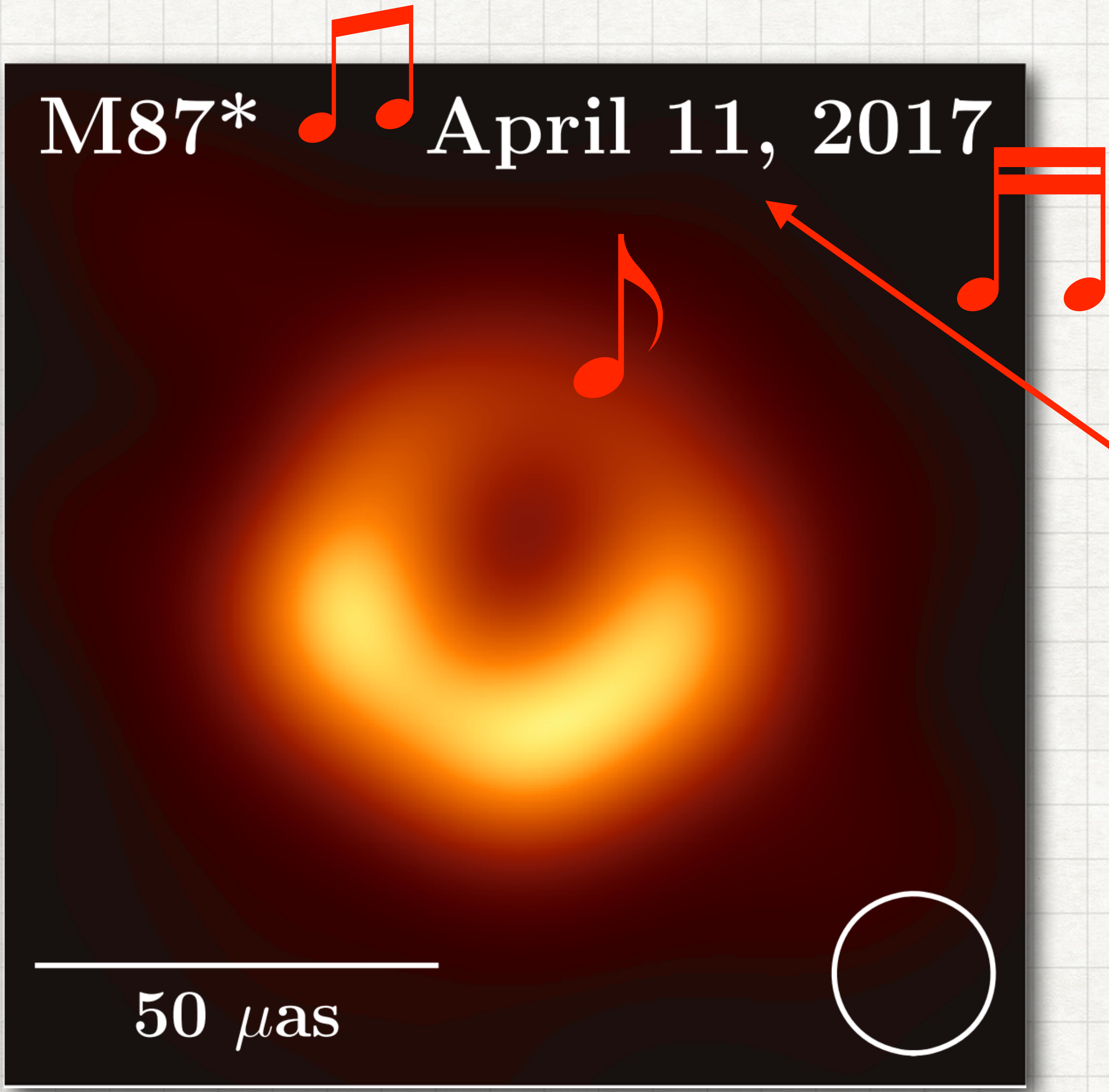
MULTI-FLUID ACTION PRINCIPLE*



My Birthday!

G. L. Comer
Saint Louis University

*: Parental discretion advised.

