NUMERICAL GR HYDRODYNAMICS

A BRIEF SURVEY

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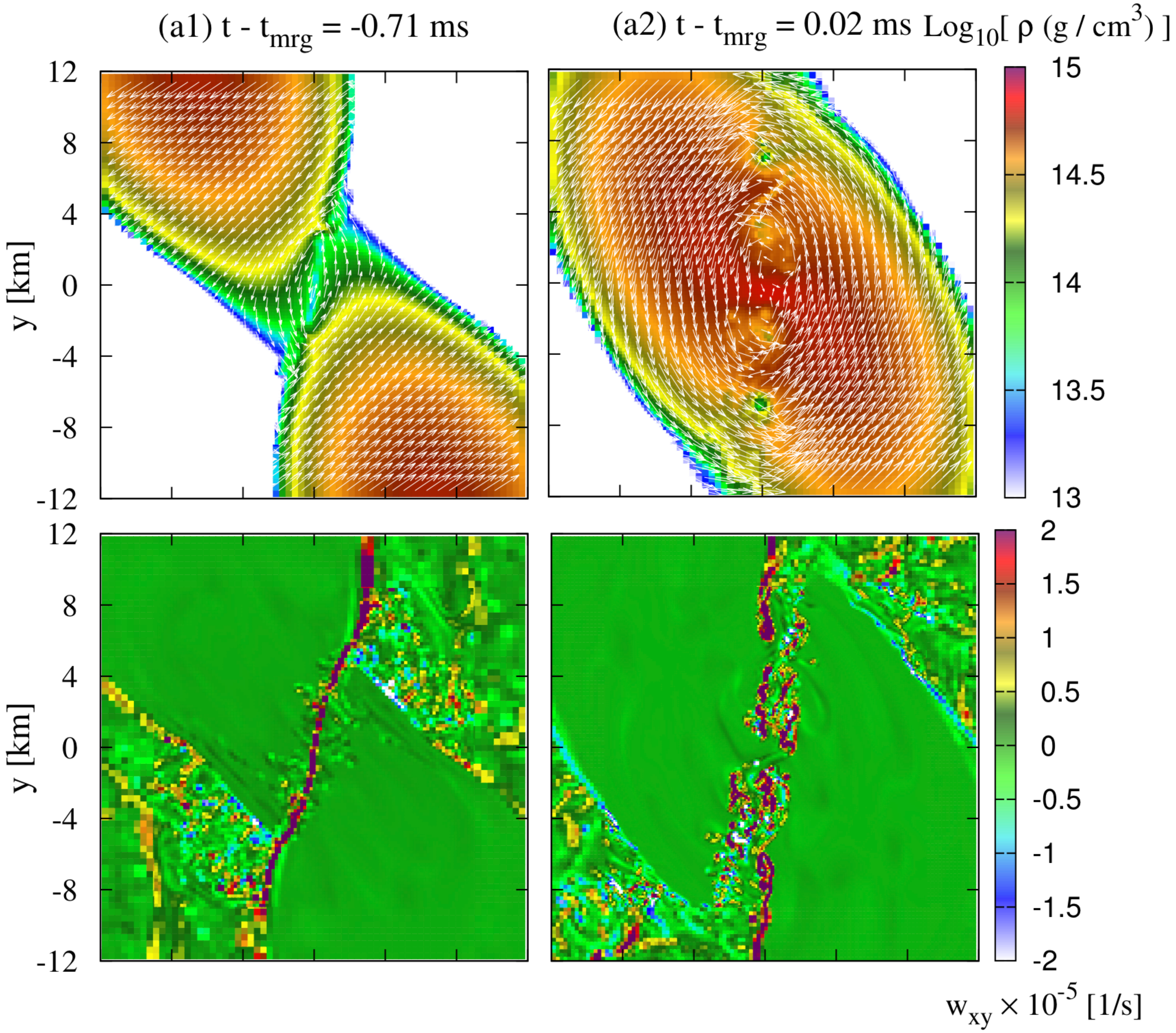
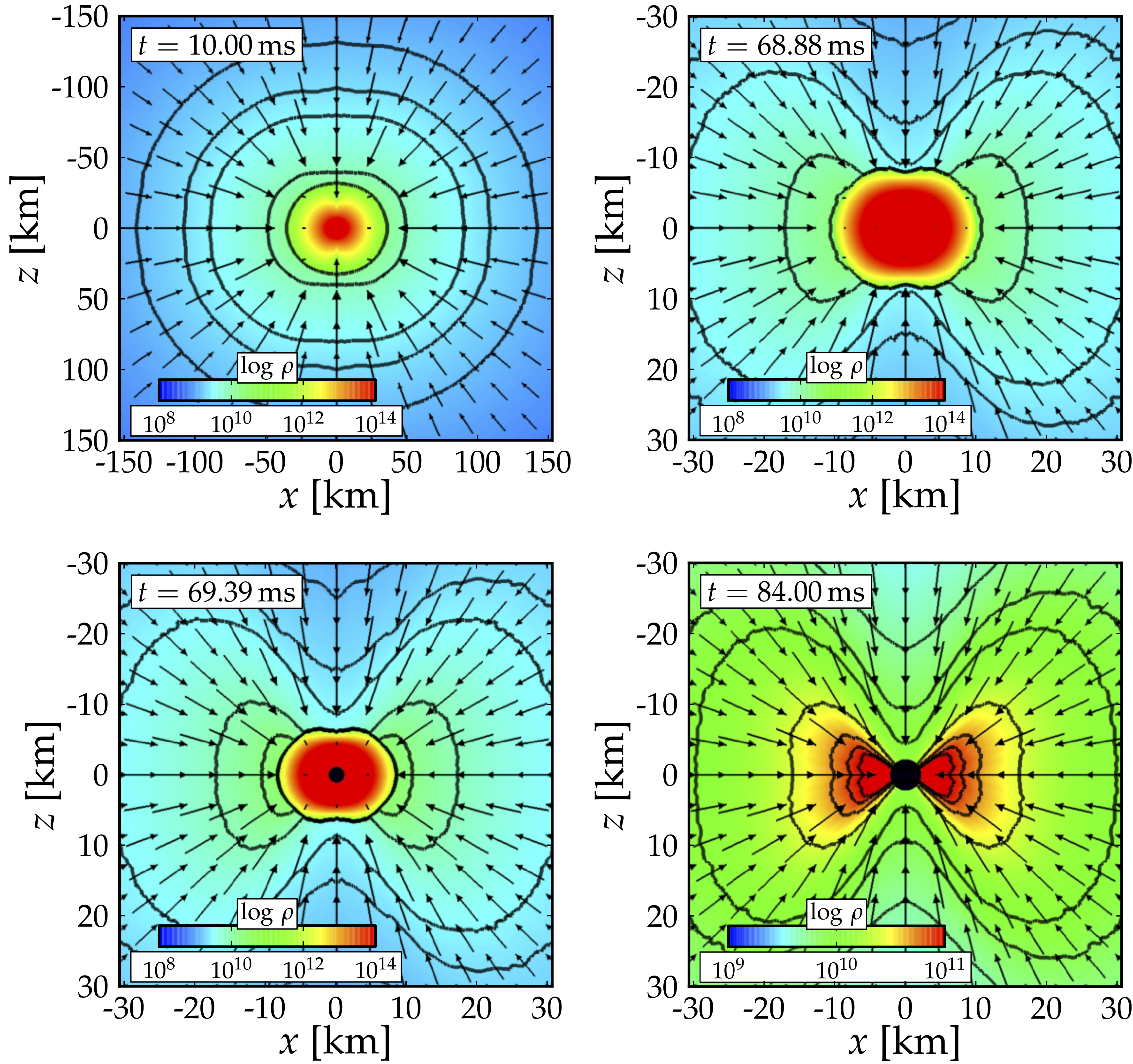


