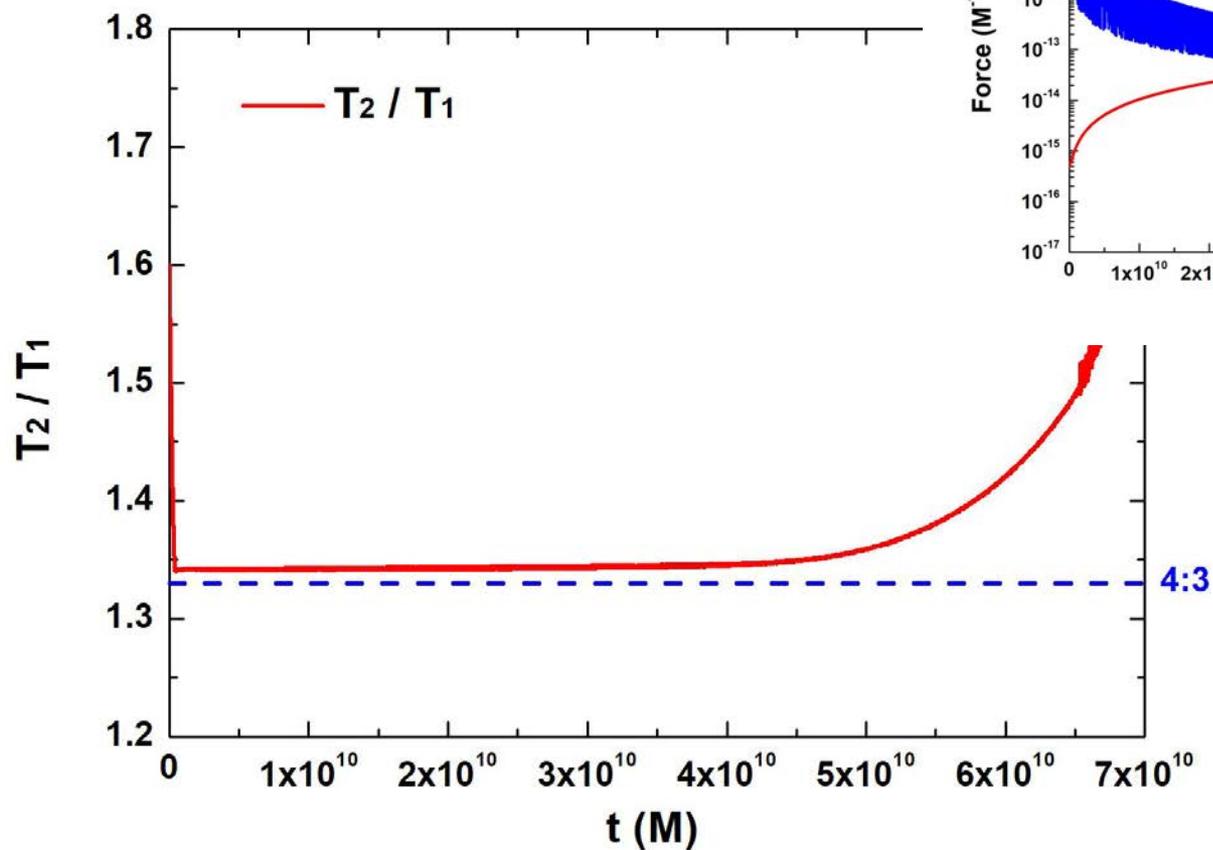
Sustained resonance in planetary systems: $\eta > C (\tau_e / \tau_a)^{3/2}$



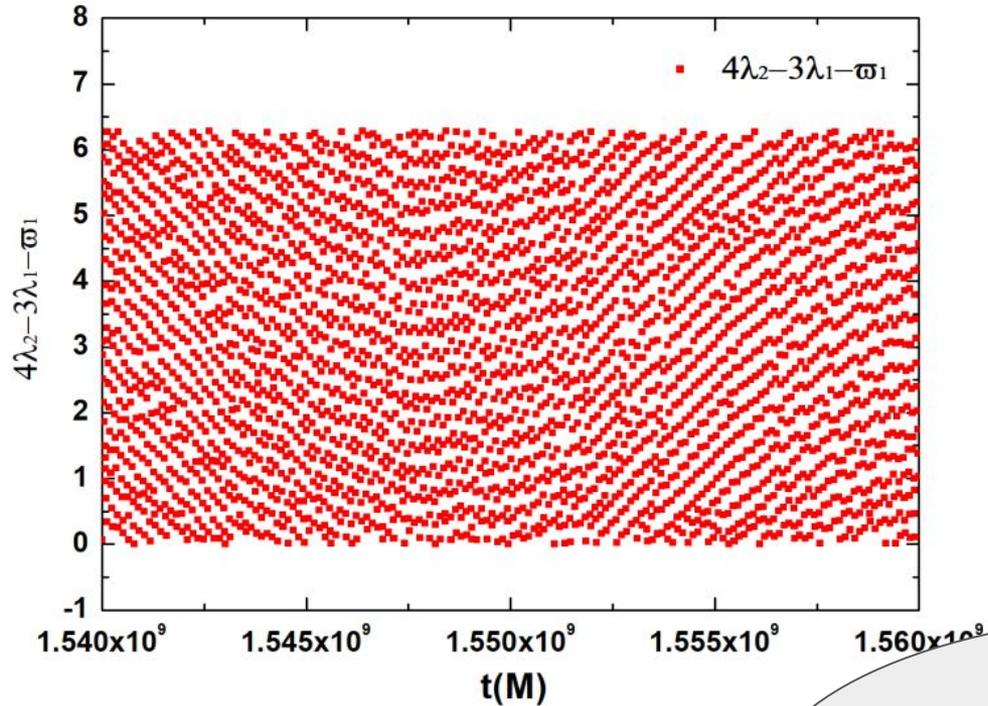
Gravitational radiation alone cannot lead to sustained resonance



Numerical evolution (disk + post-Newtonian effects)

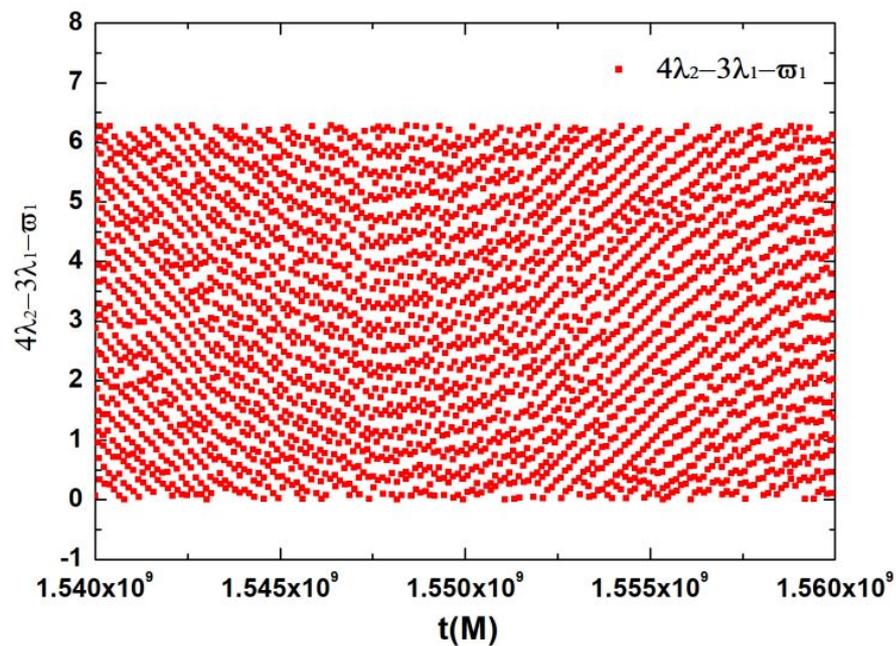
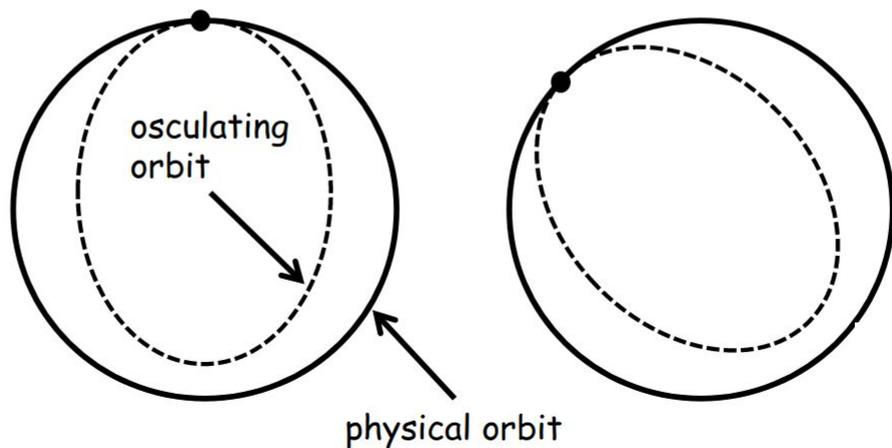


