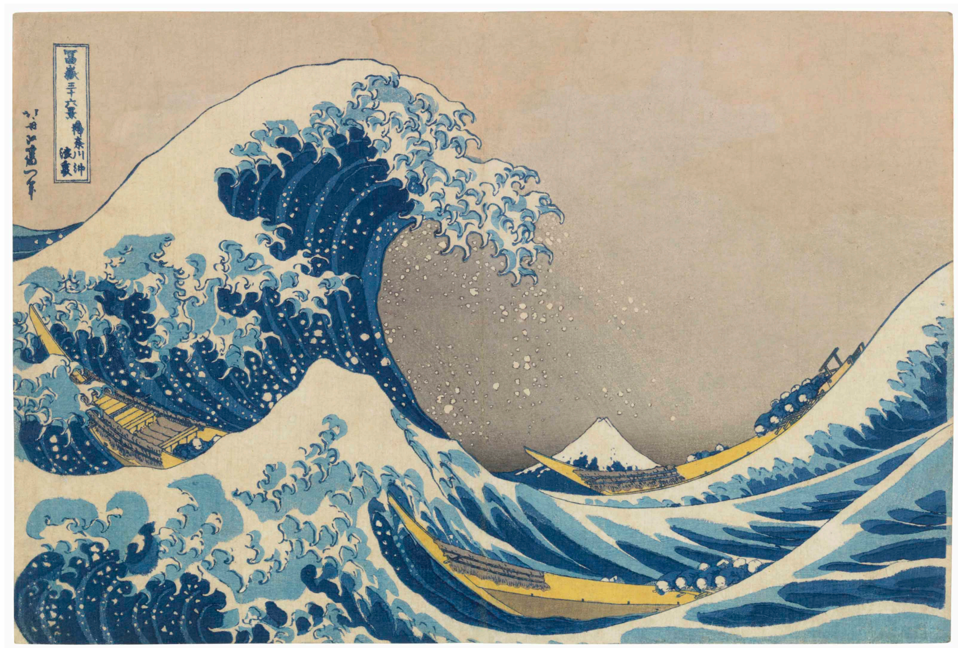
Relativistic fluid dynamics

from formulation to simulation



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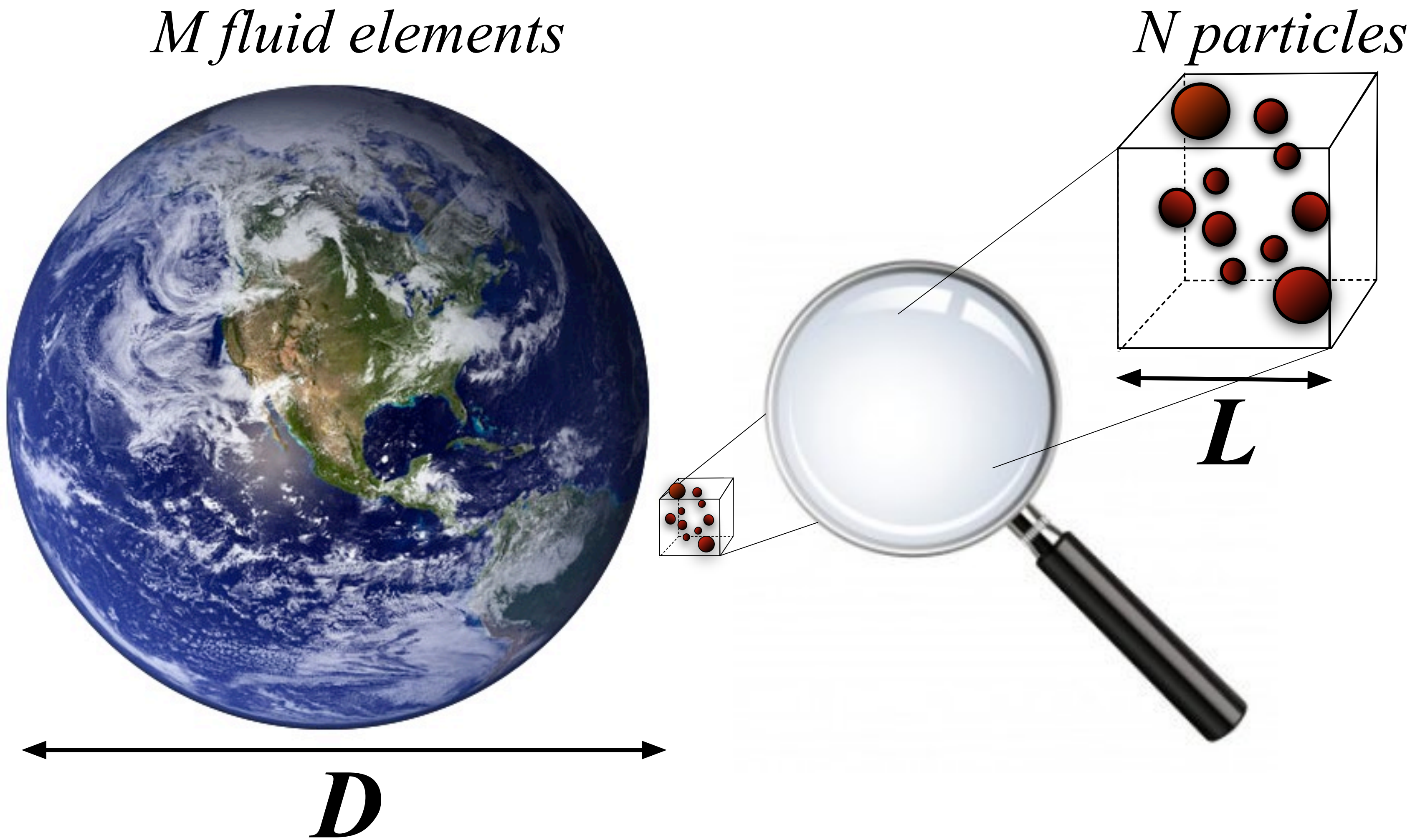