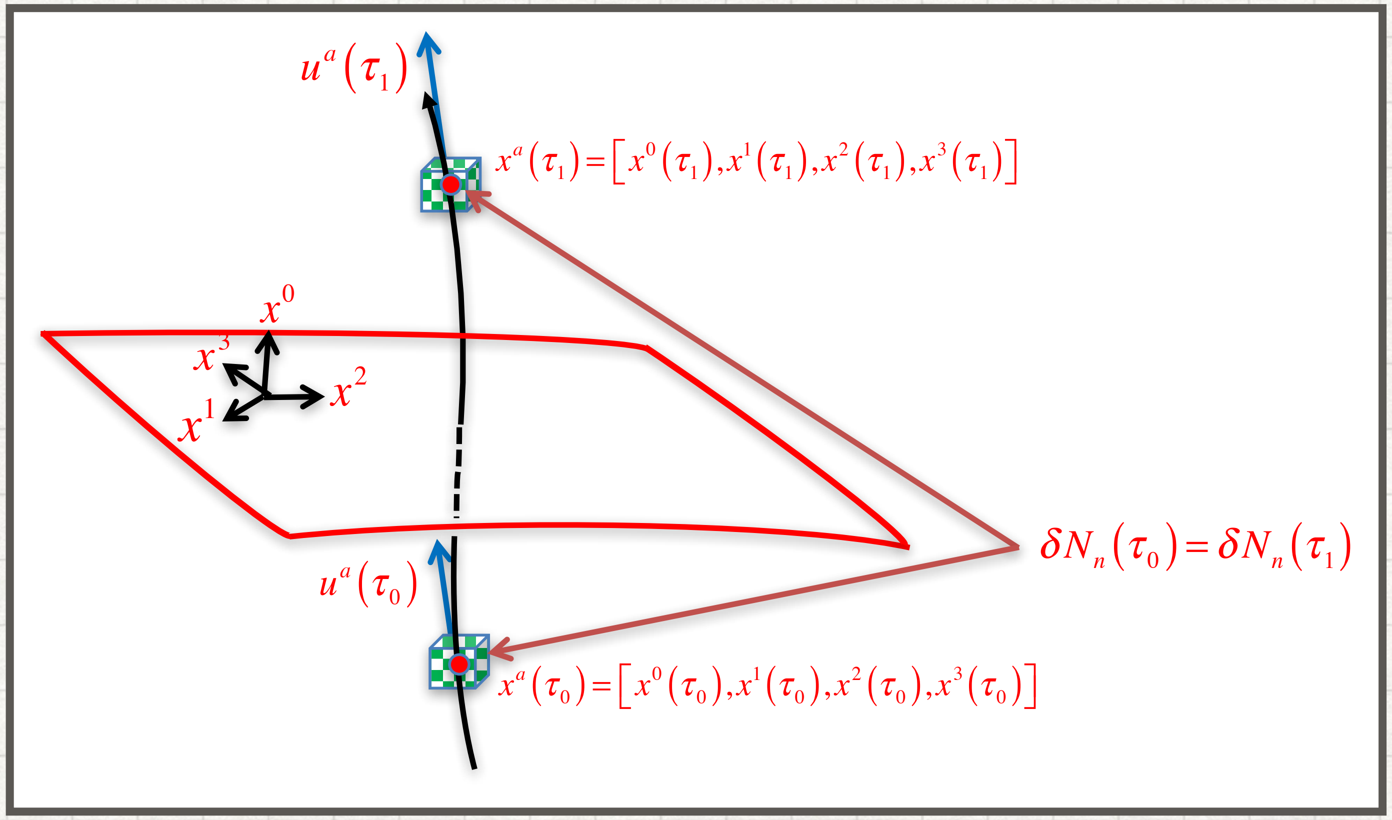
PROFESSOR KOMPRESSOR RE-CAP

1. Consider a conservative single fluid system with flux n^a:

   1.1. It is a field describing the worldlines of fluid elements in spacetime, where each element contains a certain number of particles δN_a; i.e. if $x^a(\tau)$ are the points on the worldline, where τ is the proper time, then

$$\text{Worldline:} \quad \bigcup_{\tau=\tau_1}^{\tau_2} x^a(\tau) \quad , \quad x^a(\tau) = \left\{ x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau) \right\}$$

   1.2. Pictorially,



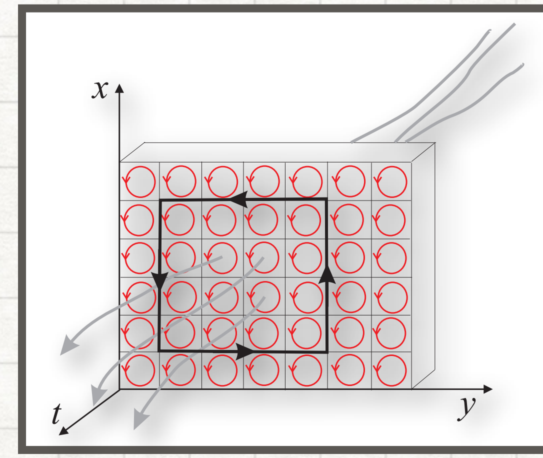
   1.3. The unit tangent vector u^a is

$$u^a = \frac{dx^a}{d\tau} \quad , \quad -d\tau^2 = ds^2 = g_{ab}dx^a dx^b = \left(g_{ab} \frac{dx^a}{d\tau}\frac{dx^b}{d\tau} \right) d\tau^2 \quad \Rightarrow \quad u_a u^a = -1$$