Cautionary tale

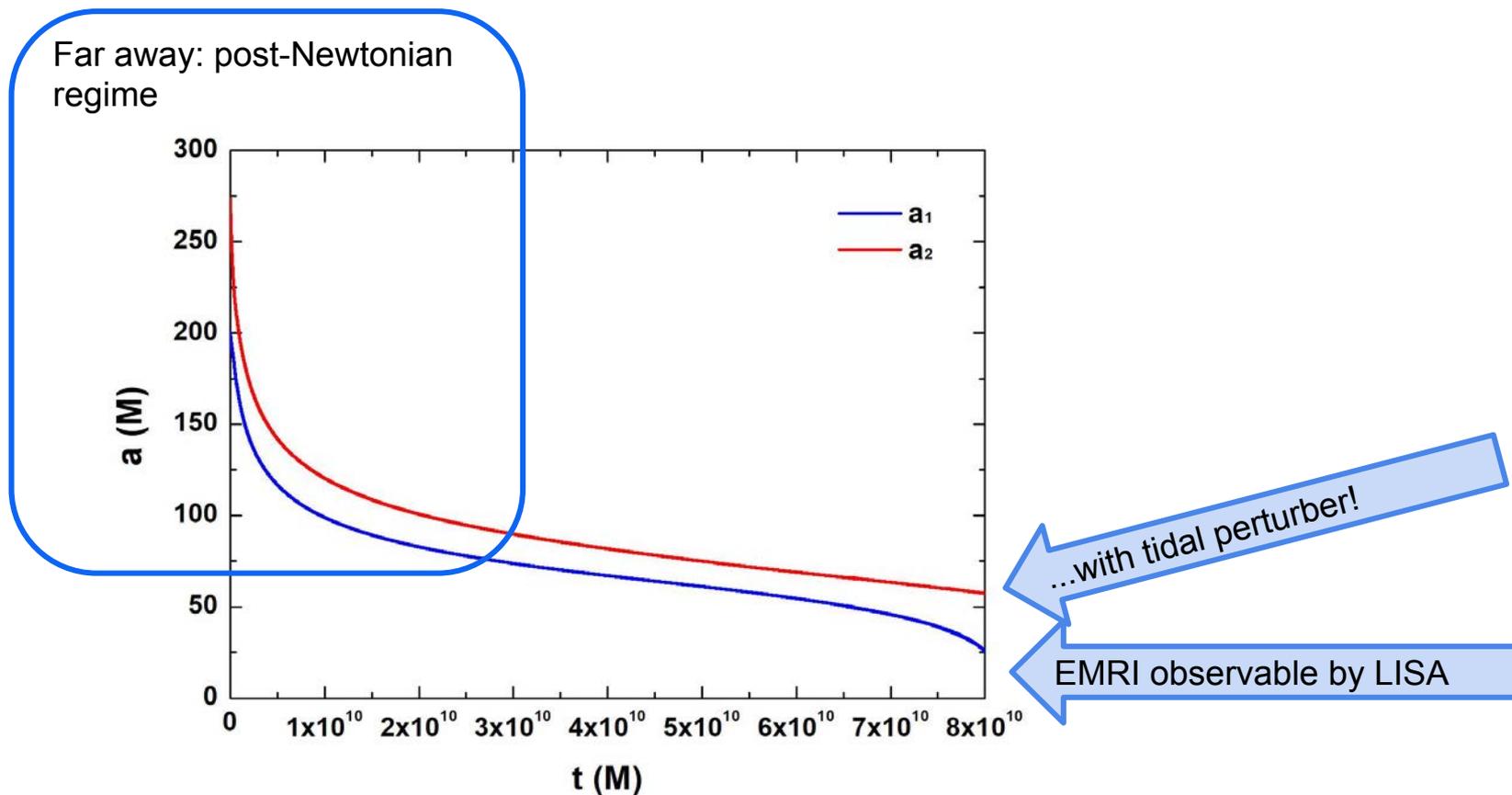


Hold on...
Resonant angle should be
locked if true resonance!

“Nature is perfectly happy with a circle!”



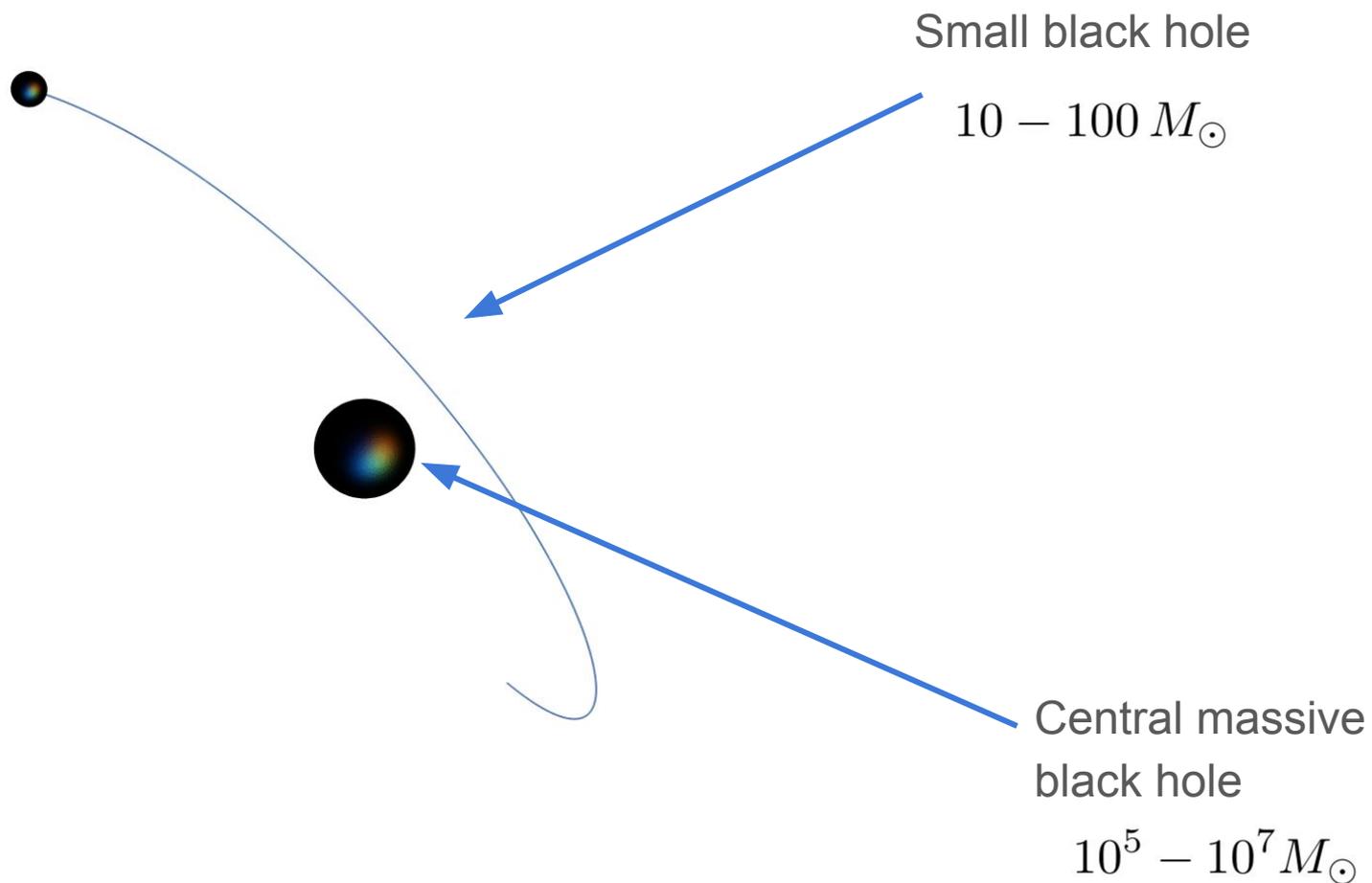
Multiple black holes close to central black hole



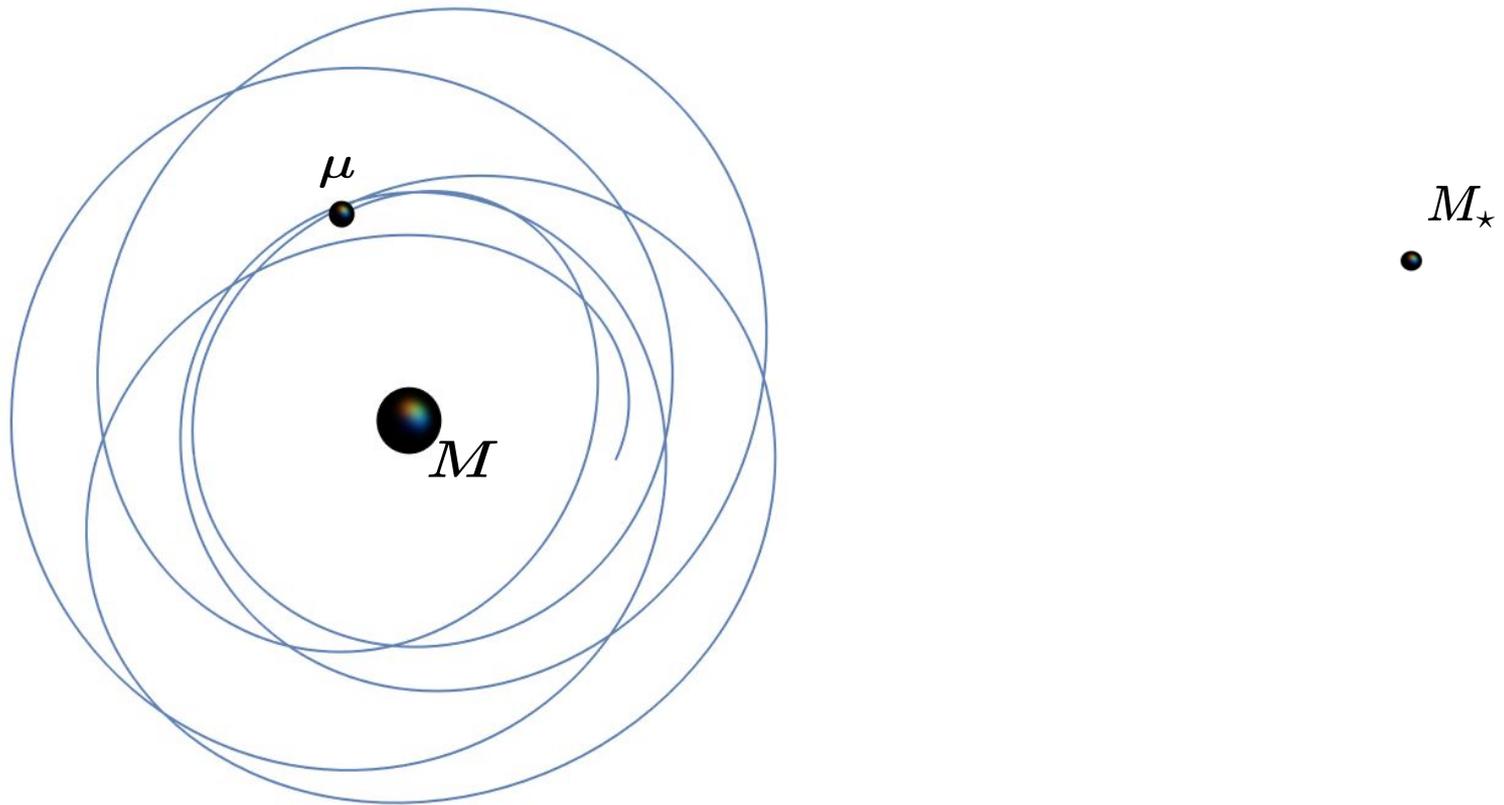
Mean motion resonance summary

- ❖ Mean motion resonances are due to the interaction of two small bodies in the gravitational field of a massive body
- ❖ Observed in many planetary systems
- ❖ During mean motion resonances the two small bodies evolve in sync and their motion is locked together

