Dynamics of superfluid-superconducting neutron stars

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• It is generally accepted that baryons (neutrons and protons) in the internal layers of neutron stars undergo transition into superfluid/superconducting state at $T \lesssim 10^8 \div 10^{10}$ K.

• Thus, to study dynamics of neutron stars at sufficiently low temperatures one has to develop a system of equations describing superfluid-superconducting mixtures.

• Generally, such mixture can be magnetized, relativistic, and can contain both neutron (Feynman-Onsager) and proton (Abrikosov) vortices.
Introduction

• Dynamics of superfluid-superconducting mixtures has been studied, both in the non-relativistic (e.g., Vardanyan & Sedrakyan’81; Holm & Kupershmidt’87; Mendell & Lindblom’91; Mendell’91; Sedrakyan & Sedrakyan’95; Glampedakis, Andersson & Samuelsson’11) and in the relativistic framework (Lebedev & Khalatnikov’81; Carter & Langlois’95; Carter & Langlois’98; Langlois, Sedrakyan & Carter’98; Kantor & Gusakov’11; Dommes & Gusakov’15; Andersson, Wells & Vickers’16).

• “State of the art” paper: Glampedakis, Andersson & Samuelsson’11 (GAS11)

essentially nonrelativistic formulation approximation of vanishing temperature superfluid-superconducting mixture; type-II proton superconductivity vortices; mutual friction; correct treatment of the magnetic field ($B \neq H$);
• So, initially, our aim was to extend the results of GAS11 to relativistic framework and to include into consideration the finite-temperature effects.

• Eventually, the equations that we derived turn out to be more general than those of GAS11 (even in the non-relativistic limit)

• We have also found that our equations differ from MHD of GAS11
All these results will be discussed in my talk, which is based on the following works:

- Gusakov M.E., **PRD** (2016)  
  “Relativistic formulation of the Hall-Vinen-Bekarevich-Khalatnikov superfluid hydrodynamics”

  “Relativistic dynamics of superfluid-superconducting mixtures in the presence of topological defects and the electromagnetic field, with application to neutron stars”

- Dommes V.A., Gusakov M.E. (in preparation)  
  “Vortex buoyancy in superfluid and superconducting neutron stars”
The result:

Particle and energy-momentum conservation:

\[
\partial_{\mu} j_{(j)}^{\mu} = 0 \\
\partial_{\mu} T^{\mu\nu} = 0
\]

\[
 j_{(i)}^{\mu} = n_i u^{\mu} + Y_{ik} w_{(k)}^{\mu} \\
 j_{(e)}^{\mu} = n_e u^{\mu} \\
 T^{\mu\nu} = (P + \varepsilon) u^{\mu} u^{\nu} + P g^{\mu\nu} + Y_{ik} \left( w_{(i)}^{\mu} w_{(k)}^{\nu} + \mu_i w_{(k)}^{\mu} u^{\nu} + \mu_k w_{(i)}^{\nu} u^{\mu} \right) + \Delta T^{\mu\nu}
\]

\[
\Delta T^{\mu\nu} = \mathcal{T}_{(E)}^{\mu\nu} + \mathcal{T}_{(M)}^{\mu\nu} + \mathcal{T}_{(VE)}^{\mu\nu} + \mathcal{T}_{(VM)}^{\mu\nu}
\]

\[
\mathcal{T}_{(E)}^{\mu\nu} = \frac{1}{4\pi} \left( \mathcal{T}^{\mu\nu} - D^{\alpha} E_{\alpha} - D^{\nu} E_{\mu} \right) \\
\mathcal{T}_{(M)}^{\mu\nu} = \frac{1}{4\pi} \left( \mathcal{G}^{\mu\alpha} F_{\alpha}^{\nu} + u^{\nu} \mathcal{G}^{\mu\alpha} E_{\alpha} + u^{\mu} \mathcal{G}^{\nu\alpha} E_{\alpha} \right) \\
\mathcal{T}_{(VE)}^{\mu\nu} = \mathcal{W}_{(Ei)}^{\alpha} \mathcal{V}_{(Ei)\alpha} - \mathcal{W}_{(Ei)}^{\mu} \mathcal{V}_{(Ei)}^{\nu} \\
\mathcal{T}_{(VM)}^{\mu\nu} = \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(i)\alpha} + u^{\nu} \mathcal{W}_{(i)}^{\mu\alpha} \mathcal{V}_{(Ei)\alpha} + u^{\mu} \mathcal{W}_{(i)}^{\nu\alpha} \mathcal{V}_{(Ei)\alpha}
\]