   1.4. The flux magnitude is the particle number density $n = - u_a n^a$.

PROFESSOR KOMPRESSOR RE-CAP CONTINUED

2. For a single-fluid system the set-up is as follows:



2.1. There are four elements:

2.1.1. the flux $n^a = n\, u^a$,

2.1.2. the mass-energy density ρ,

2.1.3. the pressure p,

2.1.4. and the Equation of State; i.e. $\rho = \rho(n)$ and/or $p = p(n)$.

2.2. The Energy-Momentum-Stress tensor T^{ab} for a single-fluid system is

$$\boxed{T^{ab} = pg^{ab} + (\rho + p)u^a u^b}$$

2.3. If we introduce the chemical potential $\mu = d\rho/dn$, the chemical potential covector $\mu^a = \mu\, u^a$, and use the Euler relation $\mu n = \rho + p$, we can write

$$\boxed{T^{ab} = pg^{ab} + \mu^a n^b}$$

2.4. Non-dissipative, or conservative, flow means

$$\boxed{\text{Particle Number Creation Rate:} \quad \Gamma_a = \nabla_a n^a = 0}$$

2.5. Take the Einstein Equation ($G^b{}_a = 8\pi\, T^b{}_a$), use the Second Bianchi Identity ($\nabla_b G^b{}_a = 0$, next slide!), use the particle conservation, and the Euler Relation, *et voila!*, we get

$$\boxed{\nabla_b T^b{}_a = n^b \omega_{ba} = 0 \quad , \quad \omega_{ab} = 2\nabla_{[a}\mu_{b]}}$$

2.6. This is an integrability condition for the vorticity; pictorially, the fluid flux worldlines do not "pierce" their own vorticity world-tube.

COORDINATE (DIFFEOMORPHISM) INVARIANCE AND ENERGY-MOMENTUM-STRESS TENSOR CONSERVATION

1. ***Coordinate (Diffeomorphism) Invariance*** is an essential assumption of General Relativity and it says the form of physical laws must be invariant under arbitrary differentiable coordinate transformations:

$$\boxed{\bar{x}^a = f^a\left(x^b\right) \quad \Leftrightarrow \quad x^a = g^a\left(\bar{x}^b\right)}$$

2. Like any invariance, identities result; here, it is for combinations of the Riemann Tensor R_{abcd}, the so-called ***Bianchi Identities***:

$$\boxed{\begin{cases} \text{First Bianchi Identity (Algebraic):} \quad R_{abcd} + R_{acdb} + R_{adbc} = 0 \\ \text{Second Bianchi Identity (Differential):} \quad R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0 \end{cases}}$$

3. The Einstein Equation is

$$\boxed{G^b{}_a = 8\pi T^b{}_a}$$

4. Because of the 2nd Bianchi Identity and the Einstein Equation it is an identity that

$$\boxed{\begin{cases} \text{Bianchi} \quad \Rightarrow \quad \nabla_b G^b{}_a = 0 \\ \text{Bianchi Plus Einstein} \quad \Rightarrow \quad \nabla_b T^b{}_a = 0 \end{cases}}$$

5. This was something that always troubled me about dissipative fluid models; not that they were wrong, but that something was missing.

COORDINATE (DIFFEOMORPHISM) INVARIANCE AND ENERGY-MOMENTUM-STRESS TENSOR CONSERVATION CONTINUED

5. In fact, regardless of the Einstein Equations, we have the following:
    5.1. Use the definition of T_{ab} as a variation of the action with respect to the metric:

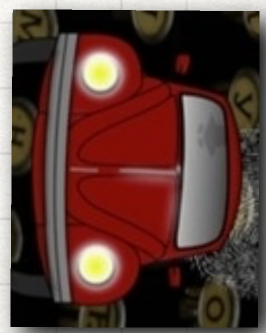
$$\delta_{metric} S_{matter} = \delta_{metric}\left(\int_M \sqrt{-g}\, d^4x \cdot \Lambda_{matter}\right) = \int_M \sqrt{-g}\, d^4x \left[\frac{2}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\,\Lambda_{matter}\right)}{\partial g_{ab}}\right]\delta g_{ab} \equiv \int_M \sqrt{-g}\, d^4x \left(T^{ab}\delta g_{ab}\right)$$

