Séminaire Université de Southampton

THE WONDERS OF THE POST-NEWTONIAN

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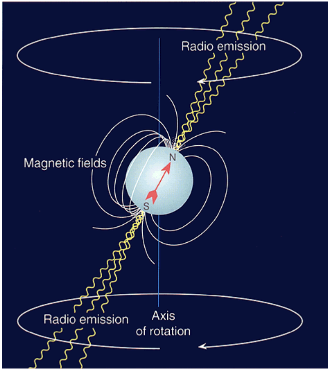
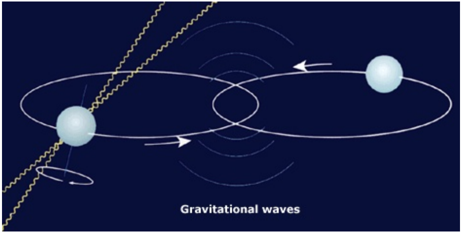
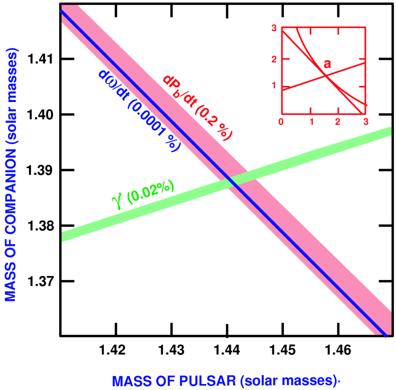
The binary pulsar PSR 1913+16 [Hulse & Taylor 1974]





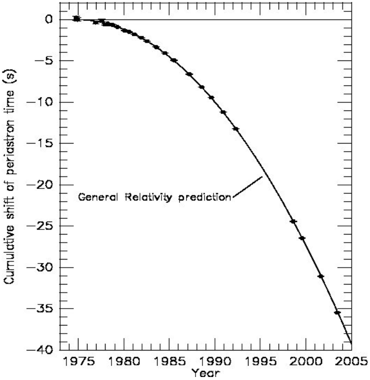
- The pulsar PSR 1913+16 is a rapidly rotating neutron star emitting radio waves like a lighthouse toward the Earth.
- This pulsar moves on a (quasi-)Keplerian close orbit around an unseen companion, probably another neutron star

Measurement of general relativistic effects



1. $\dot{\omega} = 4.2\,°/\mathsf{yr}$ relativistic advance of periastron
2. $\gamma = 4.3\,\mathsf{ms}$ gravitational red-shift and second-order Doppler effect
3. $\dot{P} = -2.4 \times 10^{-12}\mathsf{s/s}$ secular decrease of orbital period

The orbital decay of the binary pulsar [Taylor & Weisberg 1982]

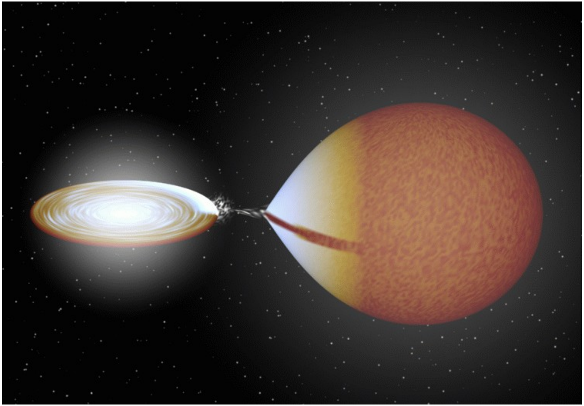


(Post-)Newtonian prediction from general relativity theory is

$$\boxed{\dot{P} = -\frac{192\pi}{5c^5}\,\frac{\mu}{M}\left(\frac{2\pi G\,M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\times 10^{-12}}$$

[Peters & Mathews 1963, Esposito & Harrison 1975; Wagoner 1975; Damour & Deruelle 1983]

Cataclysmic variables



- An evolved normal star — the Secondary, with mass M_2 — fills its Roche lobe and transfers mass to a more massive companion — the Primary, with mass $M_1 > M_2$ — which is a white dwarf
- An accretion disk of heated matter forms around the Primary and UV and X rays are emitted because of the high temperature

Loss of angular momentum in cataclysmic variables

1. The orbital angular momentum is $J = G M_1 M_2 (a/GM)^{1/2}$ so we deduce

$$\frac{\dot{a}}{a} = \frac{2\dot{J}}{J} + \frac{2(-\dot{M}_2)}{M_2}\left(1 - \frac{M_2}{M_1}\right)$$

where $-\dot{M}_2$ is the mass transfer from M_2 to M_1

2. The mass transfer tends to increase the distance a between the two stars (since $M_2 < M_1$) so to explain the long-lived cataclysmic binaries we need a mechanism of loss of angular momentum
3. When $P \lesssim 2\,\mathrm{hours}$ there is only one mechanism: gravitational radiation