Second law of thermodynamics:

\[
d\varepsilon = T \, dS + \mu_i \, dn_i + \mu_e \, dn_e + \frac{Y_{ik}}{2} \, d\left( w_{(i)}^{\alpha} w_{(k)}^{\alpha} \right) + d\varepsilon_{\text{add}}
\]

\[
d\varepsilon_{\text{add}} = \frac{1}{4\pi} \, E_\mu dD^{\mu} + \frac{1}{4\pi} \, H_\mu dB^{\mu} + \mathcal{V}_{(Ei)}^{\mu} d\mathcal{V}_{(Ei)\mu} + \mathcal{V}_{(Mi)}^{\mu} d\mathcal{V}_{(Mi)\mu}
\]
The result:

“Superfluid” equations for neutrons and protons:

\[ u^\nu \mathcal{V}(i)_{\mu\nu} = \mu_i n_i f_{(i)\mu} \]
\[ u_\mu w_{(i)}^\mu = 0 \]

\[ f_{(i)}^\mu = \alpha_i \mathcal{V}(i)_{\nu\lambda} W_{(i)\delta} \mathcal{V}(i)_{\lambda}\delta + \frac{\beta_i - \gamma_i}{\mathcal{V}_{(Mi)}} \mathcal{V}(i)_{\eta\sigma} \mathcal{V}(i)_{\lambda\nu} W_{(i)\delta} \mathcal{V}(i)_{\lambda}\delta + \gamma_i \mathcal{V}_{(Mi)} \mathcal{V}_{(i)\delta} \mathcal{V}(i)_{\lambda}\delta \]

\[ W_{(i)}^\mu = \frac{1}{n_i} \left[ Y_{ik} w_{(k)}^\mu + \partial_\alpha \mathcal{V}_{(i)\mu\alpha} \right] \]

Maxwell’s equations in the medium:

\[ \partial_\alpha F^{\alpha\beta} = 0 \]
\[ \partial_\alpha G^{\alpha\beta} = -4\pi J^{\beta}_{(\text{free})} \]

\[ J^{\mu}_{(\text{free})} = e_p (n_p - n_e) u^\mu + e_i Y_{ik} w_{(k)}^\mu \]
Idea of derivation

Initial idea [Bekarevich & Khalatnikov’61]: consistency between conservation laws and entropy equation.

• Consider a system in the absence of dissipation

• Assume that we know the form of the expressions for particle current densities as well as the form of the second law of thermodynamics

\[ d\varepsilon = T dS + \mu_i d n_i + \ldots \]

• Then it is possible to constrain the system energy-momentum tensor from the requirement that the entropy is not produced in the system (which means that the entropy density is subject to continuity equation)

\[ \partial_\mu (S u^\mu) = 0 \]

entropy density

\( S \)

four-velocity of normal excitations

\( u^\mu \)
That is, by specifying, for example, vortex contribution

\[ d\varepsilon = T dS + \mu_i d\nu_i + \ldots + d\varepsilon_{\text{vortex}} \]

\[ T^{\mu\nu} = (P + \varepsilon) u^{\mu} u^{\nu} + Pg^{\mu\nu} + \ldots + \Delta T^{\mu\nu}_{\text{vortex}} \]

one finds the correction to the energy-momentum tensor
What physics is included (brief account)?

- fully relativistic formulation
- npe-composition (additional particle species can be easily included)
- neutrons are superfluid, protons are superconducting
- entrainment and finite temperature effects
- both types (I and II) of proton superconductivity
- electromagnetic effects
- neutron and proton vortices (or magnetic domains for type-I proton SP)
- dissipation (e.g., mutual friction)
In what follows I will discuss some of these physical “ingredients” in more detail
Importance of finite-temperature effects

• Zero-temperature approximation is justified only if $T \ll T_{cn}, T_{cp}$ everywhere in the star.