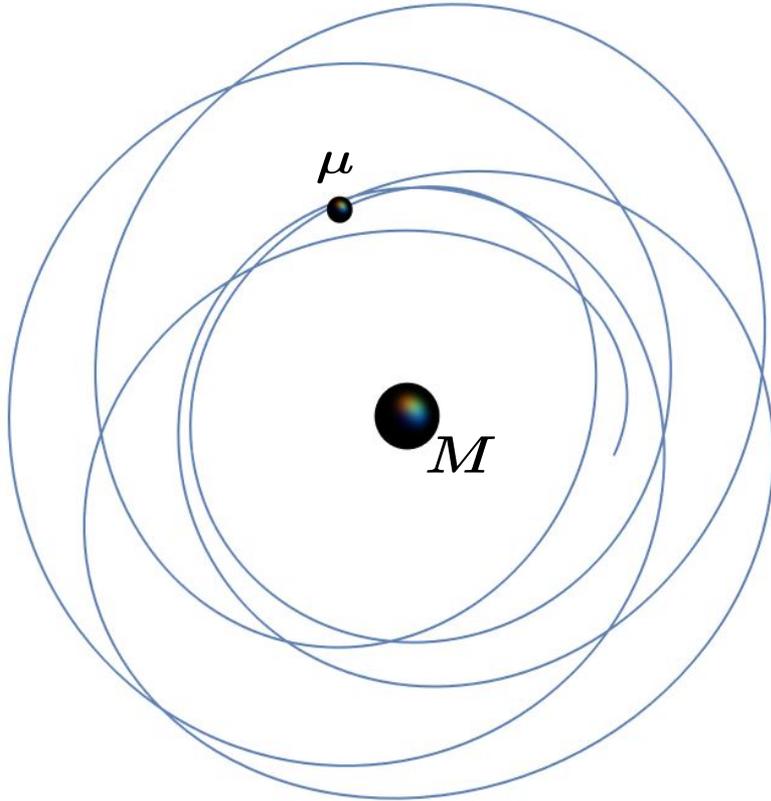
Extreme Mass Ratio Inspiral = EMRI



EMRIs as isolated systems ?



Zeroth order description



Four constants of motion:
 $\{\mu, E, L_z, Q\}$

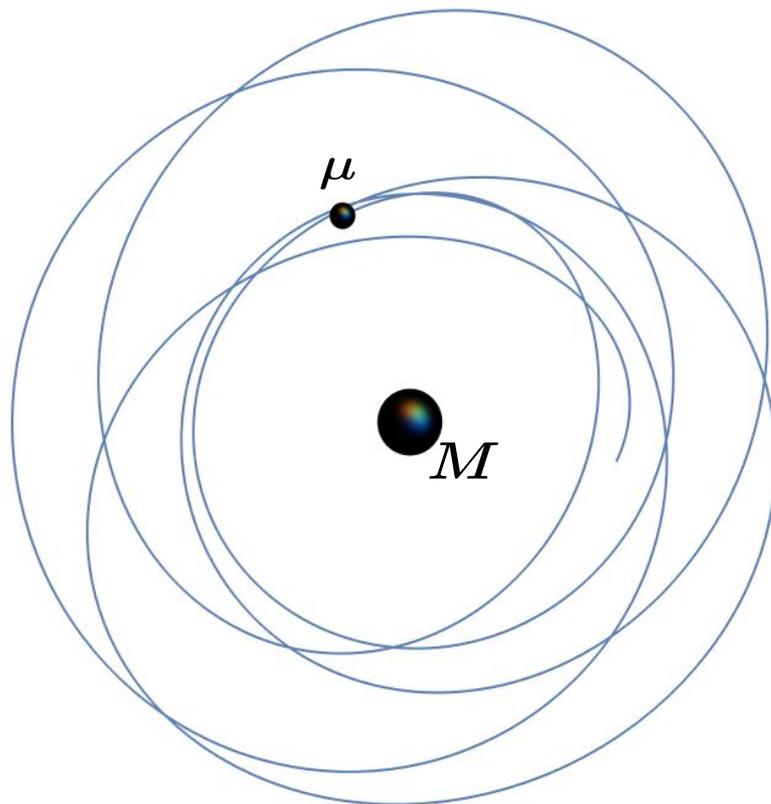
→ Geodesic equation is integrable

→ Action-angle variables are useful

$$\frac{dq_i}{d\tau} = \omega_i(\mathbf{J})$$
$$\frac{dJ_i}{d\tau} = 0$$

$$\mathbf{J} := \{J_r, J_\theta, J_\phi\}$$

Before introducing a perturber....

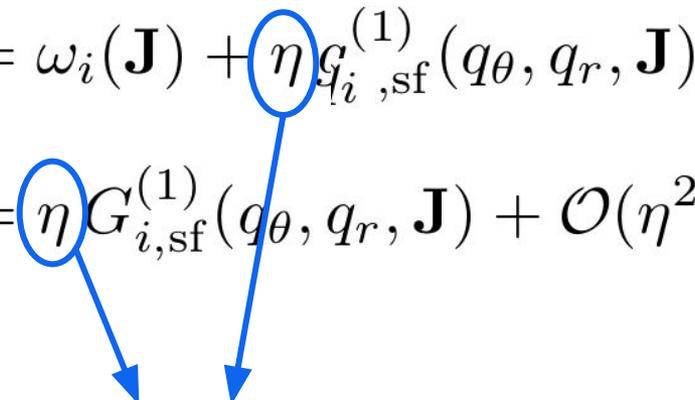


... motion is not geodesic!

Gravitational radiation changes description

$$\frac{dq_i}{d\tau} = \omega_i(\mathbf{J}) + \eta G_{i,\text{sf}}^{(1)}(q_\theta, q_r, \mathbf{J}) + \mathcal{O}(\eta^2)$$
$$\frac{dJ_i}{d\tau} = \eta G_{i,\text{sf}}^{(1)}(q_\theta, q_r, \mathbf{J}) + \mathcal{O}(\eta^2)$$

Mass ratio: μ/M

The diagram consists of two equations. In the first equation, the coefficient η is circled in blue. In the second equation, the coefficient η is also circled in blue. Two blue arrows originate from these circles: one points from the circle in the first equation down to the text 'Mass ratio: μ/M ', and the other points from the circle in the second equation down to the same text.

Self force 101: adiabatic approximation

$$\frac{dq_i}{d\tau} \approx \omega_i(\mathbf{J})$$
$$\frac{dJ_i}{d\tau} \approx \eta \left\langle G_{i,\text{sf}}^{(1)}(q_\theta, q_r, \mathbf{J}) \right\rangle$$

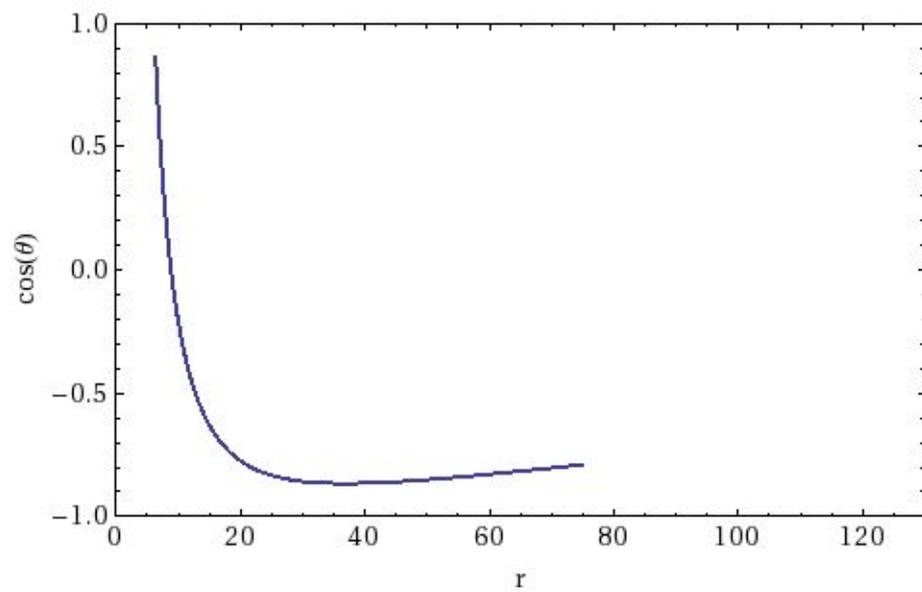
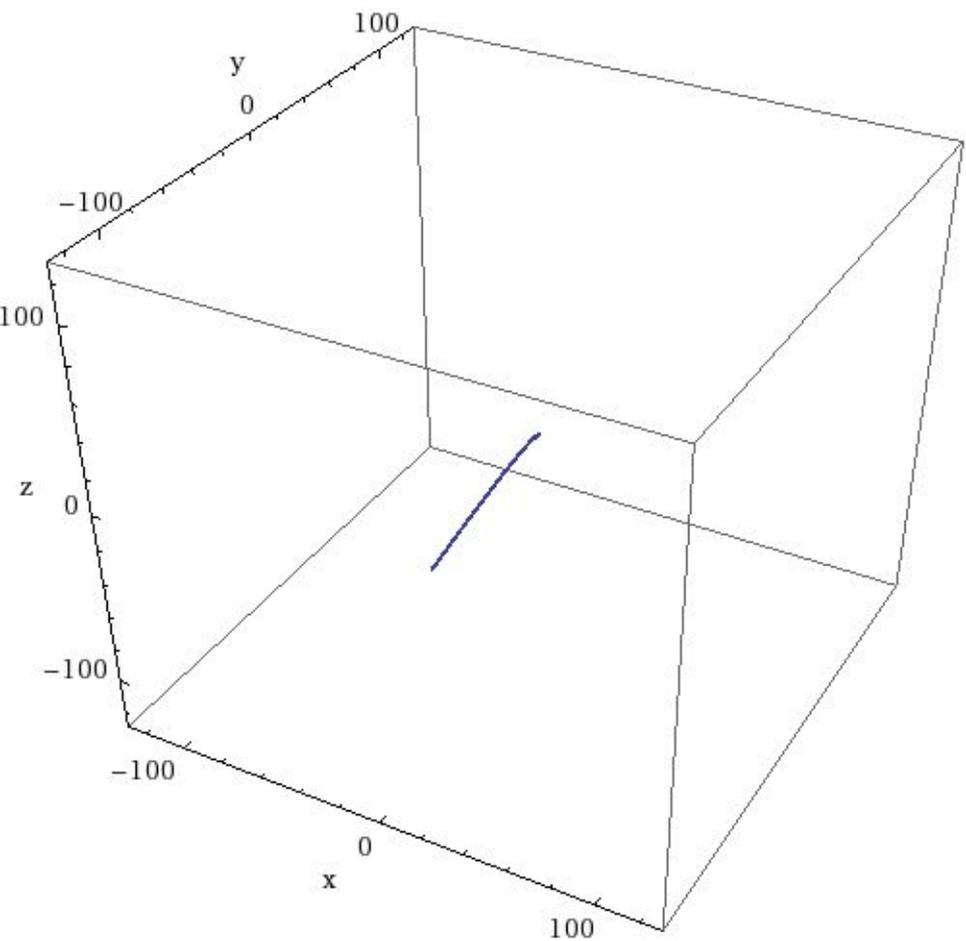
Averaging

$$G_{\text{sf}}^{(1)}(q_\theta, q_r, \mathbf{J}) = \sum_{k,n} G_{\text{sf},kn}^{(1)}(\mathbf{J}) e^{i(kq_\theta + nq_r)}$$

$$\Rightarrow \langle G_{\text{sf}}^{(1)}(q_\theta, q_r, \mathbf{J}) \rangle = G_{\text{sf},00}^{(1)}(\mathbf{J})$$