    5.2. Impose a "small" coordinate transformation on the metric:

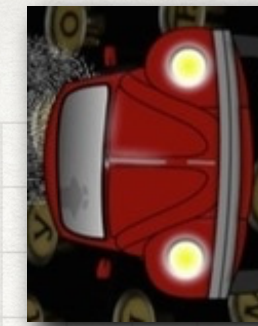
$$\delta_{DI} g_{ab} = \nabla_a \delta\xi_b + \nabla_b \delta\xi_a$$

    5.3. Coordinate invariance means the action variation must be zero:

$$\delta_{DI} S_{matter} = \int_M \sqrt{-g}\, d^4x \left[T^{ab}\left(\nabla_a \delta\xi_b + \nabla_b \delta\xi_a\right)\right] = \int_M \sqrt{-g}\, d^4x \left(2\nabla_b T^{ab}\right)\delta\xi_a = 0$$
$$\therefore \nabla_b T^b{}_a = 0$$

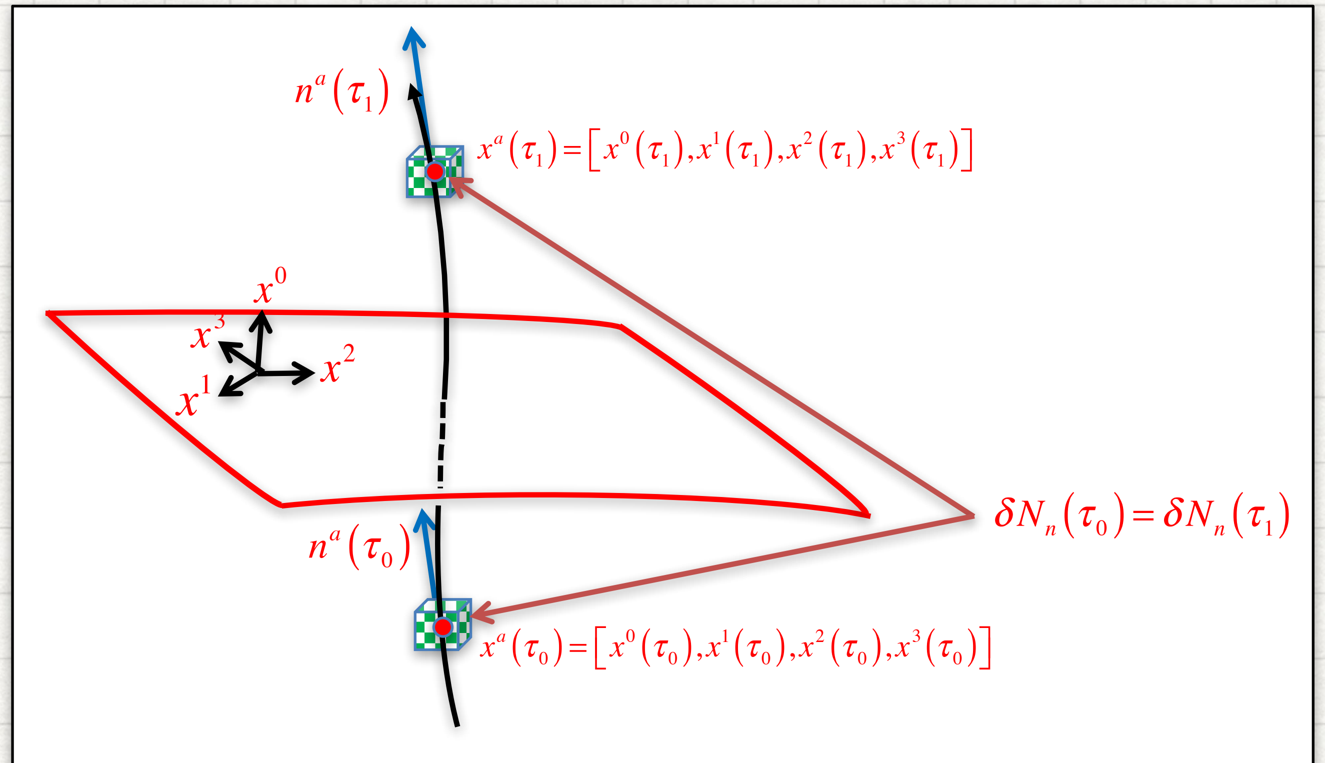


    5.4. Therefore, strictly speaking, $\nabla_b T^b{}_a = 0$ is *NOT* the fluid field equation; it is an identity — true *because* of the field equations (as we will see).
6. Also, the Bianchi Identity is four equations:
    6.1. When there are two or more fluids, it is not enough.
    6.2. A non-zero temperature superfluid compact object can have four of more independent fluxes — n^a (neutrons), p^a (protons), e^a (electrons), s^a (entropy), *etc*.
7. Also, when new physics is introduced, such as electromagnetism, the identity is not enough.