In many interesting situations (e.g., in magnetars, LMXBs) this is not the case.

• Note that the condition $T \ll \mu_n, \mu_p$ does not justify the use of the zero-temperature hydrodynamics.

---

superfluid density is a strong function of temperature!

\[
\rho_s = \rho \text{ at } T = 0 \\
\rho_s = 0 \text{ at } T = T_c
\]
What physics is included: Type I/II proton superconductivity

\[
\kappa = \frac{\xi_p}{\sqrt{2}\delta_p} > 1 \Rightarrow \text{I type} \\
\kappa = \frac{\xi_p}{\sqrt{2}\delta_p} < 1 \Rightarrow \text{II type}
\]

Coherence length:
\[
\xi_p \approx 5 \times 10^{-12} \left( \frac{m_p}{m^*_p} \right) \rho_{14}^{1/3} \left( \frac{x_p}{0.1} \right)^{1/3} \left( \frac{10^9 \text{K}}{T_{cp}} \right) \text{ cm}
\]

London penetration depth:
\[
\delta_p \approx 9 \times 10^{-12} \left( \frac{m^*_p}{m_p} \right)^{1/2} \rho_{14}^{-1/2} \left( \frac{0.1}{x_p} \right)^{1/2} \text{ cm}
\]

\[H_c \sim 10^{15} \text{ G}\]

Credit: Glampedakis et al.'11
What is the difference between neutron star interiors with type-I and type-II superconductors?

• Transition to superconducting state occurs at constant magnetic flux (Baym et al. 1969; typical cooling timescale is much shorter than the magnetic flux expulsion timescale)

• Under these conditions type-I superconductor undergoes transition into an “intermediate” state, while type-II superconductors – into mixed state.

**Intermediate state of type-I superconductor:**
- consists of alternating domains of superconducting (field-free) regions and normal regions hosting magnetic field

**Mixed state of type-II superconductor:**
- consists of Abrikosov vortices (fluxtubes)
Intermediate vs mixed state

**Huebener’00**

Typical “open topology” intermediate state domain structure

Normal regions are dark

**Distance between neighboring flux tubes:**

\[ b \sim \sqrt{R \delta} \sim 2 \times 10^{-3} \text{ cm} \]

(Huebener’13, Sedrakian’05, DeGennes’66)

**Flux tube radius:**

\[ a \approx b (B/H_c)^{1/2} \sim 6 \times 10^{-5} \text{ cm} \]

**Number of flux quanta in a flux tube:**

\[ N_\phi \approx \pi a^2 H_c/\phi_{p0} \approx 6 \times 10^{13} \]

**Hess et al’89**

Mixed state: Abrikosov vortices

**Distance between neighboring vortices:**

\[ b \sim \sqrt{\frac{\phi_{p0}}{\pi B}} \approx 2.6 \times 10^{-10} \sqrt{\frac{10^{12} \text{ G}}{B}} \text{ cm} \]

**“Vortex radius”:**

\[ \delta_p \approx 9 \times 10^{-12} \left( \frac{m^*}{m_p} \right)^{1/2} \rho_{14}^{-1/2} \left( \frac{0.1}{x_p} \right)^{1/2} \text{ cm} \]

**Number of flux quanta in a vortex:**

1

\[ B = 10^{12} \text{ G} \quad H_c = 10^{15} \text{ G} \]
What physics is included: vortices

**Neutron Vortices**

Neutron vortices appear in neutron stars in order to imitate solid-body rotation with a non-superfluid component.