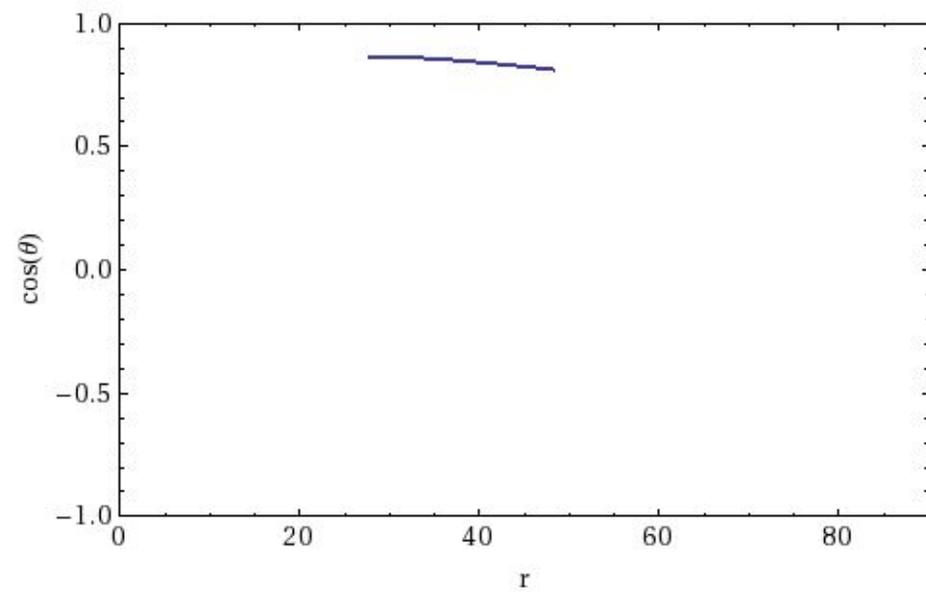
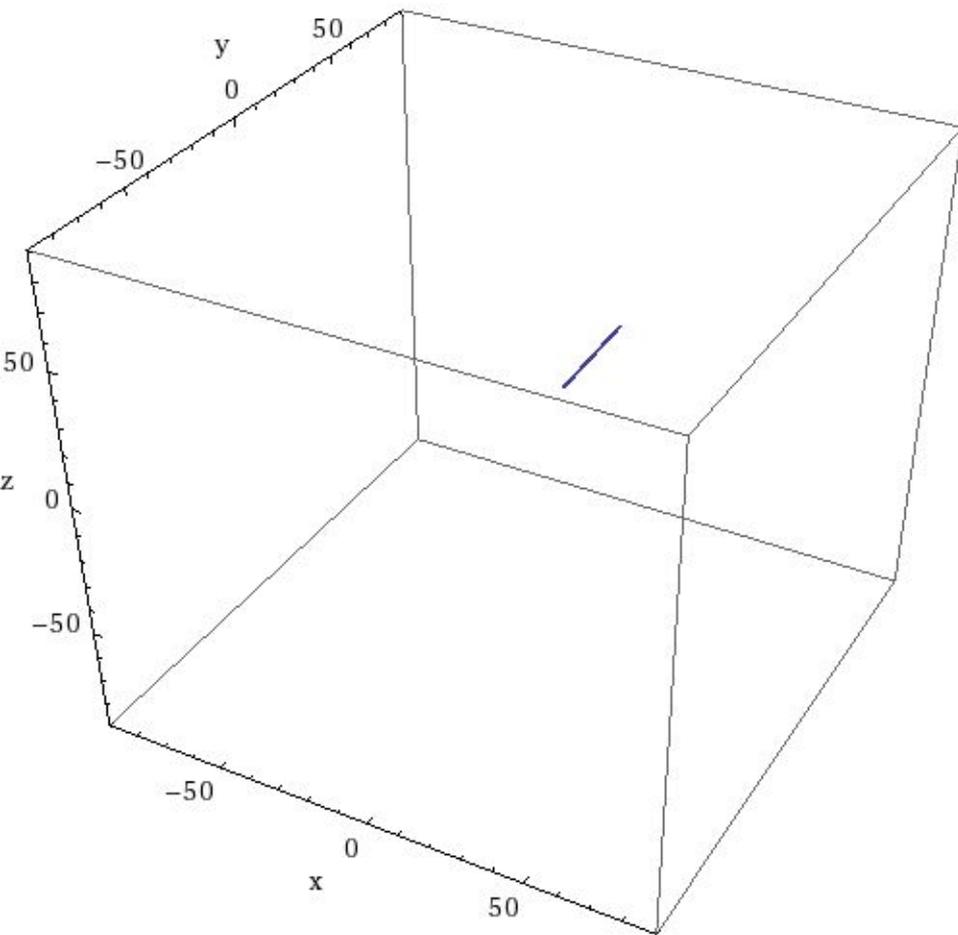
Not true when $k\omega_\theta + n\omega_r \approx 0$

Generic evolution



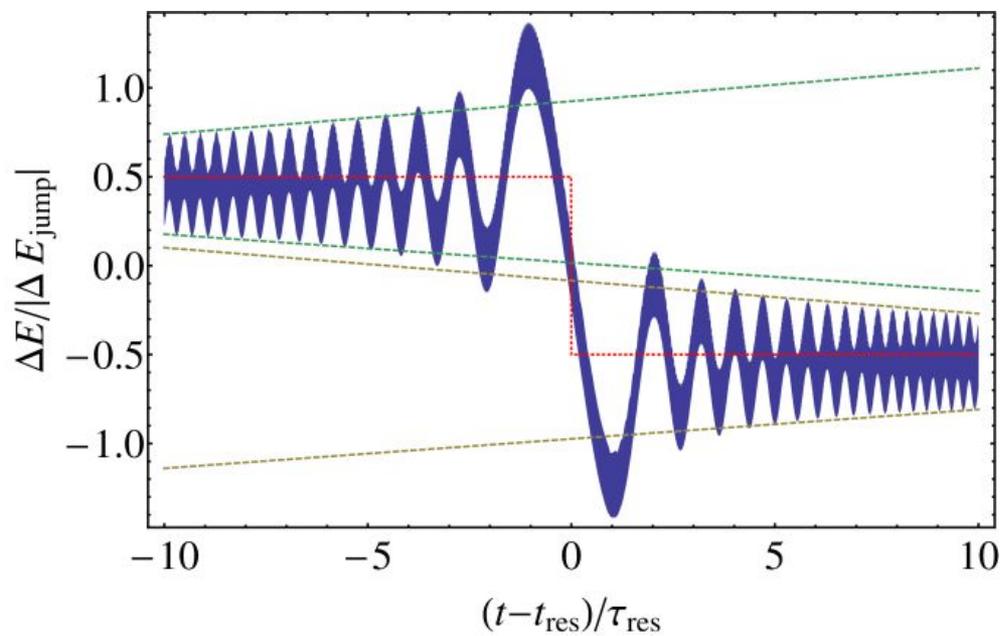
Credit: Rob Cole

Resonant orbit



Credit: Rob Cole

Self-force resonance [Flanagan & Hinderer, PRL 2012]



Are self-force resonances astrophysically important?

- They happen generically for EMRIs in the LISA band.
- Kick size is typically small but if early in the inspiral → significant dephasing of the waveform.
- Bigger kicks when
 - (1) Lower order resonances (=small n and k)
 - (2) More eccentric orbits

**Only lose a few percent for detection purposes,
but important for parameter estimation!**

[Berry, Cole, Canizares, Gair, PRD 2016]

How about the tidal perturber?

Action-angle variables *with* tidal perturber

$$\frac{dq_i}{d\tau} = \omega_i(\mathbf{J}) + \epsilon g_{i,\text{td}}^{(1)}(\underline{q_\phi}, q_\theta, q_r, \mathbf{J}) + \eta g_{k,\text{sf}}^{(1)}(q_\theta, q_r, \mathbf{J}) + \mathcal{O}(\eta^2, \epsilon^2, \eta\epsilon)$$

$$\epsilon = M_\star M^2 / a^3$$

$$\frac{dJ_i}{d\tau} = \epsilon G_{i,\text{td}}^{(1)}(\underline{q_\phi}, q_\theta, q_r, \mathbf{J}) + \eta G_{i,\text{sf}}^{(1)}(q_\theta, q_r, \mathbf{J}) + \mathcal{O}(\eta^2, \epsilon^2, \eta\epsilon)$$

Tidal resonance

$$G_i^{(1)}(q_\phi, q_\theta, q_r, \mathbf{J}) = \sum_{m,k,n} G_{i,mkn}^{(1)}(\mathbf{J}) e^{i(mq_\phi + kq_\theta + nq_r)}$$

$$\omega_{mkn} := m\omega_\phi + k\omega_\theta + n\omega_r = 0$$

- ❖ More general condition than for self-force resonance
- ❖ Also occurs for low eccentricity orbits

Size of the effect?

$$a^\alpha = -\frac{1}{2}(g_{\text{Kerr}}^{\alpha\beta} + u^\alpha u^\beta)(2h_{\beta\lambda;\rho} - h_{\lambda\rho;\beta})u^\lambda u^\rho$$

Metric of tidally perturbed Kerr $h_{\mu\nu}$ from [Gonzales + Yunes, 2005]