LOCAL FLUID CONTINUITY/PARTICLE CONSERVATION

1. Recall that the fundamental field of fluid dynamics is the flux n^a:



2. The heart of the fluid model action principle is the basic picture of flux, and how to keep it conserved:
   2.1. There are fluid elements having, say, volume $\delta^{(3)}V$.
   2.2. Into each fluid element we place, at some initial time, δN_n particles.
   2.3. The fluid element itself traces out a worldline with unit tangent u^a.
   2.4. These elements are combined to give

$$\left\{\begin{array}{l} \text{Worldline:}\quad x^a(\tau)\quad,\quad u^a=\dfrac{dx^a}{d\tau} \\[2ex] \text{Density:}\quad n=\dfrac{\delta N_n}{\delta^{(3)}V} \end{array}\right. \Rightarrow \quad \text{Flux:}\quad n^a=nu^a\quad,\quad n=-u_a n^a$$

LOCAL FLUID CONTINUITY/PARTICLE CONSERVATION CONTINUED

3. Consider a fluid element moving from spacetime point (t,x^i) to $(t+\delta t, x^i+\delta x^i)$:

   3.1. The number of particles at the different points are, respectively,

$$\delta N_n(t,x^i) = n(t,x^i)\,\delta^{(3)}V(t,x^i)$$

$$\begin{aligned}\delta N_n(t+\delta t, x^i+\delta x^i) &= n(t+\delta t, x^i+\delta x^i)\,\delta^{(3)}V(t+\delta t, x^i+\delta x^i)\\ &\approx n(t,x^i)\,\delta^{(3)}V(t,x^i) + \left[\delta^{(3)}V(t,x^i)\frac{\partial n}{\partial t}(t,x^i) + n(t,x^i)\frac{\partial \delta^{(3)}V}{\partial t}(t,x^i)\right]\delta t\\ &\quad + \left[\delta^{(3)}V(t,x^i)\frac{\partial n}{\partial x^i}(t,x^i) + n(t,x^i)\frac{\partial \delta^{(3)}V}{\partial x^i}(t,x^i)\right]\delta x^i\end{aligned}$$

   3.2. For small enough fluid three-velocity, and cheap gravity, we have

$$\delta\tau \approx \delta t \quad \Rightarrow \quad u^0 \approx \frac{\delta t}{\delta \tau} \approx 1 \quad , \quad u^i \approx \frac{\delta x^i}{\delta t} \quad \Rightarrow \quad x^i(t+\delta t) \approx x^i(t) + u^i \delta t$$

   3.3. The fluid volume takes the two values

$$\delta^{(3)}V_t = \varepsilon^t_{ijk}\,\delta x^i_t \delta x^j_t \delta x^k_t \quad , \quad \delta^{(3)}V_{t+\delta t} = \varepsilon^{t+\delta t}_{ijk}\,\delta x^i_{t+\delta t}\delta x^j_{t+\delta t}\delta x^k_{t+\delta t} \quad , \quad \varepsilon^t_{ijk} = \varepsilon^{t+\delta t}_{ijk} = \pm 1, 0$$

   3.4. We can think of an "active" coordinate transformation, or that the fluid element coordinates are carried along (Lie-Dragged), so as to write

$$\frac{\partial x^i_{t+\delta t}}{\partial x^l_t} \approx \delta^i_j + \frac{\partial u^i}{\partial x^j_t} \quad \rightarrow \quad \delta^{(3)}V(t+\delta t, x^i_{t+\delta t}) \approx \varepsilon^t_{ijk}\frac{\partial x^i_{t+\delta t}}{\partial x^l_t}\frac{\partial x^j_{t+\delta t}}{\partial x^m_t}\frac{\partial x^k_{t+\delta t}}{\partial x^n_t}\delta x^l_t \delta x^m_t \delta x^n_t \approx \left(1+\frac{\partial u^i}{\partial x^i_t}\right)\delta^{(3)}V(t,x^i_t)$$

   3.5. Putting everything back together

$$\delta N_n(t+\delta t, x^i_t+\delta x^i) = \delta N_n(t,x^i_t) \quad \rightarrow \quad \therefore \nabla_a n^a = 0$$

THE ACTION PRINCIPLE: FIRST ATTEMPT

1. A fluid element is small enough as to be infinitesimal with respect to the whole fluid, but having enough particles where a thermodynamic description is warranted:
   1.1. Let's assume there is one free parameter and take it to be the particle number density n.
   1.2. Let's also assume that an equation of state is known: $\rho = \rho(n)$.
   1.3. From it all dependent thermodynamic functions can be determined:

$$\boxed{\rho = \rho(n) \quad \rightarrow \quad \begin{cases} \text{Chemical Potential:} \quad \mu(n) = \dfrac{\partial \rho}{\partial n} \\ \\ \text{Pressure:} \quad p(n) = n\dfrac{\partial \rho}{\partial n} - \rho \end{cases}}$$