VACUUM VS MATTER

- Scale (Planck length vs particle and turbulence lengths). Inherent approximations (EOS, interactions, ...).
- Range of models considered (GR vs (M)HD, non-ideal, νs, multifluids, ...). Parameter extraction harder.
- Form of solutions (C^∞ or C^4 vs C^0). Restricts useful numerical methods.



CONSERVATION

EFEs imply $\nabla_a T^{ab} = 0$. Pick a tetrad, $e_b^{(j)}$ to get

$$\nabla_a \left[e_b^{(j)} T^{ab} \right] = \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} e_b^{(j)} T^{ab} \right) = -T^{ab} \nabla_a e_b^{(j)}$$

$$\implies \qquad \partial_t \mathbf{q} + \partial_i \mathbf{f}^{(i)}(\mathbf{q}) = \mathbf{s}.$$

Balance law form. Only four equations: need other constituitive equations for, eg, EM, particle number, etc.

KEY TAKEAWAY

Need numerical methods for

- complex, interlinked models
- some described by balance laws
- *may* have non-conserved constituents
- small (unresolved) scale effects through closure relations.

SHOCK FORMATION

Advection equation

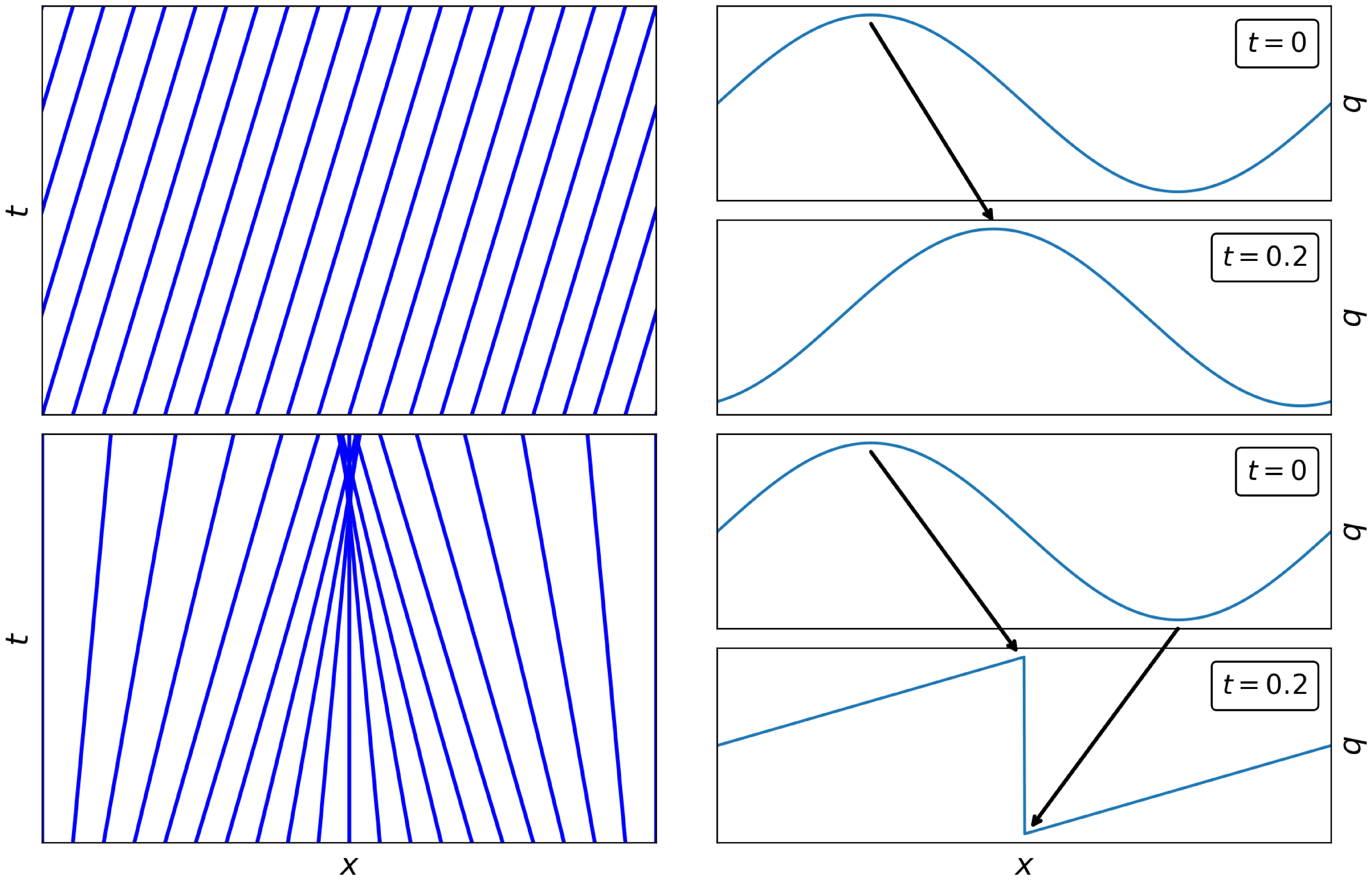
$$\partial_t q + \partial_x (vq) = 0.$$

Information moves right, speed v.