Vortex density: \[ \frac{2m_n \Omega}{\pi \hbar} \approx \frac{6 \times 10^3}{P} \text{ vortices cm}^{-2} \]

**Proton Vortices**

Vortex density: \[ \frac{B}{\hat{\phi}_{p0}} \approx 4.8 \times 10^{18} \left( \frac{B}{10^{12} \text{ G}} \right) \text{ cm}^{-2} \]

Total number of vortices: \[ \sim \frac{B \pi R^2}{\hat{\phi}_{p0}} \approx 1.5 \times 10^{31} \left( \frac{B}{10^{12} \text{ G}} \right) \]

Intervortex spacing: \[ \sim \sqrt{\frac{\hat{\phi}_{p0}}{\pi B}} \approx 2.6 \times 10^{-10} \sqrt{\frac{10^{12} \text{ G cm}}{B}} \]

Magnetic flux: \[ \frac{\pi \hbar c}{e_p} \frac{Y_{np}}{Y_{pp}} \sim 4 \times 10^{-8} \text{ G cm}^2 \]

Magnetic flux: \[ \frac{\pi \hbar c}{e_p} \sim 2 \times 10^{-7} \text{ G cm}^2 \]
The suggested dynamic equations naturally account for:

- Both neutron and proton vortex energies

\[ E_V = \frac{\rho_s \kappa^2}{4\pi} \ln \left( \frac{b}{a} \right) \]

- Mutual friction (as well as Magnus force etc.)

\[ F_D = -\kappa \rho_s \mathcal{R}(V_L - V_{norm}) \]

- Vortex tension (appears when vortex is bent)

\[ F_T = E_V (e \nabla) e \quad \text{vortex energy per unit length divided by curvature radius } R \]

- Vortex buoyancy

\[ F_B = -\nabla E_V = -E_V \frac{\nabla \perp \rho_s}{\rho_s} \]

\[ \nabla \perp = \nabla - e(e \nabla) \]
Vortex buoyancy in more detail

\[ F_B = -\nabla E_V = -E_V \frac{\nabla \rho_s}{\rho_s} \]

acts to push a vortex out into the region with **smaller** superfluid density

- usually it is either ignored (as in the Hall-Vinen hydrodynamics) or introduced “by hands” in the form (e.g., Muslimov & Tsygan’85, Elfritz et al.’16, ...)

\[ F_B = -E_V \frac{g}{c_s^2} \]

where \( g \) is the gravitation acceleration and \( c_s \) is the speed of sound

which is popular in studies of the magnetic flux expulsion.

- The latter expression reduces to the correct one only for a one-component liquid at zero temperature.

- It should be noted that the correct buoyancy force is contained implicitly in the Bekarevich & Khalatnikov superfluid hydrodynamics and its multifluid extensions.
The next interesting feature of the dynamic equations that we propose is that they consider a superfluid-superconducting mixture as a medium in which \( H \neq B \) and \( D \neq E \). Thus they are coupled with the standard Maxwell’s equations in the medium.

Maxwell’s equations in the medium:

\[
\begin{align*}
\text{div } D &= 4\pi \rho_{\text{free}}, \\
\text{rot } E &= -\frac{1}{c} \frac{\partial B}{\partial t}, \\
\text{div } B &= 0, \\
\text{rot } H &= \frac{4\pi}{c} J_{\text{free}} + \frac{1}{c} \frac{\partial D}{\partial t}
\end{align*}
\]

- \( B \) magnetic induction
- \( H \) magnetic field
- \( E \) electric field
- \( D \) electric displacement
SP-SFL mixture as a medium with $H \neq B$ and $D \neq E$

• Why $H \neq B$?

  Carter, Prix, Langlois’00; Glampedakis et al.’11

• Short answer: Because $H = B - 4\pi M$ and $M \neq 0$ (textbook result)

• Each vortex has a magnetic field supported by superconducting currents

• These “molecular” currents contribute to magnetization $M$
  (magnetic moment of the unit volume)

$$B_V(r)$$

• It is straightforward to show: $|M| = \frac{1}{4\pi} \phi_0 N_V$
  (vortex magnetic flux)

  areal vortex density
Using \( \cdot \), one can calculate the electric polarization vector \( \mathbf{P} \) (or the electric dipole moment of a unit volume) and find:

- Why \( \mathbf{D} \neq \mathbf{E} \)?