Introduction

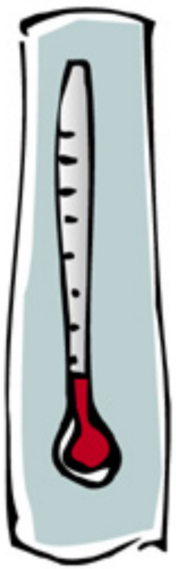

M fluid elements
N particles
L
D

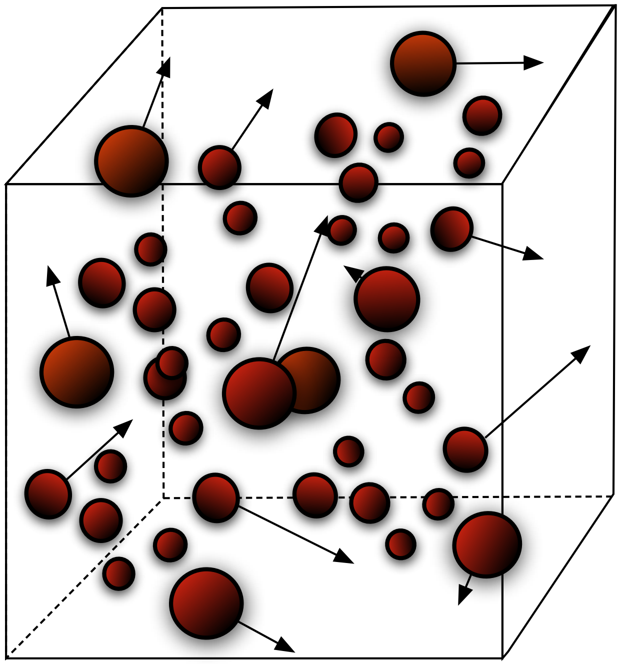
Phenomenological approach to complex systems.