> Teukolsky equation + metric reconstruction

> Takes as input \mathcal{E}_{ij}

> Assumes tidal field is stationary

> Caveat: only takes into account $m=\pm 1$ and $m=\pm 2$

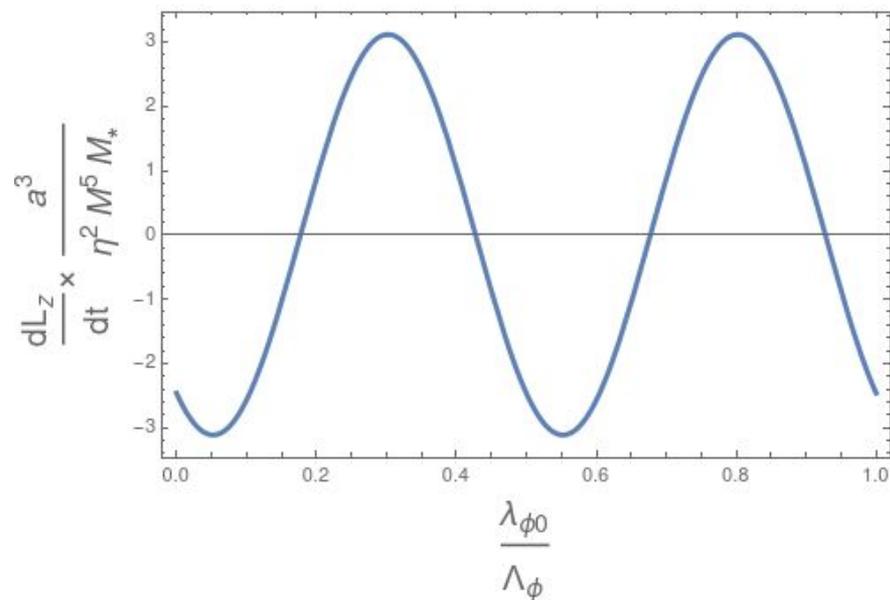
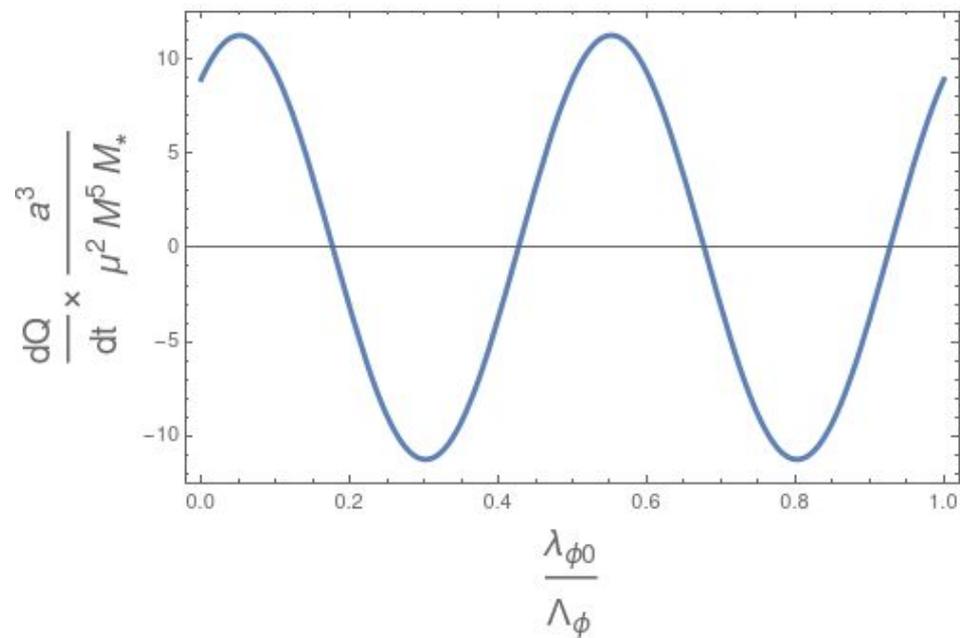
```
(* Created with the Wolfram Language : www.wolfram.com *)  
{{(m2*Sin[ϕ]^2*((-30*a^8 + 4*a^4*(2*m1 - 9*r)*r^3 +  
96*r^6*(-2*m1 + r)^2 + a^6*(44*m1^2 + 84*m1*r - 78*r^2) +  
16*a^2*r^4*(14*m1^2 - 16*m1*r + 3*r^2) +  
a^2*(-45*a^6 + 32*a^2*m1*(2*m1 - r)*r^2 +  
a^4*(62*m1^2 + 70*m1*r - 99*r^2) + 16*r^4*(10*m1^2 - 24*m1*r +  
9*r^2))*Cos[2*ϕ] - 2*a^4*(9*a^4 - 2*(8*m1 - 9*r)*(2*m1 - r)*  
r^2 + a^2*(-10*m1^2 + 10*m1*r + 9*r^2))*Cos[4*ϕ] +  
a^6*(-3*a^2 + 2*m1^2 - 6*m1*r + 3*r^2)*Cos[6*ϕ] +  
Cos[2*ϕ] - 4*a*(a^2 + 2*r^2 + a^2*Cos[2*ϕ]))*  
(4*a^2*r^2*(-11*m1 + 6*r) + a^4*(-13*m1 + 9*r) +  
8*r^3*(4*m1^2 - 8*m1*r + 3*r^2) + 4*a^2*(a^2*(-4*m1 + 3*r) +  
r^2*(-13*m1 + 6*r))*Cos[2*ϕ] + 3*a^4*(-m1 + r)*  
Cos[4*ϕ]) + Sin[2*ϕ]) /  
(128*b^3*(r^2 + a^2*Cos[ϕ]^2)^3),  
(m2*Sin[ϕ]^2*((3*a^6 + 16*a^2*(m1 - r)*r^3 + 24*(2*m1 - r)*r^5 -  
a^4*r*(10*m1 + r) + 4*a^2*(a^4 + 2*(2*m1 - 3*r)*r^3 -  
a^2*r*(2*m1 + r))*Cos[2*ϕ] + a^4*(a^2 + (2*m1 - 3*r)*r)*  
Cos[4*ϕ]) + Cos[2*ϕ] +  
8*a*r*(a^2 + 2*r^2 + a^2*Cos[2*ϕ])*(a^2 + 2*r*(-m1 + r) +  
a^2*Cos[2*ϕ]) * Sin[2*ϕ]) /  
(16*b^3*(r^2 + a^2*Cos[ϕ]^2)^2),  
(m2*Sin[2*ϕ]*((3*a^6*(4*m1 - r) + 8*(2*m1 - r)*r^6 +  
a^4*r*(-2*m1^2 + 32*m1*r - 11*r^2) + 8*a^2*r^3*(m1^2 + 6*m1*r -  
2*r^2) + 4*a^2*(a^4*(4*m1 - r) + 3*a^2*(4*m1 - r)*r^2 -  
2*(m1 - r)^2*r^3)*Cos[2*ϕ] + a^4*(a^2*(4*m1 - r) + 2*m1^2*r -  
r^3)*Cos[4*ϕ]) + Cos[2*ϕ] +  
2*a*(a^2 + 2*r^2 + a^2*Cos[2*ϕ])*(a^4 - 4*m1^2*r^2 + 2*r^4 +  
a^2*(-2*m1^2 - 4*m1*r + 3*r^2) + a^2*(a^2 - 2*m1^2 + 4*m1*r + r^2))*  
Cos[2*ϕ] + Sin[2*ϕ]) /  
(32*b^3*(r^2 + a^2*Cos[ϕ]^2)^2),  
(m2*Sin[ϕ]^2*  
(a*(2*(5*a^8 + 8*r^6*(-9*m1^2 + 2*m1*r + 2*r^2) +  
2*a^2*r^4*(-26*m1^2 - 5*m1*r + 20*r^2) +  
a^6*(-11*m1^2 - 9*m1*r + 23*r^2) + a^4*r^2*(-25*m1^2 - 15*m1*r +  
42*r^2)) + (15*a^8 + 16*m1*(5*m1 - 2*r)*r^6 +  
16*a^2*r^4*(-5*m1^2 + m1*r + 3*r^2) + a^6*(-31*m1^2 - 3*m1*r +  
63*r^2) + a^4*r^2*(-97*m1^2 - 15*m1*r + 96*r^2))*Cos[2*ϕ] +  
2*a^2*(3*a^6 + 2*m1*r^4*(-2*m1 + r) + a^2*r^2*(-23*m1^2 + 23*m1*r +
```

Proof of principle with $m=-2, k=2, n=1$

$$\left\langle G_i^{(1)}(q_\phi, q_\theta, q_r, \mathbf{J}) \right\rangle \approx G_{i,-2,2,1}^{(1)}(\mathbf{J}) e^{-2iq_\phi 0} + \text{cc}.$$

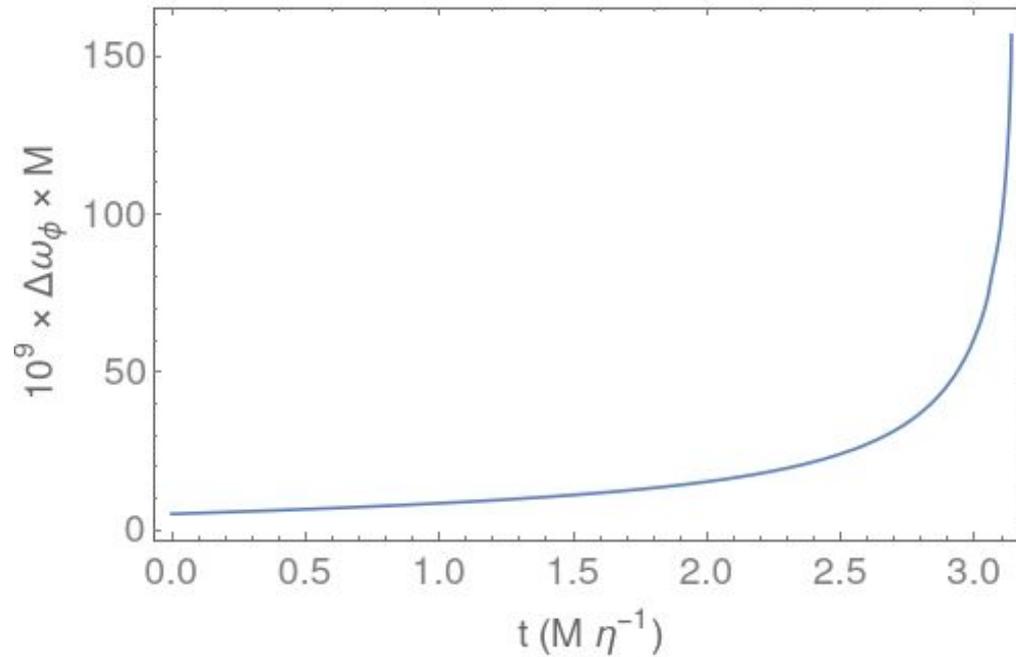
χ	r_{\min}	r_{\max}	θ_{\min}	$\dot{Q}_{-2,2,1}$	$\dot{L}_{z-2,2,1}$
0.7	3.5	5.1628033	$\pi/3$	$1.66 + 2.27i$	$-0.35 - 0.47i$
0.9	3	6.6159726	$\pi/4$	$6.60 + 7.70i$	$-1.72 - 2.01i$
0.99	3	5.3718120	$\pi/4$	$4.46 + 3.43i$	$-1.23 - 0.95i$