  Gusakov & Dommes'16

  - Short answer: Because \( \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \) and \( \mathbf{P} \neq \mathbf{0} \) (textbook result)

- Each moving vortex induces an electric field

  \[
  \mathbf{E}_V = -\frac{1}{c} \mathbf{V}_L \times \mathbf{B}_V
  \]

  and electric charge:

  \[
  \rho_c = \frac{1}{4\pi} \text{div} \mathbf{E}_V
  \]

- Using \( \rho_c \), one can calculate the electric polarization vector \( \mathbf{P} \) (or the electric dipole moment of a unit volume) and find:

  \[
  \mathbf{P} = -\frac{1}{c} \mathbf{V}_L \times \mathbf{M} \neq \mathbf{0}
  \]
EM + vortex energy density

• By specifying the energy density we specify the energy-momentum tensor

• What is the contribution to the system energy density from the electromagnetic field and vortices?

\[ d\varepsilon_{EM+vortex} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}^\mu_{(Ei)} d\mathcal{W}_{(Ei)\mu} + \mathcal{W}_{(Mi)\mu} d\mathcal{V}^\mu_{(Mi)} \]
EM + vortex energy density

• By specifying the energy density we specify the energy-momentum tensor

• What is the contribution to the system energy density from the electromagnetic field and vortices?

\[ d\varepsilon_{\text{EM+vortex}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}_{(Ei)}^\mu d\mathcal{W}_{(Ei)}(Ei)\mu + \mathcal{W}_{(Mi)}d\mathcal{V}_{(Mi)}^\mu \]

“electromagnetic” contribution

• has a standard form

• depends on the four-vectors which reduce to

\[
E^\mu = (0, E) \quad D^\mu = (0, D) \\
B^\mu = (0, B) \quad H^\mu = (0, H)
\]

in the comoving frame \( u^\mu = (1, 0, 0, 0) \), moving with the normal liquid component
EM + vortex energy density

- By specifying the energy density we specify the energy-momentum tensor.

- What is the contribution to the system energy density from the electromagnetic field and vortices?

\[ d\varepsilon_{\text{EM+vortex}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}^{\mu}_{(Ei)} d\mathcal{W}_{(Ei)\mu} + \mathcal{V}^{\mu}_{(Mi)} d\mathcal{V}^{\mu}_{(Mi)} \]

“electromagnetic” contribution

- Generally, one can say that it depends on two tensors:

\[
F^{\alpha\beta} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & B_3 & -B_2 \\
-E_2 & -B_3 & 0 & B_1 \\
-E_3 & B_2 & -B_1 & 0
\end{pmatrix}
\]

\[
G^{\alpha\beta} = \begin{pmatrix}
0 & D_1 & D_2 & D_3 \\
-D_1 & 0 & H_3 & -H_2 \\
-D_2 & -H_3 & 0 & H_1 \\
-D_3 & H_2 & -H_1 & 0
\end{pmatrix}
\]
EM + vortex energy density

• By specifying the energy density we specify the energy-momentum tensor

• What is the contribution to the system energy density from the electromagnetic field and vortices?

\[ d\varepsilon_{\text{EM+vortex}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}_\mu^{(Ei)} d\mathcal{W}_{(Ei)\mu} + \mathcal{W}_{(Mi)\mu} d\mathcal{V}_\mu^{(Mi)} \]

“vortex” contribution

• Depends on the vorticity tensor

\[ \mathcal{V}_\mu^{(i)} = \partial^\mu \mathcal{V}_\nu^{(si)} - \partial^\nu \mathcal{V}_\mu^{(si)} + \epsilon_{i} F_{\mu\nu} \]

which is related to the density of vortices

(non-relativistic analogue: \[ m_i \text{ curl } \mathbf{V}_{(si)} + \frac{e_i}{c} \mathbf{B} \])

• Depends on a complementary tensor \( \mathcal{W}_\mu^{(i)} \)

\[
\begin{align*}
\mathcal{V}_\mu^{(Ei)} &\equiv u_\nu \mathcal{V}_\mu^{\nu (i)} \\
\mathcal{V}_\mu^{(Mi)} &\equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{V}_{(i)\alpha\beta} \\
\mathcal{W}_\mu^{(Ei)} &\equiv u_\nu \mathcal{W}_\mu^{\nu (i)} \\
\mathcal{W}_\mu^{(Mi)} &\equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \mathcal{W}_{(i)\alpha\beta}
\end{align*}
\]
Electromagnetic and vortex contributions to the second law of thermodynamics induce corrections to the energy-momentum tensor.