- averaging over suitable scale (mean free path)
- long wavelength limit of a field theory
- superfluids (coherence length)

Realistic physics requires several distinct “flows”.

Why “relativistic” fluids?

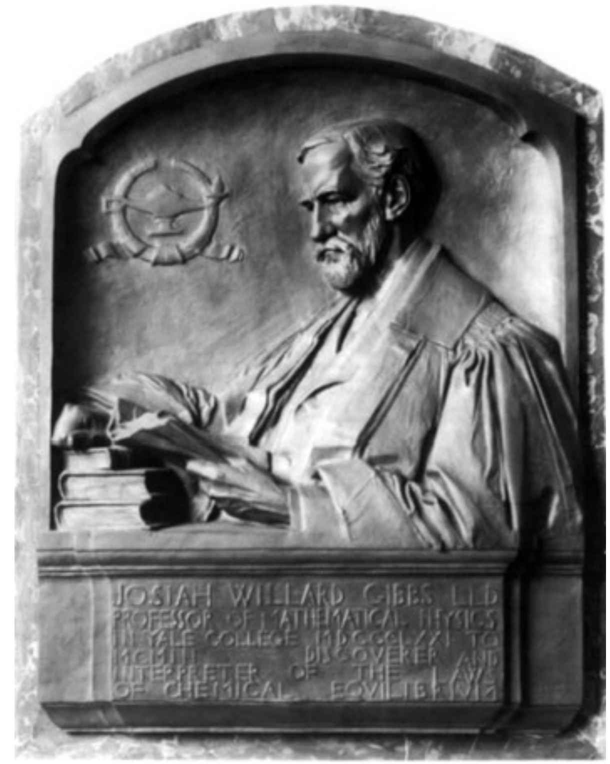




- at high temperatures, individual particle velocities may be large
- the average velocity of the fluid elements may be large
- the curved spacetime may play a role

“**Thermodynamics** is a branch of physics concerned with heat and temperature and their relation to energy and work. It defines macroscopic variables, such as internal energy, entropy and pressure that partly describe a body of matter or radiation.”

(the wisdom of Wikipedia)