2. Since the equation of state contains all the thermodynamic "knowledge", it is a reasonable guess that a fluid action principle could be based on $\rho = \rho(n)$, provided that the independent parameter is relatable to the flux n^a, which of course it is because

$$\boxed{n^2 = \left(-u_a u^a\right) n^2 = -n_a n^a = -g_{ab} n^a n^b}$$

3. For variations which fix the metric, $u_a \delta u^a = 0$, so that the definition of the chemical potential allows us to write

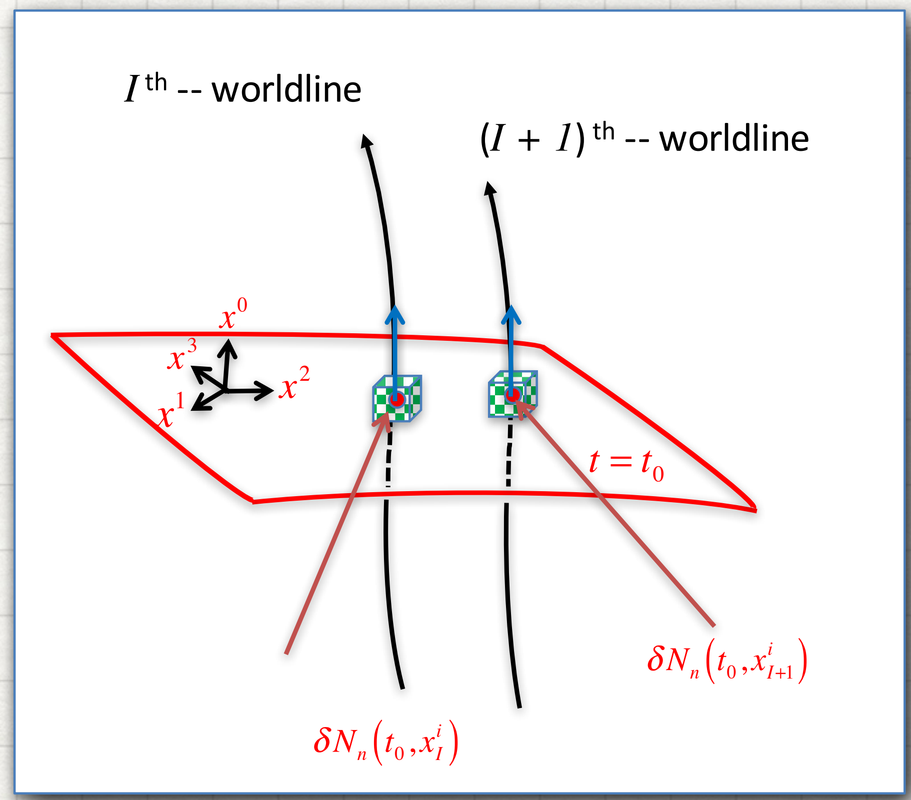
$$\boxed{\mu = \frac{\partial \rho}{\partial n} \quad \Rightarrow \quad \delta\rho = \frac{\partial \rho}{\partial n}\delta n = \mu\left(-u_a u^a\right)\delta n = -\mu u_a \delta\left(n u^a\right) = -\mu_a \delta n^a}$$

4. Unfortunately, if we take this to be our Lagrangian variation then the equation of motion is

$$\boxed{\delta\rho = 0 \quad \Rightarrow \quad \mu_a = 0}$$

THE ACTION PRINCIPLE: FIRST ATTEMPT CONTINUED

5. We don't need to give up on using the thermodynamic "master function" as the basis for the fluid action principle, rather, we need to be more careful about the flux variation:
   5.1. It needs to be constrained in the since that even though n^a has four components, it has only three dynamical degrees of freedom.
   5.2. Obviously, the fact that $u_a u^a = -1$ means the four-velocity is constrained.
   5.3. The fluid element particle numbers depend on only the initial spatial coordinates, as in the picture below:



6. So, it seems that both the initial worldline location and the number of particles imparted depend on only the ***3*** initial spatial coordinate values.
7. In the flux conservation demonstration we argued that the initial fluid element locations could be carried along with the fluid element.
8. We will now show that this feature can be exploited to the point that "worldline labels" become the degrees of freedom.