\[
d\varepsilon_{\text{EM+vortex}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}_{(E_i)}^\mu d\mathcal{W}_{(E_i)\mu} + \mathcal{W}_{(M_i)}^\mu d\mathcal{V}_{(M_i)}^\mu
\]

- **Electromagnetic correction**

\[
\mathcal{T}_{(E)}^\mu = \frac{1}{4\pi} \left( \|G^{\mu\alpha} F_{\nu\alpha} + u^\mu u^\gamma \downarrow_{\nu\beta} F^{\alpha\beta} G_{\alpha\gamma} + g^\mu_\nu D_\alpha E^\alpha \right)
\]

- **Vortex correction**

\[
\mathcal{T}_{(M)}^\mu = \frac{1}{4\pi} \left( \perp G^{\mu\alpha} F_{\nu\alpha} - u^\mu u^\gamma \downarrow_{\nu\beta} G^{\alpha\beta} F_{\alpha\gamma} \right)
\]

Related to Abraham tensor of ordinary electrodynamics.
“Closing” the system of equations

• We have found that the second law of thermodynamics and energy-momentum tensor depend on the electromagnetic and vorticity tensors $F^{\mu\nu}$, $\mathcal{V}^{\mu\nu}_{(i)}$, as well as on the complementary tensors $G^{\mu\nu}$, $\mathcal{W}^{\mu\nu}_{(i)}$.

• To close the system of equations we need to express the tensors $G^{\mu\nu}$, $\mathcal{W}^{\mu\nu}_{(i)}$ through $F^{\mu\nu}$, $\mathcal{V}^{\mu\nu}_{(i)}$. The relation between these tensors will depend on a detailed microphysics model of a mixture.

• This is in full analogy with the ordinary electrodynamics where, in order to close the system one needs to specify the relation between the tensors $F^{\mu\nu}$ and $G^{\mu\nu}$

\[ F^{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad \text{and} \quad G^{\alpha\beta} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix} \]

\[ D = \varepsilon E \quad \text{and} \quad B = \mu H \]
• In the next slides we will discuss the simplified dynamic equations in the so called “MHD” approximation. In that case the complementary tensors can be expressed as:

\[
\begin{align*}
G^{\mu\nu} &= 0 \\
\mathcal{W}^{\mu\nu}_{(n)} &= 0 \\
\mathcal{W}^{\mu\nu}_{(p)} &= \frac{H_c c_1 c}{4\pi e_p B} \downarrow_{\mu\alpha} \downarrow_{\nu\beta} F_{\alpha\beta} \\
\downarrow^{\mu\nu} &\equiv g^{\mu\nu} + u^{\mu}u^{\nu}
\end{align*}
\]

energy of neutron vortices is neglected
“MHD” approximation

• protons form type-II superconductor

Assumptions:  • vortex interactions are neglected

• diffusion of normal thermal excitations is suppressed

In neutron stars: \( B \approx B_{Vp} \gg B_{Vn}, H, E, D \)

\[ D \sim \nabla \mu_e/e \]

magnetic field stored in proton vortices

This allows one to simplify substantially general equations describing superfluid-superconducting mixture
1. One can discard the Maxwell’s equations:

\[
\begin{align*}
\text{div } \mathbf{D} &= 4\pi \rho_{\text{free}} \\
\text{curl } \mathbf{H} &= \frac{4\pi}{c} \mathbf{J}_{\text{free}} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}
\end{align*}
\]

\[
\partial_{\alpha} G^{\alpha\beta} = -4\pi J_\beta^{(\text{free})}
\]

and set to zero the four-current density of free charges in other equations:

\[
J_\mu^{(\text{free})} = e_j j_\mu^{(j)} = e_p (n_p - n_e) u^\mu + e_i Y_{ik} w_{(k)}^\mu = 0
\]