Rate of change depends on the phase



Compare two orbits

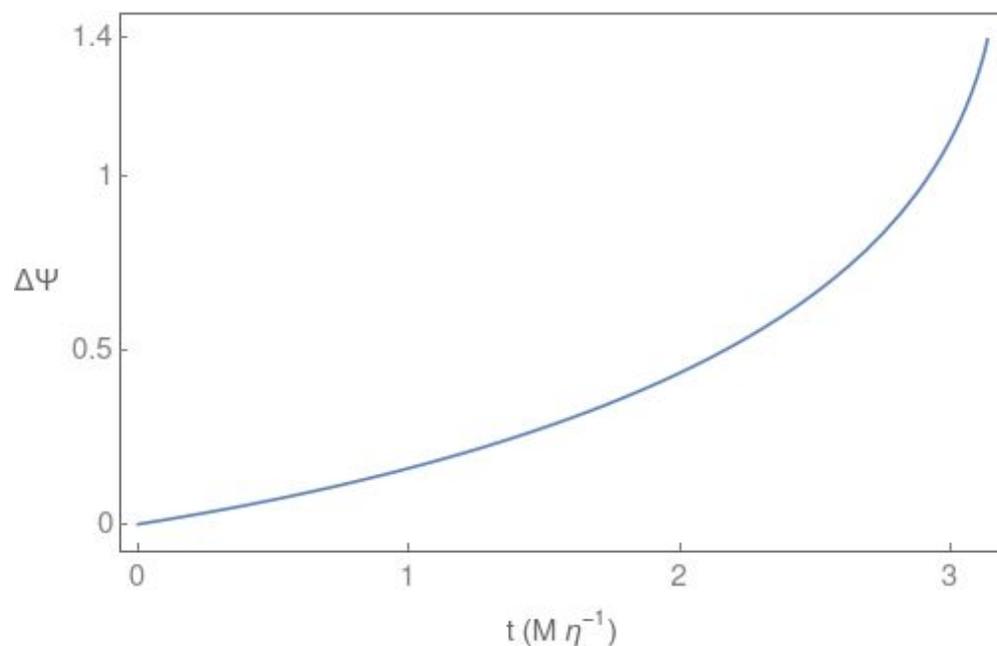
$$\{E, Q, L_z\} \rightarrow \omega_\phi^{(1)} \quad \text{versus} \quad \{E, Q + \Delta Q, L_z + \Delta L_z\} \rightarrow \omega_\phi^{(2)}$$



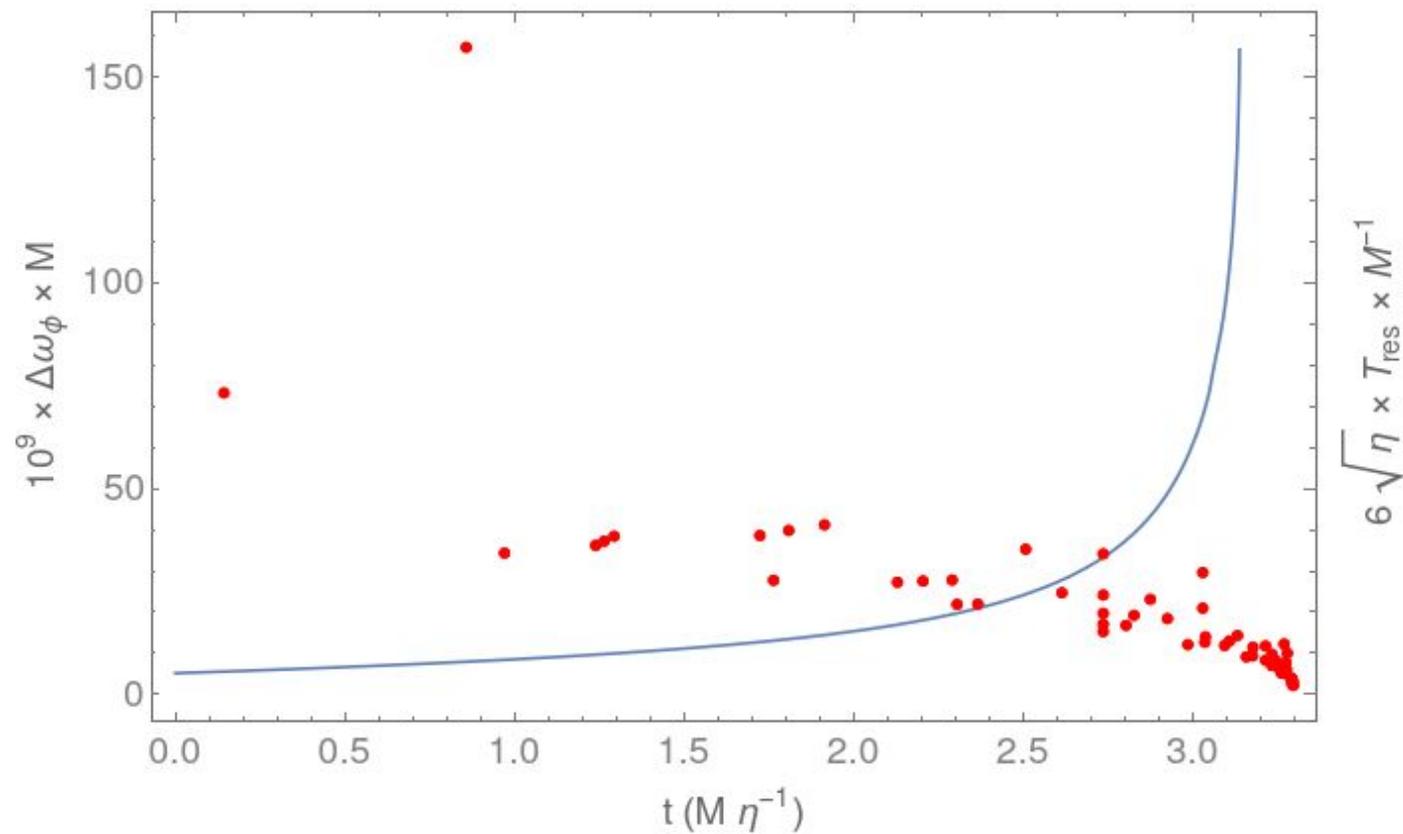
Influence on phase gravitational waveform

$$\Delta\Psi := \int_0^{T_{\text{plunge}}} 2\Delta\omega_\phi dt$$

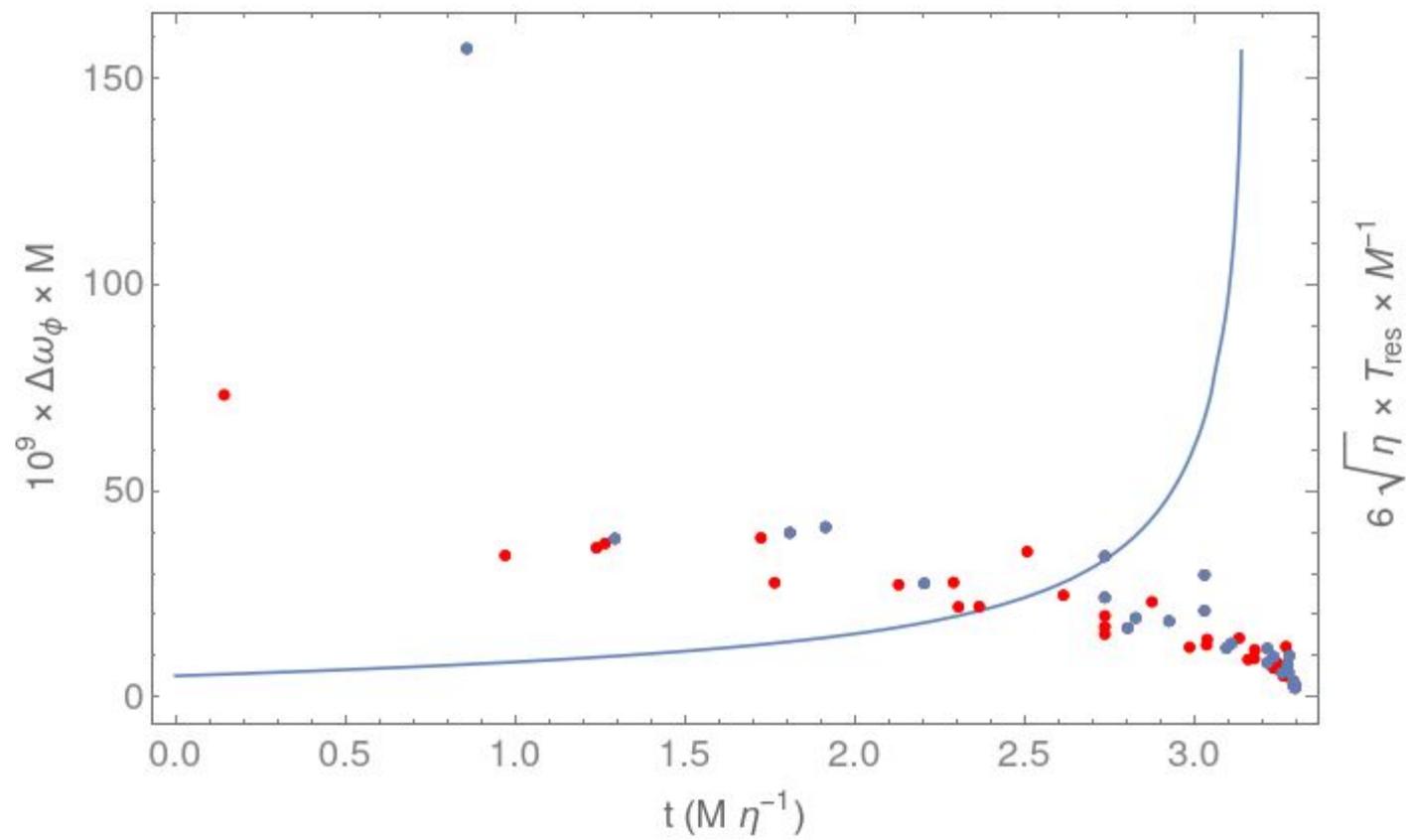
$$= 1.4 \left(\frac{\mu}{10M_\odot} \right)^{-\frac{1}{2}} \left(\frac{M}{M_{\text{SgrA}^*}} \right)^{\frac{7}{2}} \left(\frac{M_*}{10M_\odot} \right) \left(\frac{a}{4.3 \text{ AU}} \right)^{-3}$$