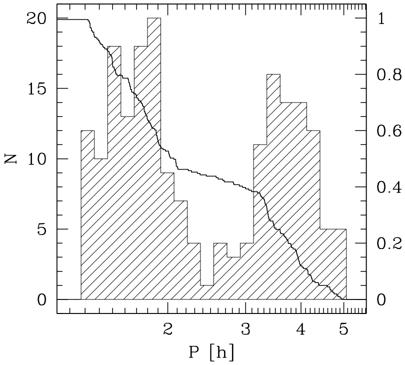
$$\left(\frac{\dot{J}}{J}\right)^{\mathrm{GW}} = -\frac{32G^2}{5c^5}\frac{M_1 M_2}{a^4}$$

4. With $\dot{a} = 0$ we get an estimate for $-\dot{M}_2$ and the result is in good agreement with the mass tranfer infered from X-ray observations

Histogram of cataclysmic variables

The presence of this peak (corresponding to orbital periods $P \lesssim 2$ hours) is only explained by gravitational radiation

$\Downarrow$



Ground-based laser interferometric detectors

LIGO

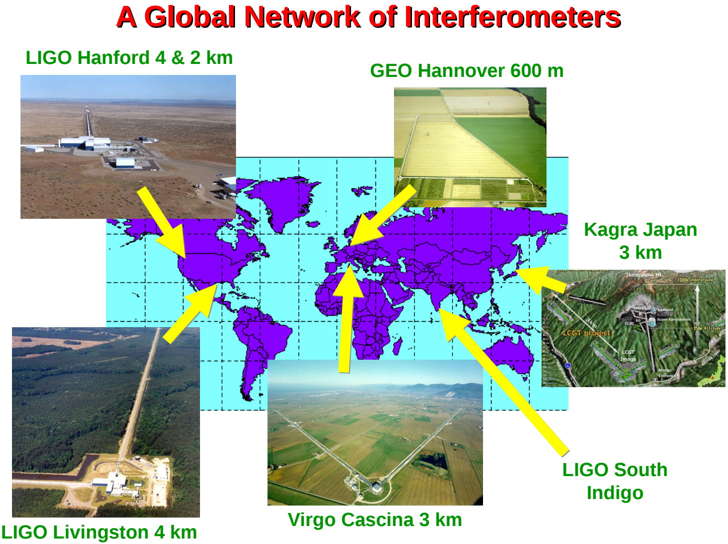
GEO

VIRGO

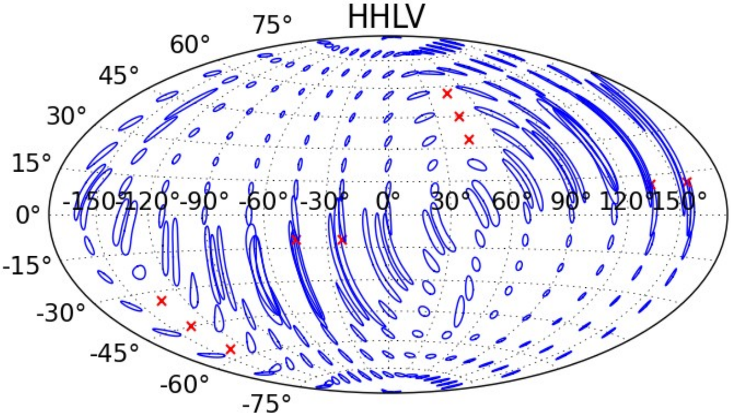
LIGO/VIRGO/GEO observe the GWs in the high-frequency band