THE PULL-BACK FORMALISM

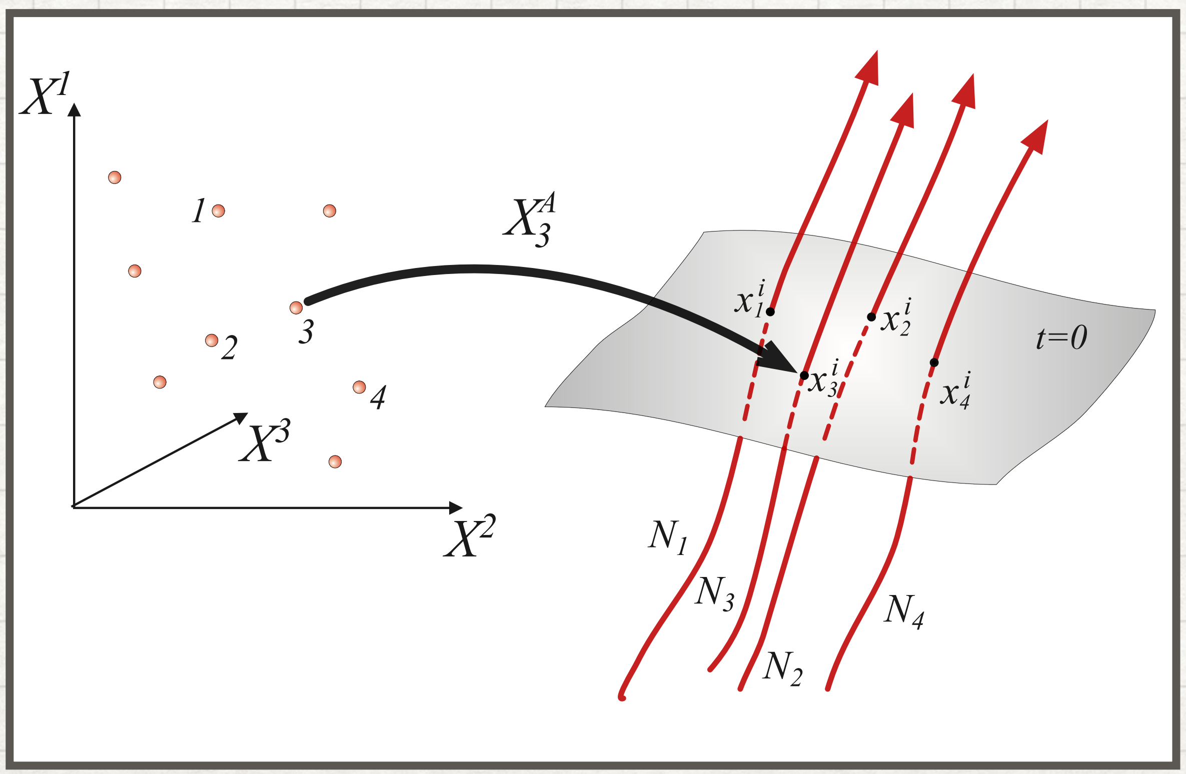
1. There is a dual formalism to particle flux conservation; namely, closed three-forms:
   1.1. The three-form which is dual to the particle number flux is given by

$$\boxed{n_{abc} = \varepsilon_{abcd} n^d \quad \leftrightarrow \quad n^a = \frac{1}{3!} \varepsilon^{abcd} n_{bcd}}$$

   1.2. There is a one-to-one correspondence between the vanishing covariant divergence of the flux and the closure of the dual three-form:

$$\boxed{\nabla_{[a} n_{bcd]} = 0 \quad \Rightarrow \quad \frac{1}{3!} \varepsilon^{abcd} \nabla_{[a} n_{bcd]} = \nabla_a \left(\frac{1}{3!} \varepsilon^{abcd} n_{bcd} \right) = \nabla_a n^a \quad \Rightarrow \quad \therefore \nabla_a n^a = 0}$$

2. The pull-back formalism allows one to construct three-forms which are automatically closed:
   2.1. Introduce an abstract, 3D matter space where the points are labeled by X^A, $A = 1,2,3$.
   2.2. Each worldline in spacetime is identified with a matter space point, as in the following picture:



THE PULL-BACK FORMALISM CONTINUED

3. In the matter space we can construct a volume-form; i.e., it is a three-index, totally antisymmetric object which is a function of the matter space labels:

$$\boxed{n_{ABC} = n_{[ABC]}\left(X^D\right)}$$

4. The matter space labels, can be "imported" into spacetime and thereby become scalar functions $X^A(x^a)$.
5. Using the maps $\nabla_a X^A$ and the volume form n_{ABC}, we can build the spacetime three-form

$$\boxed{n_{abc} = \left(\nabla_{[a} X^A\right)\left(\nabla_b X^B\right)\left(\nabla_{c]} X^C\right) n_{ABC}\left(X^D\right)}$$