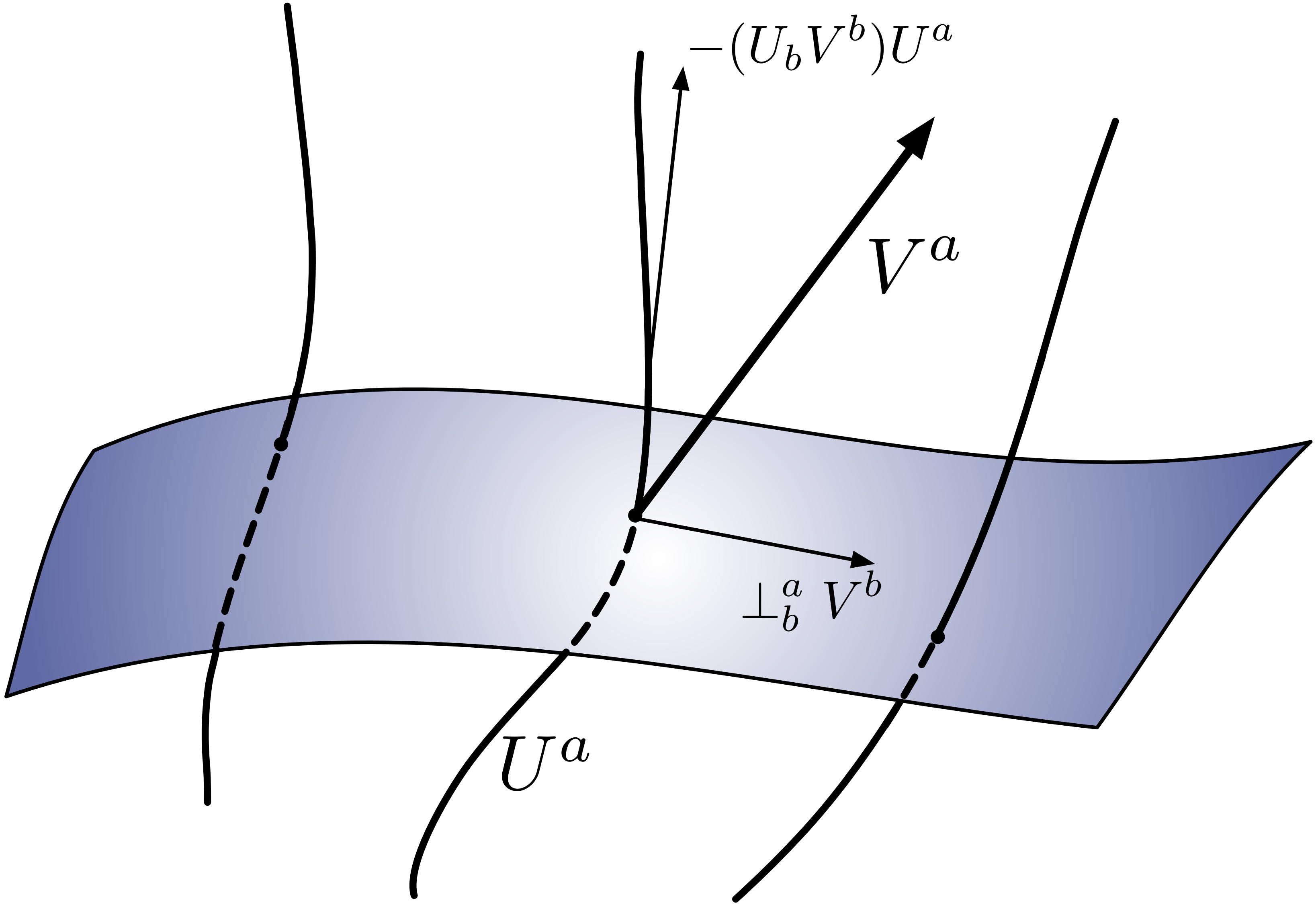
Euler relation (extensive system)

$$E = TS - pV + \mu N$$

or, in terms of densities

$$p + \varepsilon = Ts + \mu n$$

Key question: **Who** measures **what**?


$-(U_b V^b)U^a$
V^a
$\perp^a_b V^b$
U^a

General observer measures:

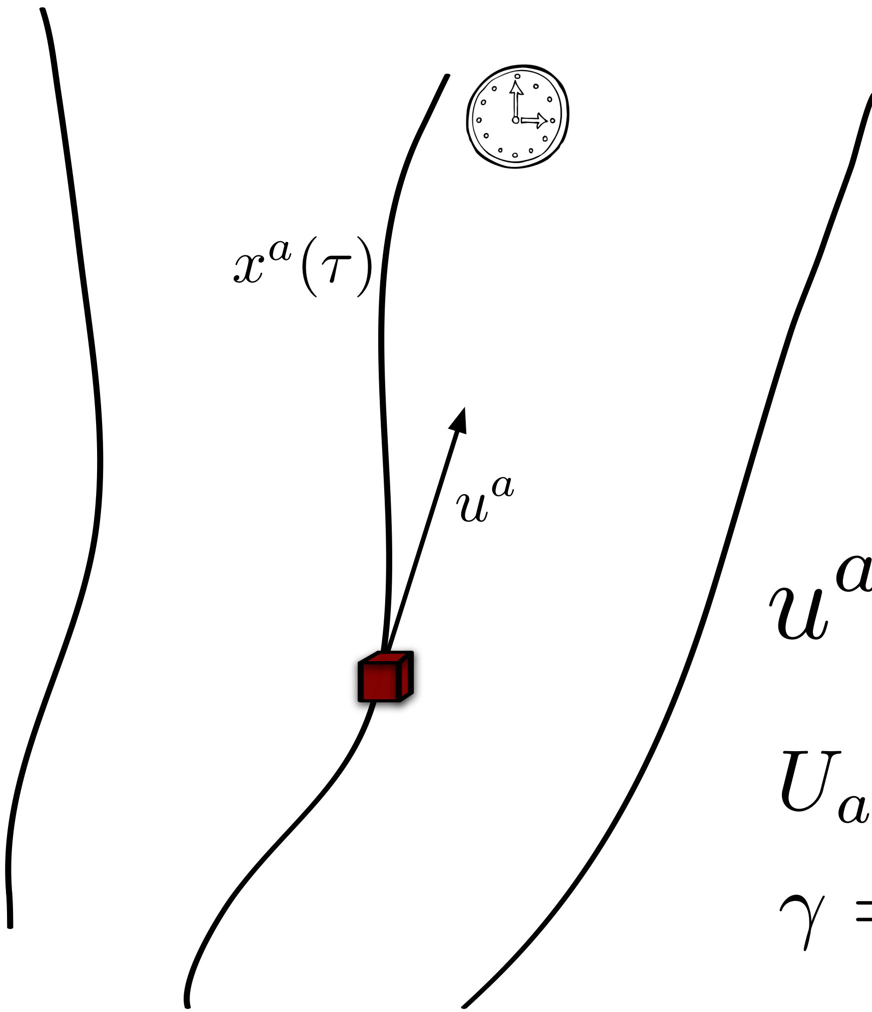
$$\varepsilon = U^a U^b T_{ab}$$

$$\mathcal{P}_a = - \perp_a^b U^c T_{bc}$$

$$\mathcal{S}_{ab} = \perp_a^c \perp_b^d T_{cd}$$

$$\boxed{T_{ab} = \varepsilon U_a U_b + 2U_{(a}\mathcal{P}_{b)} + \mathcal{S}_{ab}}$$

In a coordinate frame moving with the fluid, the four velocity measures the progression of (proper) time. This allows us to “fibrate” spacetime.



$$u^a = \gamma(U^a + v^a)$$

$$U_a v^a = 0$$

$$\gamma = (1 - v^2)^{-1/2}$$

For fluid observer, we have (assume isotropic):

$$T_{ab} = (\varepsilon + p)\, u_a u_b + p g_{ab}$$

And the equations of motion follow from (although… see later)

$$\nabla_b T^b{}_a = 0$$

From energy to **baryon number**:

$$u^a \nabla_a \varepsilon + (\varepsilon + p) \nabla_a u^a = 0$$

$$\mu n = p + \varepsilon \qquad d\varepsilon = \frac{\partial \varepsilon}{\partial n} dn \equiv \mu \, dn$$

$$\mu u^a \nabla_a n + \mu n \nabla_a u^a = 0$$

$$\boxed{\nabla_a n^a = 0}$$

From momentum to **vorticity**:

$$(\varepsilon + p)u^b \nabla_b u_a = - \perp_a^b \nabla_b p$$

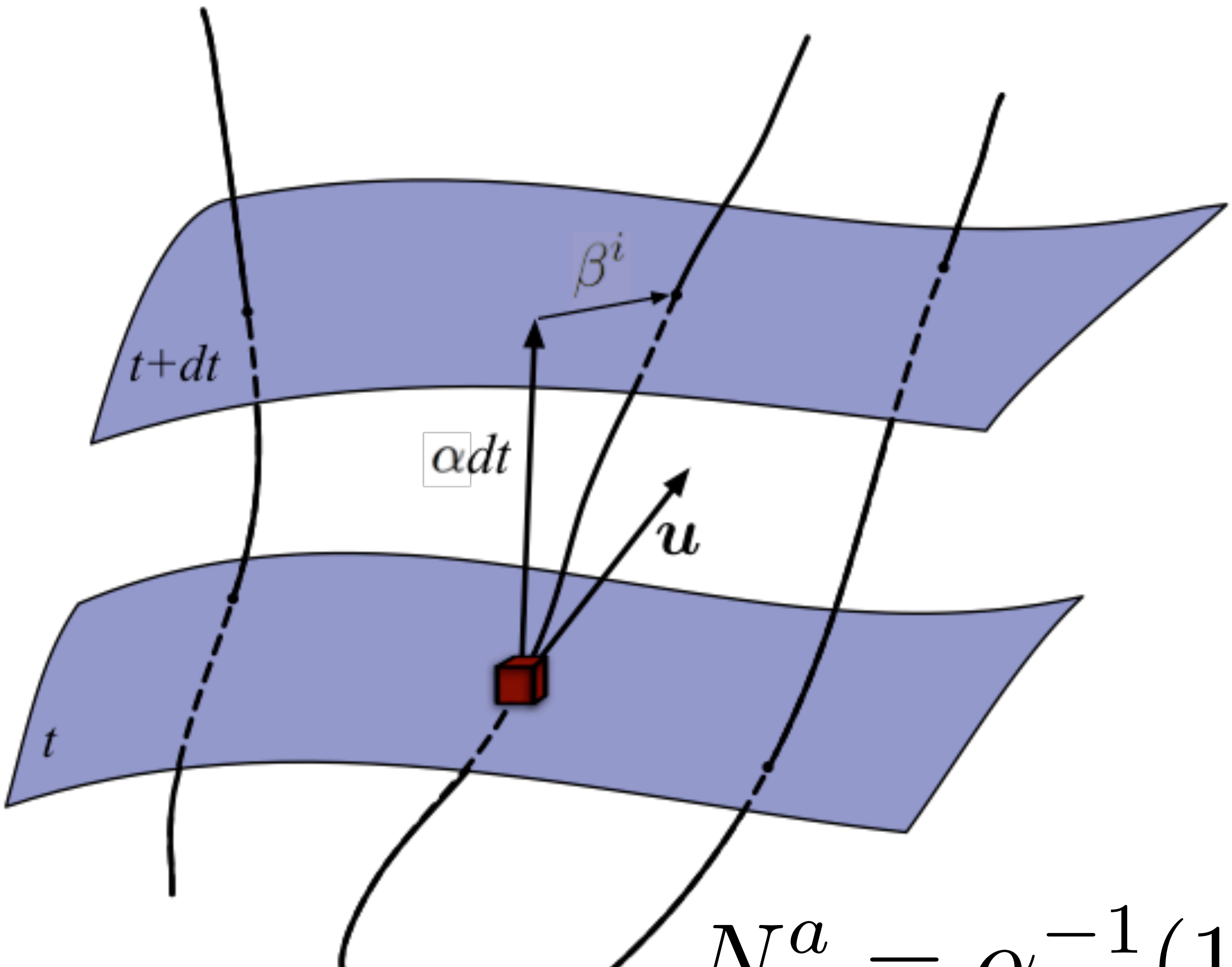
$$\mu_a = \mu u_a$$

$$f_a + \left(\nabla_b n^b\right) \mu_a = 0$$

$$\boxed{f_a = n^b \omega_{ba} = 0}$$

$$\omega_{ab} \equiv 2\nabla_{[a}\mu_{b]} = \nabla_a \mu_b - \nabla_b \mu_a$$

For simulations, we need to replace the fibration with a **foliation** of spacetime.



$$N^a = \alpha^{-1}(1, -\beta^i)$$

Baryon number (again):

$$u^a = W(N^a + \hat{v}^a)$$

$$W = (1 - \hat{v}_i \hat{v}^i)^{-1/2}$$

$$\hat{n} = -N_a n u^a = nW$$

$$\nabla_a (n u^a) = \nabla_a [W n (N^a + \hat{v}^a)] = 0$$

$$\partial_t \left(\gamma^{1/2} \hat{n} \right) + D_i \left[\gamma^{1/2} \hat{n} (\alpha \hat{v}^i - \beta^i) \right] = 0$$

At each “time step” we need to connect:

$$\hat{n} = nW$$

$$S^i = (p + \varepsilon) W^2 \hat{v}^i$$

to

$$p + \varepsilon = \mu n$$

This inversion (conservative to primitive) becomes more complicated for more “realistic” physics.