Many resonances

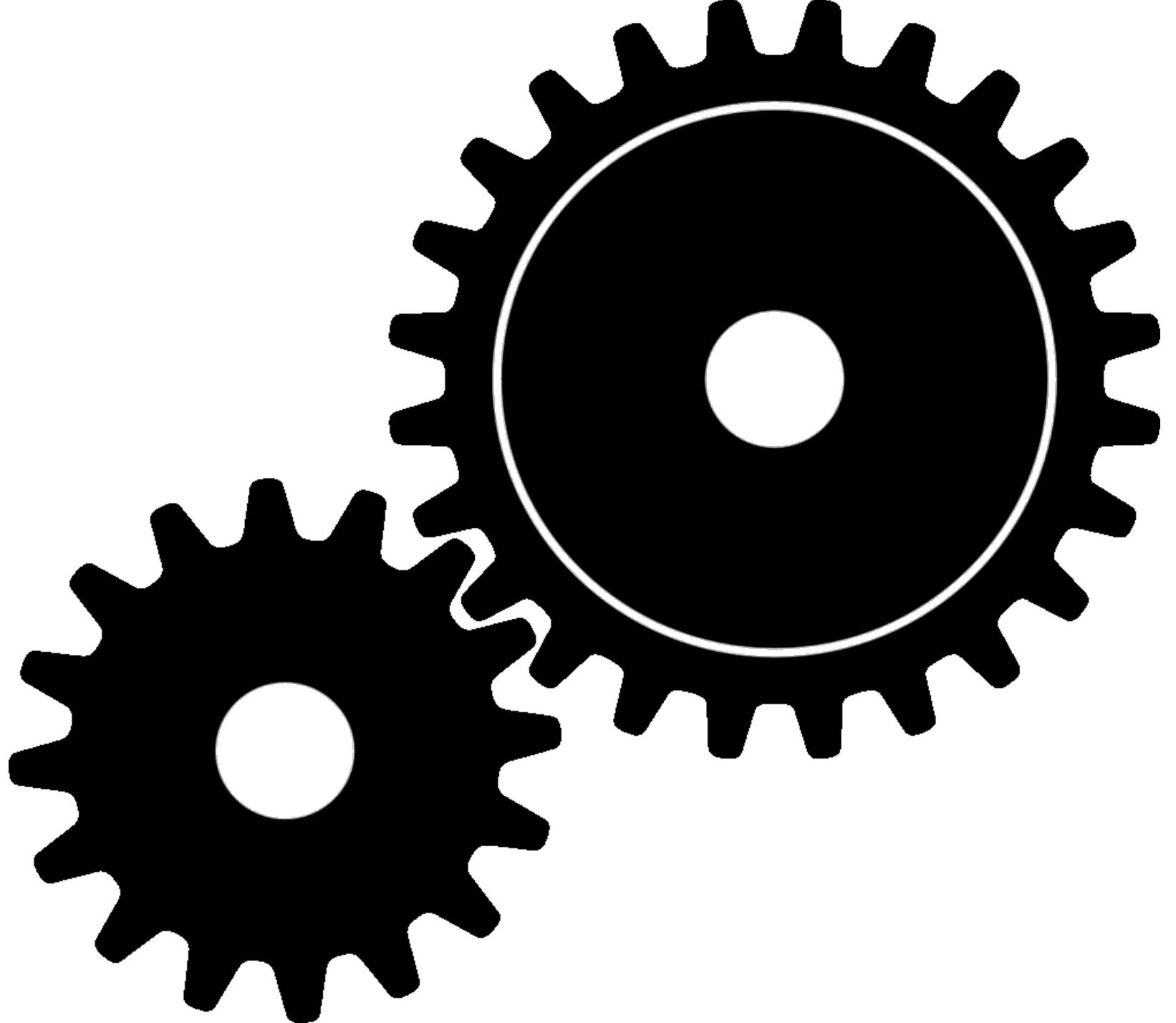


Many resonances



Tidal resonances summary

- ❖ Tidal field can change EMRI waveforms appreciably depending on the distance and mass of the tidal perturbers
- ❖ Opportunity to learn about “light” black holes or stars “close” to massive black holes
- ❖ Critical to understand when constraining deviations from General Relativity



How do we constrain Equation of State (EOS) of neutron stars?

1

M_{max} Maximum observed mass
(Radio astronomy)

2

M vs R
(X-ray astronomy)

3

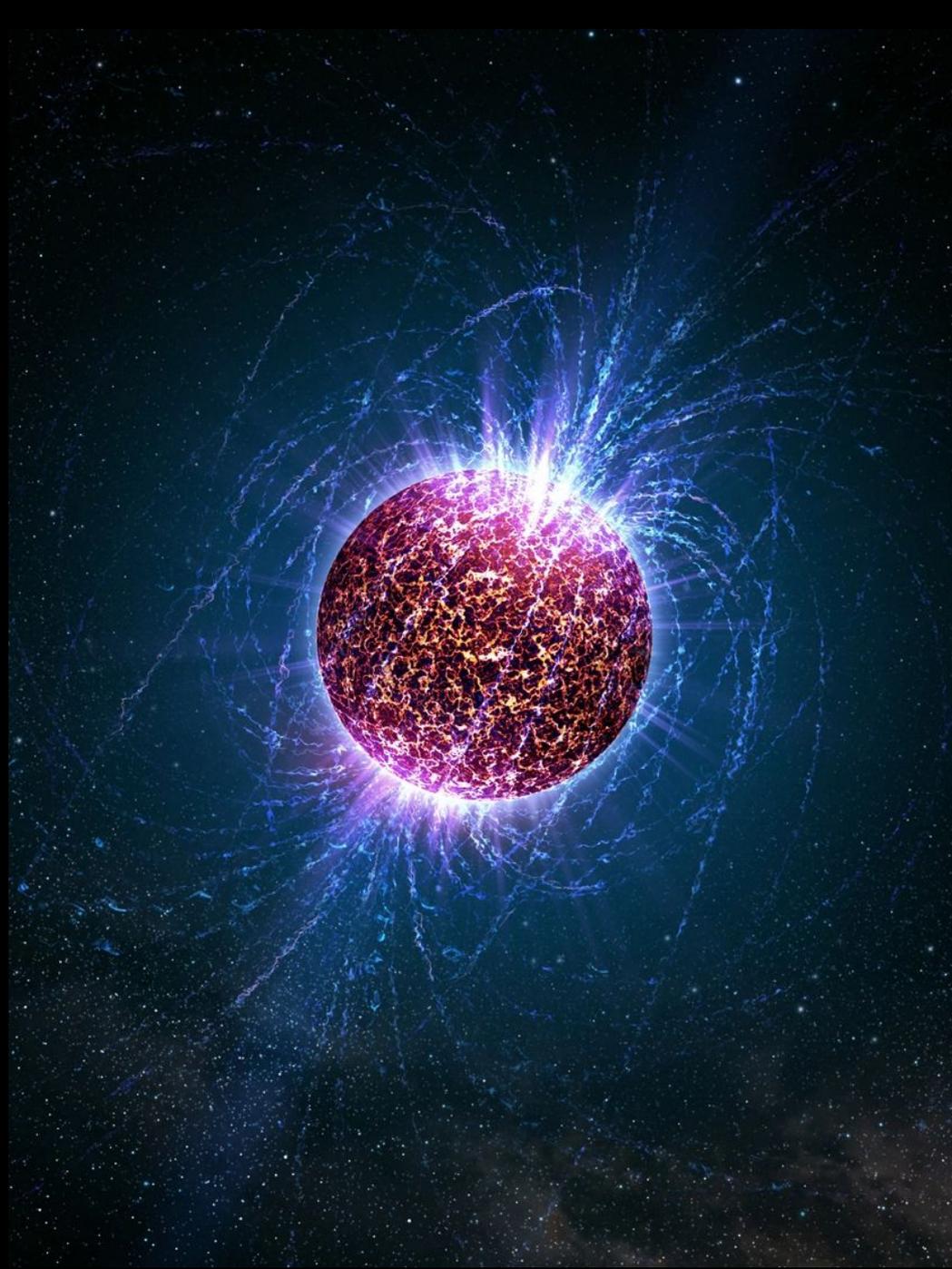
Deformability
(Gravitational wave astronomy)

4

NS seismology
(Light curves)

5

NS seismology
(Gravitational wave astronomy)