In the absence of entrainment this means that protons approximately co-move with electrons.
2. The electromagnetic + vortex contribution to the second law of thermodynamics simplifies

\[ d\varepsilon_{\text{EM+vortex}} \approx \frac{H_{c1}}{4\pi B} B_\mu dB^\mu \]

(neglect contribution from neutron vortices)

which is simply the statement: \( d\varepsilon_{\text{EM+vortex}} \approx E_V dN \), because \( N \propto |B^\mu| \)

\[ \mathcal{V}^\mu_{(M_p)} \approx \frac{e_p}{c} B^\mu \]
\[ H_{c1} = \frac{4\pi E_{\nu p}}{\phi_{p0}} \]

\[ \mathcal{W}_{(n)} = 0 \]
\[ \mathcal{W}_{(p)} = \frac{H_{c1} c}{4\pi e_p B} \perp^{\mu\alpha}_\perp^{\nu\beta} F_{\alpha\beta} \]

\[ \perp^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu \]

**NOTE**: This expression for EM+vortex energy density corresponds to the following choice of complementary tensors.
Full system of MHD equations

Particle and energy-momentum conservation:

\[ \partial_{\mu} j_{(j)}^{\mu} = 0 \]
\[ \partial_{\mu} T^{\mu\nu} = 0 \]

\[ j_{(i)}^{\mu} = n_i u^{\mu} + Y_{ik} w_{(k)}^{\mu} \]
\[ j_{(e)}^{\mu} = n_e u^{\mu} \]

\[ T^{\mu\nu} = (P + \varepsilon) u^{\mu} u^{\nu} + P g^{\mu\nu} + Y_{ik} \left( w_{(i)}^{\mu} w_{(k)}^{\nu} + \mu_i w_{(k)}^{\mu} u^{\nu} + \mu_k w_{(i)}^{\nu} u^{\mu} \right) + \Delta T^{\mu\nu} \]

\[ \Delta T^{\mu\nu} = \frac{H_{c1}}{4\pi B} \left( \nabla^{\mu\alpha} \nabla^{\nu\alpha} E_{\alpha} + u^{\mu} \nabla^{\nu\alpha} E_{\alpha} \right) \]

Second law of thermodynamics:

\[ d\varepsilon = T \, dS + \mu_i \, dn_i + \mu_e \, dn_e + \frac{Y_{ik}}{2} \, d \left( w_{(i)}^{\alpha} w_{(k)}^{\alpha} \right) + \frac{H_{c1}}{4\pi B} \, B_{\mu} \, dB^{\mu} \]

“Superfluid” equations for neutrons and protons:

\[ u^{\nu} V_{(n)\mu\nu} = \mu_n n_n \, \bar{f}_{(n)\mu} \]
\[ E_{\mu} = \mu_p n_p \, \bar{f}_{(p)\mu} \]

\[ \partial_{\alpha} F^{\alpha\beta} = 0 \]
\[ J_{(\text{free})}^{\mu} = e_j j_{(j)}^{\mu} = e_p (n_p - n_e) u^{\mu} + e_i Y_{ik} w_{(k)}^{\mu} = 0 \]

Electromagnetic sector
Evolution equation for the magnetic field

The MHD approximation discussed above allows one to obtain a simple nonrelativistic evolution equation for the magnetic field (see also Konenkov & Geppert’01):

$$\frac{\partial B}{\partial t} + \text{curl} (B \times \mathbf{v}_{LP}) = 0$$

magnetic field transport by vortices

$$\mathbf{v}_{LP} = \mathbf{V}_{\text{norm}} - \alpha_p m_p n_p \mathbf{W}_p - \frac{\beta_p}{B} m_p n_p \mathbf{B} \times \mathbf{W}_p$$

velocity of proton vortices

$$\mathbf{W}_p = \frac{c}{4\pi e n_p} \text{curl} \left( \frac{H_{c1} B}{B} \right)$$