Burgers equation

$$\partial_t q + \frac{1}{2}\partial_x q^2 = 0.$$

Information moves right, speed q.
Shocks form.



SHOCKS AND UNIQUENESS

Shock speed V_s from Rankine-Hugoniot, $V_s\,[q] = [f]$. Burgers equation

$$\partial_t q + q \partial_x q = 0$$

is equivalent to *any* of

$$\partial_t q^n + \tfrac{n}{n+1} \partial_x q^{n+1} = 0,$$

but they all have different shock speeds. Total derivative form crucial: otherwise need more information to fix solution.

WEAK FORMS

Move the derivatives off the fields using C^∞ test function ϕ:

$$\int_V \phi \partial_t \mathbf{q} + \oint_{\partial V} \phi \mathbf{f} - \int_V \mathbf{f} \cdot \nabla \phi = \int_V \phi \mathbf{s}.$$

- Split domain into volumes/elements/cells.
- Approximate fields, including ϕ, leads to method.
- Surface integral couples neighbours; other terms *local*.

EULER EQUATIONS

Newtonian:

$$\partial_t \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (E + p) v \end{pmatrix} = \mathbf{0}.$$

SR:

$$\partial_t \begin{pmatrix} \rho W \\ \rho h W^2 v \\ E \end{pmatrix} + \partial_x \begin{pmatrix} \rho W v \\ \rho h W^2 v^2 + p \\ (E + p) v \end{pmatrix} = \mathbf{0}.$$

GR EULER EQUATIONS

$$\partial_t \sqrt{\gamma} \begin{pmatrix} \rho W \\ \rho h W^2 v_j \\ \tau \end{pmatrix} + \partial_i \sqrt{-g} \begin{pmatrix} \rho W \left(v^i - \frac{\beta^i}{\alpha} \right) \\ \rho h W^2 v_j \left(v^i - \frac{\beta^i}{\alpha} \right) + p \delta^i_j \\ \tau \left(v^i - \frac{\beta^i}{\alpha} \right) + p v^i \end{pmatrix} = \mathbf{s},$$

$$\mathbf{s} = \sqrt{-g} \begin{pmatrix} 0 \\ T^{\mu\nu} \left(\partial_\mu g_{\nu j} - \Gamma^\mu_{\nu j} \right) \\ \alpha \left(T^{\mu 0} \partial_\mu \log \alpha - T^{\mu\nu} \Gamma^0_{\mu\nu} \right) \end{pmatrix}.$$

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FINITE VOLUME METHODS

Spatially constant $q \simeq \hat{q}_i(t)$ within cell:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{q}_i + \frac{1}{|V|}\oint_{\partial V} f(q) = s(\hat{q}_i) \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{q}_i + \frac{1}{\Delta x}\left[f_{i+1/2} - f_{i-1/2}\right] = 0.$$

On ∂V, q is multi-valued: solve *Riemann Problem* to get unique q.

RECONSTRUCTION

Go from $\hat{q}_i$ to $q(x_{i\pm1/2})$.

- Piecewise constant is stable at shocks.
- Higher order leads to Gibbs' oscillations.
- Need limiting.
- WENO: expensive but accurate.

RIEMANN SOLVER

Go from two values for $q_{i-1/2}$ to $f_{i-1/2}$. Example (Rusanov):

$$f = \frac{1}{2}\left(f_L + f_R + \frac{\Delta x}{\Delta t}(q_L - q_R) \right).$$

- Full solution of Riemann problem expensive, not always available.
- Need care when phase transitions happen.
- Typically get more accuracy gain from reconstruction.

DISCONTINUOUS GALERKIN

Full function basis expansion: $q = \sum_m \hat{q}_m P_m(x)$. Plug in:

$$\int_V P_m P_n \hat{\phi}_m \partial_t \hat{q}_n + \oint_{\partial V} P_m P_n \hat{\phi}_m \hat{f}_n - \int_V P_m \hat{f}_m \hat{\phi}_n \nabla P_n = \int_V P_m P_n \hat{\phi}_m \hat{s}_n .$$

Simplifies to

$$M\partial_t \hat{\mathbf{q}} + S^T f(\hat{\mathbf{q}}) = -[\phi \mathbf{F}]_{x_{i-1/2}}^{x_{i+1/2}} .$$

- More accurate.
- Less communication - exascale!
- Issues with shocks.