6. The exterior derivative of this is

$$\begin{aligned}
\nabla_{[a} n_{bcd]} &= \nabla_{[a}\left[\left(\nabla_b X^B\right)\left(\nabla_c X^C\right)\left(\nabla_{d]} X^D\right) n_{BCD}\left(X^A\right)\right] \\
&= \left[\left(\nabla_{[a}\nabla_b X^B\right)\left(\nabla_c X^C\right)\left(\nabla_{d]} X^D\right) - \left(\nabla_{[b} X^B\right)\left(\nabla_a\nabla_c X^C\right)\left(\nabla_{d]} X^D\right) + \left(\nabla_b X^B\right)\left(\nabla_c X^C\right)\left(\nabla_a\nabla_{d]} X^D\right)\right] n_{BCD}\left(X^A\right) \\
&\quad + \left(\nabla_{[a} X^A\right)\left(\nabla_b X^B\right)\left(\nabla_c X^C\right)\left(\nabla_{d]} X^D\right)\frac{\partial n_{BCD}}{\partial X^A}
\end{aligned}$$

7. It vanishes identically because

$$\nabla_a\nabla_b X^A = \nabla_b\nabla_a X^A$$

$$\left(\nabla_{[a} X^A\right)\left(\nabla_b X^B\right)\left(\nabla_c X^C\right)\left(\nabla_{d]} X^D\right)\frac{\partial n_{BCD}}{\partial X^A} \equiv \left(\nabla_a X^{[A}\right)\left(\nabla_b X^B\right)\left(\nabla_c X^C\right)\left(\nabla_d X^{D]}\right) n_{BCD,A} = 0$$

THE ACTION PRINCIPLE: SECOND ATTEMPT

1. Now we are set!
   1.1. Assume that a Lagrangian is given:

$$\boxed{\Lambda = -\rho\left(n^2\right)}$$

   1.2. Assume that the n_{ABC} are given:

$$\boxed{n^a = \frac{1}{3!}\varepsilon^{abcd}\left(\nabla_b X^B\right)\left(\nabla_c X^C\right)\left(\nabla_d X^D\right)n_{BCD}\left(X^A\right)}$$

   1.3. Introduce the Lagrangian displacement

$$\boxed{\delta X^A = -\left(\nabla_a X^A\right)\xi^a}$$

   1.4. A straightforward, but tedious, calculation gives the proper flux variation:

$$\boxed{\delta n^a = n^b \nabla_b \xi^a - \xi^b \nabla_b n^a - n^a\left(\nabla_b \xi^a + \frac{1}{2} g^{bc}\delta g_{bc}\right)}$$

THE ACTION PRINCIPLE: SECOND ATTEMPT CONTINUED

2. Next, take the following steps:
   2.1. Form the action:

$$\boxed{S_{fluid} = \int_M d^4x\sqrt{-g}\,\Lambda\left(n^2\right)}$$

   2.2. Take its variation:

$$\delta S_{fluid} = \int_M d^4x\sqrt{-g}\left[-\left(n^b\omega_{ba}\right)\xi^a + \frac{1}{2}\left(\Psi g^{ab} + \mu^a n^b\right)\delta g_{ab}\right]$$

$$\mu_a = -2\frac{\partial\Lambda}{\partial n^2}n_a \quad , \quad \omega_{ab} = 2\nabla_{[a}\mu_{b]} \quad , \quad \Psi = -\Lambda + \mu_a n^a$$

   2.3. Therefore, the equation of motion is

$$\boxed{n^b\omega_{ba} = 0}$$

   2.4. The energy-momentum-stress tensor is

$$\boxed{T^{ab} = \Psi g^{ab} + \mu^a n^b}$$

CONCLUDING REMARKS

1. Want to have a multi-fluid system?
   1.1. Form all the scalars that can be made from the fluxes.
   1.2. Assume there is a Lagrangian which is a function of these independent scalars.
   1.3. Introduce an independent abstract matter space for each independent fluid.
2. Want to have electromagnetic effects? Start with 1. above and add to it the standard minimal coupling and Maxwell actions. [See, for example, Andersson et. al., CQG, v. 34, 125001 (2017).]
3. Want to have a crust? Start with 1. above, and add to the action that of an elastic material. [See, for example, Andersson et. al., CQG, v. 36, 105004 (2019).]
4. Want to have dissipation? Think about it from 1993 to 2013, and learn how to extend 1. above with non-closed three-forms. [See Andersson and Comer, CQG , v. 32, 075008 (2015).]