vanishes in the absence of vortex tension and buoyancy

$$\alpha_p = -\frac{1}{m_p n_p} \frac{1}{1 + \mathcal{R}_p^2}$$

correct at $T = 0$

$$\beta_p = \frac{1}{m_p n_p} \frac{\mathcal{R}_p}{1 + \mathcal{R}_p^2}$$

$\mathcal{R}_p$ drag coefficient
Evolution equation for the magnetic field

- This equation **differs** from the similar equation derived in Glampedakis *et al.*'11, Graber *et al.*'15 under the same assumptions:

\[
\partial_t B^i = \varepsilon^{ijk} \nabla_j \left[ \epsilon_{klm} (v^l_e B^m) - \frac{H_{c1} B}{4\pi a_p \rho_p} \frac{\mathcal{R}}{1 + \mathcal{R}^2} \left( \mathcal{R} \kappa_p^l \nabla_l \kappa_p^m + \epsilon_{klm} \kappa_p^l \kappa_p^s \nabla_s \kappa_p^m \right) \right]
\]

- Magnetic field here is not transported with the velocity of vortices (although it is the magnetic field of flux tubes) – puzzling result.

- In the weak-drag limit, \( \mathcal{R} \to 0 \), magnetic field is transported with velocity:

  **Glampedakis *et al.*'11, Graber *et al.*'15:**

  \[ V_{sp} \]

  **Our result:**

  \[ V_{sp} + \frac{c}{4\pi e n_p} \text{curl} \left( \frac{H_{c1} B}{B} \right) \]
This result is easy to understand; it follows from the balance of forces acting upon vortex in the weak-drag regime:

\[ \mathbf{v}_{Lp} = \mathbf{V}_{sp} + \frac{c}{4\pi e \nu_p} \text{curl} \left( \frac{Hc_1 \mathbf{B}}{B} \right) \]

The second term vanishes only if:

\[ F_{\text{Magnus}} + F_{\text{Tension}} + F_{\text{Buoyancy}} = 0 \]
Conclusions and some comments

• A set of fully relativistic finite-temperature equations is derived for superfluid-superconducting npe-mixture.

• Neutron and proton vortices, both types of proton SP and various dissipative corrections are allowed for; buoyancy force (i) is contained in our equations (no need to introduce it “by hands”); (ii) differ from the “standard” usually used expression.

• In comparison to MHD of Glampedakis et al’11 we:
  
  (i) take into account the relativistic and finite-temperature effects;

  (ii) provide a general framework allowing one to incorporate new physics into the existing dynamic equations (relation between $G^{\mu \nu}, \mathcal{V}^{\mu \nu}_{(i)}$ and $F^{\mu \nu}, \mathcal{V}^{\mu \nu}_{(i)}$);

  (iii) demonstrate that the displacement field is not equal to the electric field; and

  (iv) obtain a different evolution equation for the magnetic field in the MHD limit.

Our equations does not reduce to those of GAS11 in the nonrelativistic limit
• The proposed dynamic equations can be used, e.g., to study evolution of the NS magnetic field.

• However, for sufficiently hot neutron stars, for which \( T \sim T_{ci} \) the effects of particle diffusion (more precisely, diffusion of thermal excitations) may become important.

• These effects are ignored in the proposed MHD.

• Now we work to take them into account properly.  

(Dommes & Gusakov’’16, in preparation).

More details:  
• Gusakov M.E., PRD (2016)
Preliminary result:
Magnetic field evolution equation in the presence of diffusion

• Magnetic field evolution equation will remain formally unchanged

\[
\frac{\partial B}{\partial t} + \text{curl} (B \times \mathbf{v}_{Lp}) = 0
\]

But vortex velocity will be different:

\[
\mathbf{v}_{Lp} = \mathbf{V}_{\text{norm}} - \alpha_p m_p n_p \mathbf{W}_p - \frac{\beta_p}{B} m_p n_p \mathbf{B} \times \mathbf{W}_p
\]

\[
\mathbf{W}_p = \frac{c}{4\pi e n_p} \text{curl} \left( \frac{H_{c1} B}{B} \right) + \frac{c}{n_p} (\alpha_{pk} - \alpha_{ek}) \left[ \nabla \left( \frac{\mu_k}{T} \right) - \frac{e_k E}{T} \right]
\]

diffusion-induced term

Dommes & Gusakov’’16, in preparation