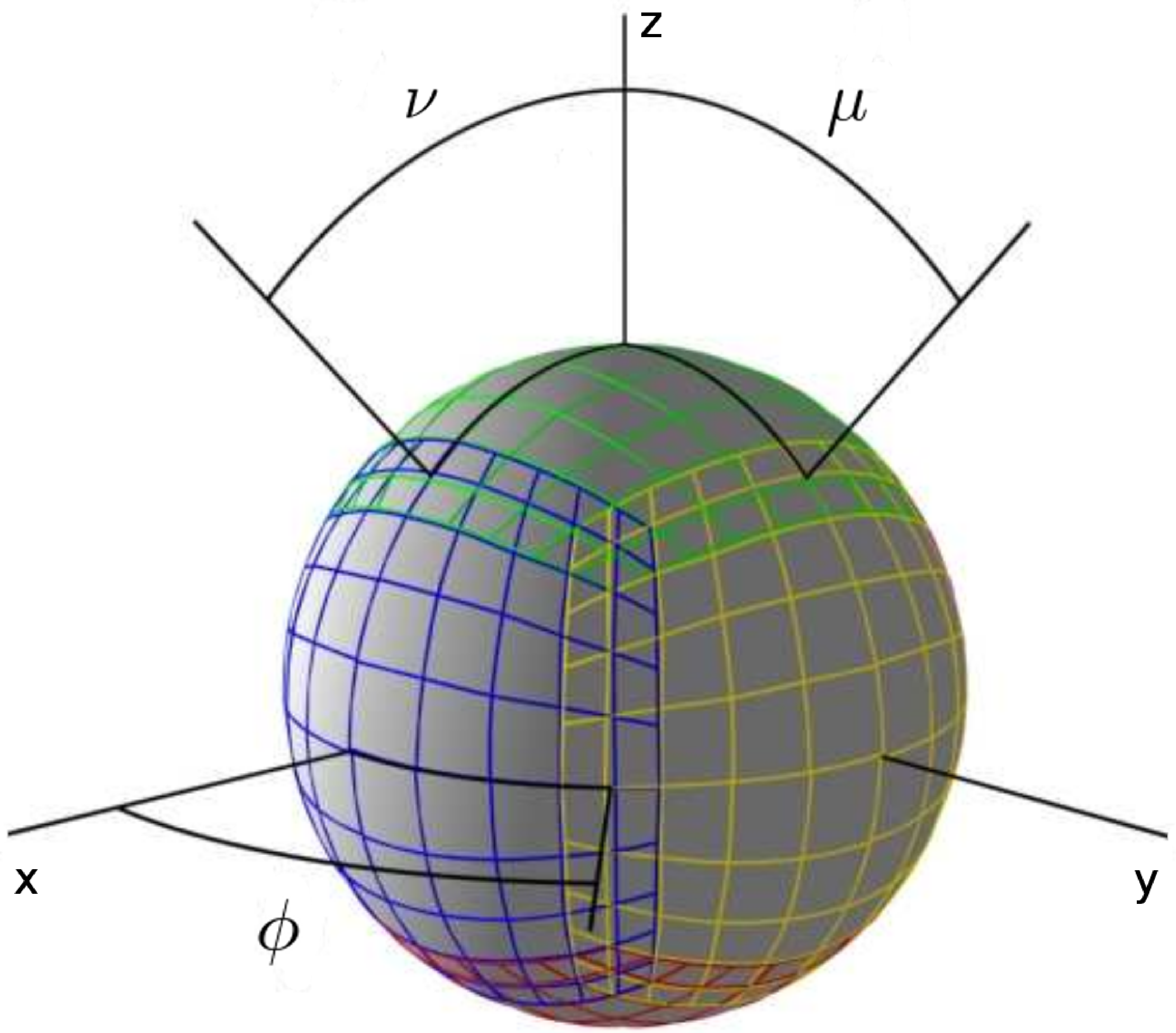
GRIDDING

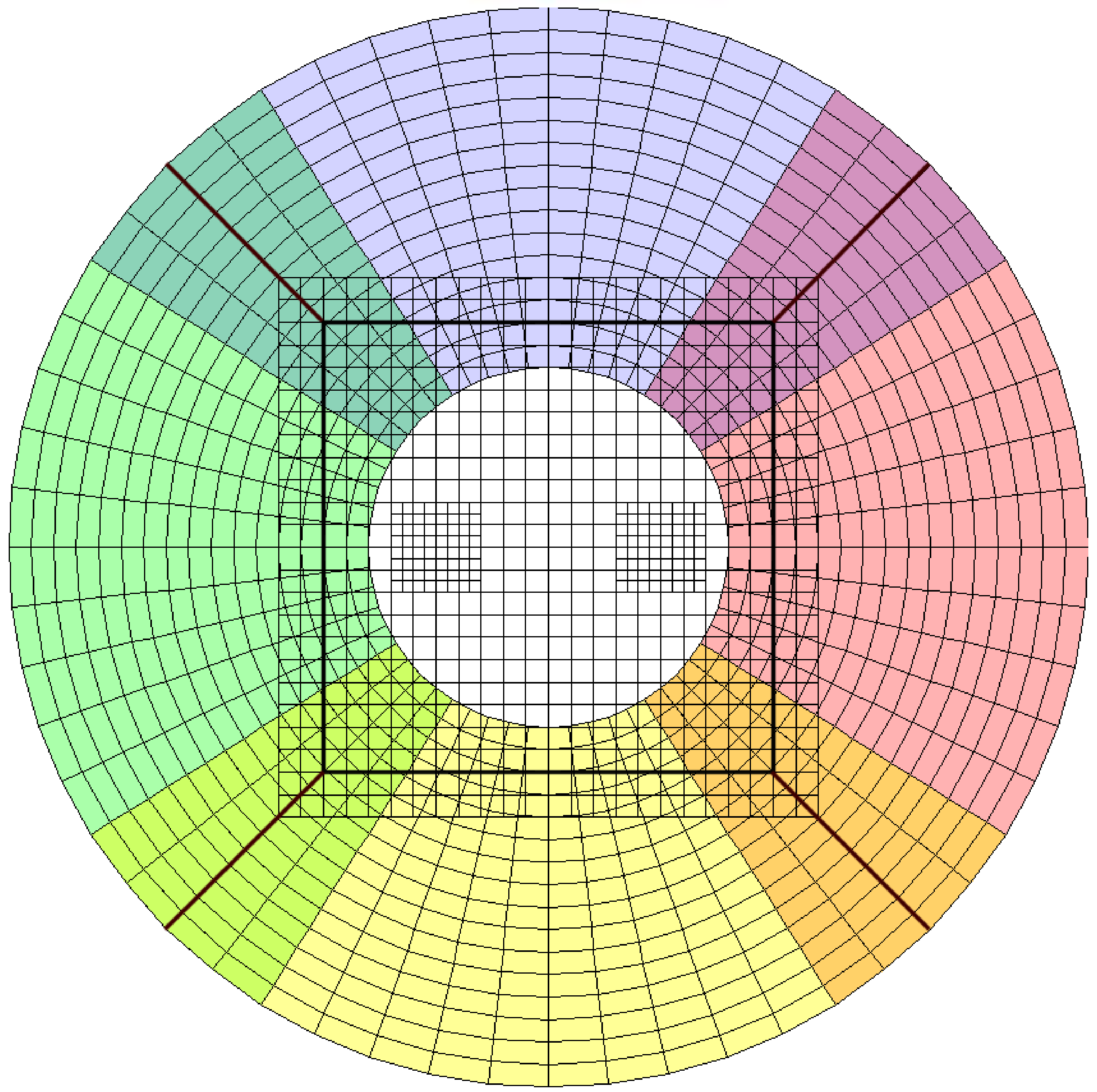
Adaptive Mesh Refinement

- Cuts resource massively.
- Builds on standard code.
- Communication and complexity issues.

Multi-patch and spherical coordinates

- Cuts resolution requirements, improves accuracy.
- Coordinate and complexity issues.





CONSTRAINTS

MHD has no monopoles: require

$$\nabla \cdot \mathbf{B} = 0.$$

Enforces numerically through

- Constrained transport;
- Changing variables (vector potential);
- Constraint damping.

GOING FURTHER

- Well balanced
- Positivity preserving
- Adaptivity (h/p, adaptive model)
- Path conservative schemes.