NS seismology in binaries with gravitational waves

Principle

Mode resonance
is excited when

$$f_0 \approx f_{\text{GW}}$$



Binary loses
orbital energy to
the mode

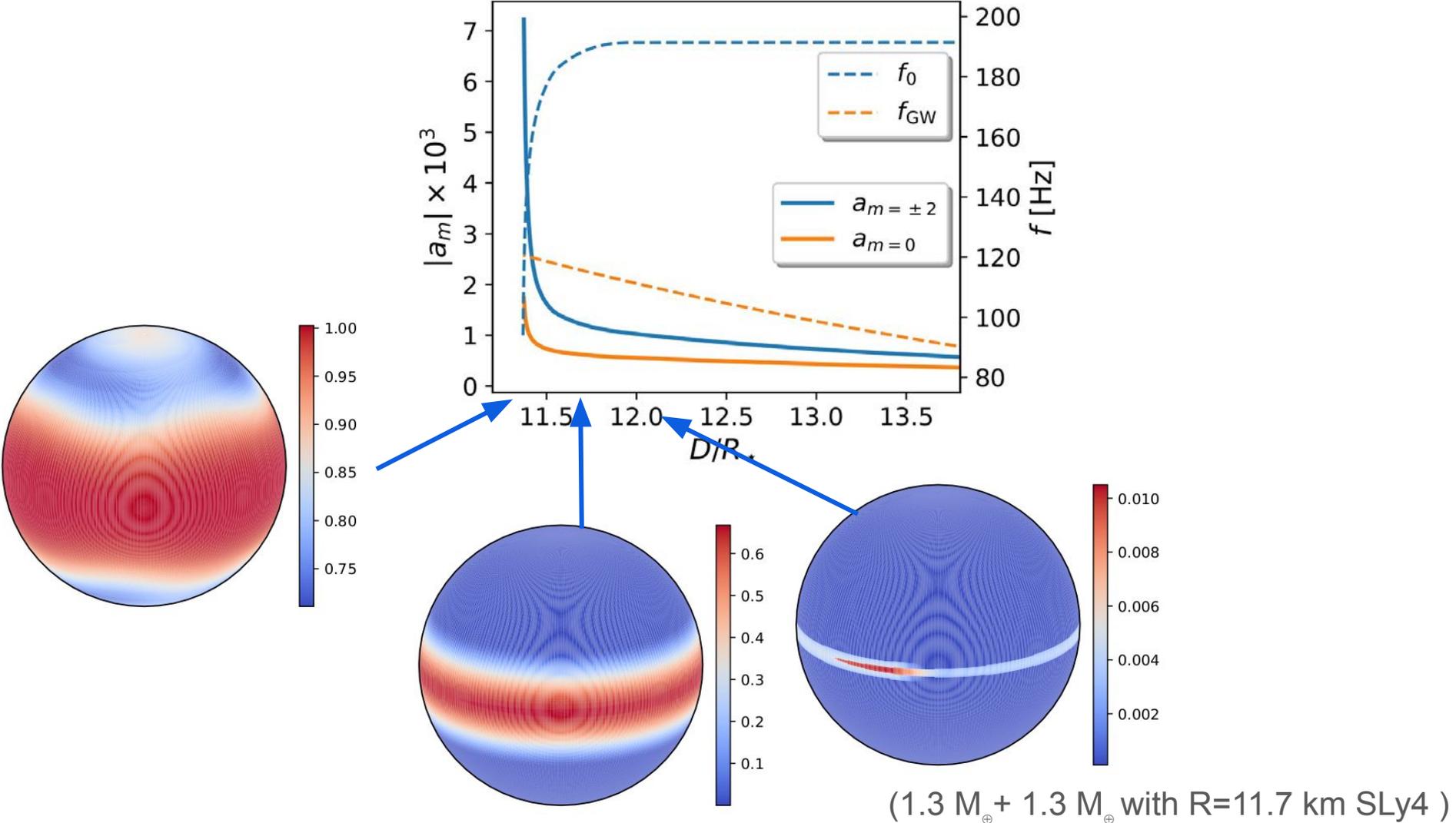


Phase shift $\delta\phi$
in GW

$$\delta\varphi \simeq 5 \cdot 10^{-3} \left(\frac{100 \text{ Hz}}{f_0} \right)^2 \left(\frac{Q}{3 \cdot 10^{-4}} \right)^2 \left(\frac{1.4 M_{\odot}}{M} \right)^4 \left(\frac{R}{10 \text{ km}} \right)^2$$

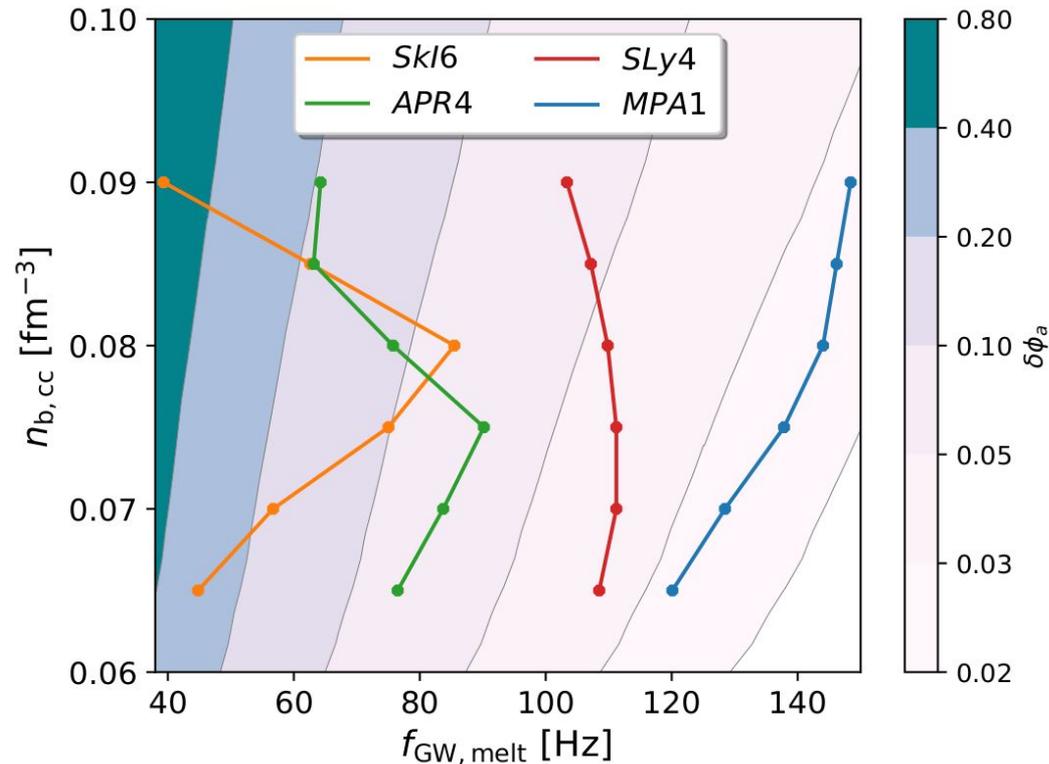
f	$f_0 \approx 2 \text{ kHz}$ $Q \approx 0.5$	Too high frequency for LIGO
p	$f_0 \geq 2 \text{ kHz}$	Too high frequency for LIGO
g	$f_0 \approx 100 \text{ Hz}$ $Q \approx 10^{-4}$	Small $\delta\varphi$
r	$f_0 \approx f_{\text{star}}$	Requires highly spinning NS ($\geq 100 \text{ Hz}$)
i	$f_0 \approx (30,200) \text{ Hz}$ $Q \approx 10^{-2}$	$\delta\varphi = (1,50)$

More interesting: the crust melts!



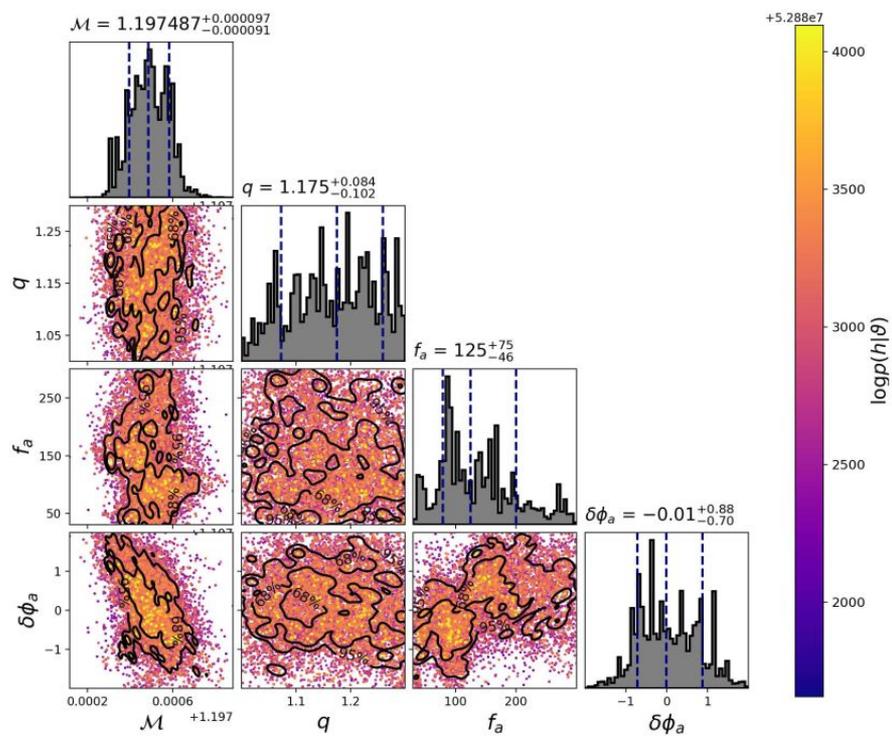
Phase shift depends on EOS

$$\delta\varphi \simeq \underbrace{\frac{\delta\varphi}{\delta t}}_{=2\omega_{\text{orbit}}} \times \underbrace{\frac{\delta t}{\delta E}}_{=P_{\text{GW}}^{-1}} \times \underbrace{\delta E}_{=E_{\text{melt}}} \simeq 0.1 \left(\frac{E_{\text{melt}}}{10^{47} \text{ ergs}} \right) \left(\frac{1.3 M_{\odot}}{M} \right)^{10/3} \left(\frac{70 \text{ Hz}}{f_{\text{GW,melt}}} \right)^{7/3}$$

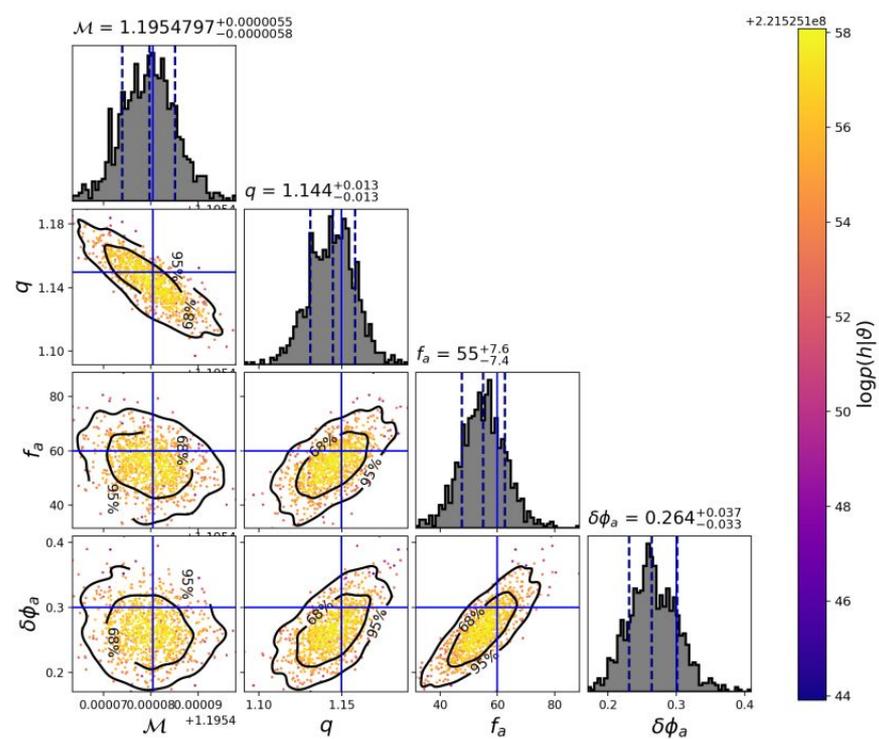


Observable by LIGO A+

GW170817



Example with LIGO A+



Summary i-mode

- Tidal interactions in binary neutron stars can excite the core-crust interface mode
- This can melt the crust and change the waveform
- Mode analysis is based on Newtonian perturbations on a relativistic background
 - work in progress: fully relativistic calculation i-mode