$$10\,\mathrm{Hz} \lesssim f \lesssim 10^3\,\mathrm{Hz}$$

World-wide network of interferometric detectors

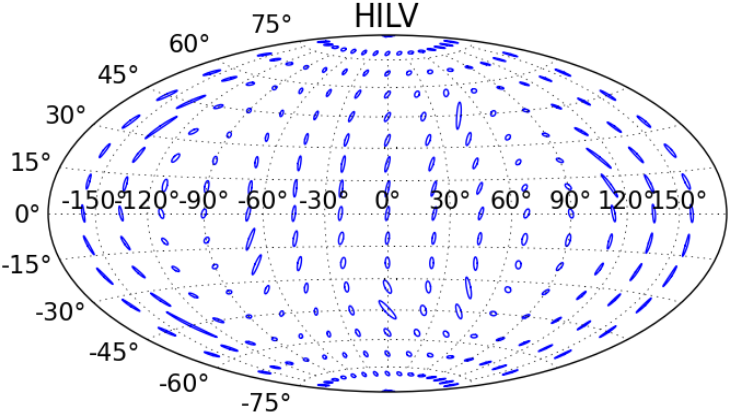


Binary neutron star merger localisation



90% localization ellipses for face-on
BNS sources @ 160 Mpc

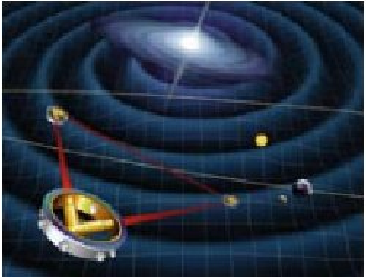
Binary neutron star merger localisation

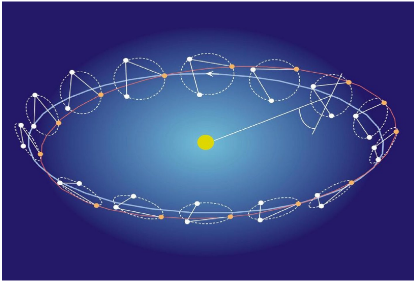


90% localization ellipses for face-on
BNS sources @ 160 Mpc

Space-based laser interferometric detector

eLISA

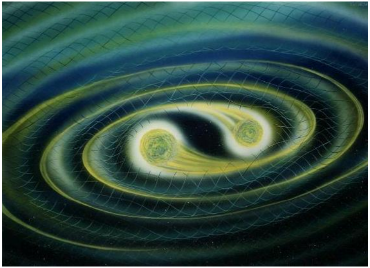




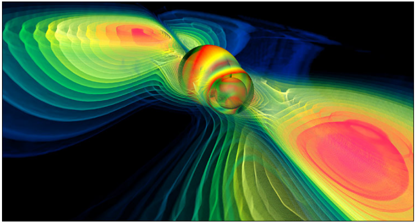
eLISA will observe the GWs in the low-frequency band

$$10^{-4}\,\mathrm{Hz} \lesssim f \lesssim 10^{-1}\,\mathrm{Hz}$$

The inspiral and merger of compact binaries



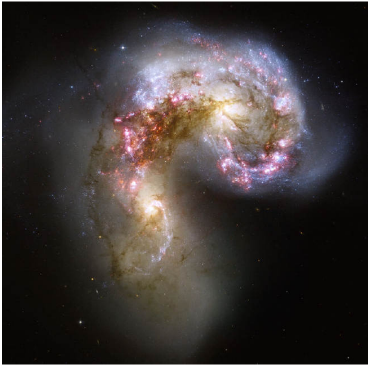
Neutron stars spiral and coalesce



Black holes spiral and coalesce

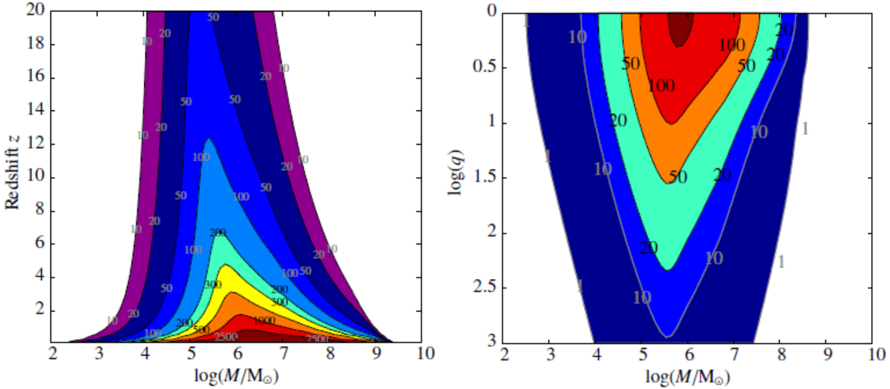
1. Neutron star ($M = 1.4\,M_\odot$) events will be detected by ground-based detectors LIGO/VIRGO/GEO
2. Stellar size black hole ($5\,M_\odot \lesssim M \lesssim 20\,M_\odot$) events will also be detected by ground-based detectors
3. Supermassive black hole ($10^5\,M_\odot \lesssim M \lesssim 10^8\,M_\odot$) events will be detected by the space-based detector eLISA

Coalescences of supermassive black-holes

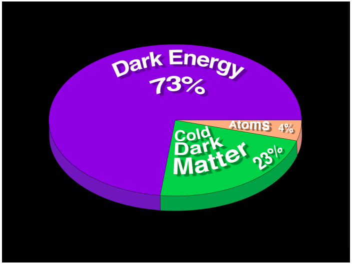


When two galaxies collide their central supermassive black holes may form a bound binary system which will spiral and coalesce. eLISA will be able to detect the gravitational waves emitted by such enormous events anywhere in the Universe

Supermassive black-holes detected by eLISA



Supermassive black-holes as dark energy probes

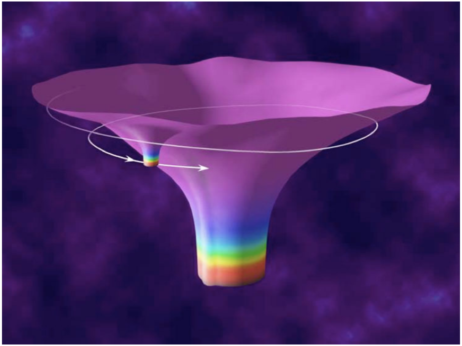


Supermassive black-hole coalescences will be observed by eLISA up to high red-shift z. In the concordance model of cosmology the distance D_L is

$$D_\mathsf{L}(z) = \frac{1+z}{H_0} \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_\mathsf{M}(1+z')^3 + \Omega_\mathsf{DE}(1+z')^{3(1+w)}}}$$

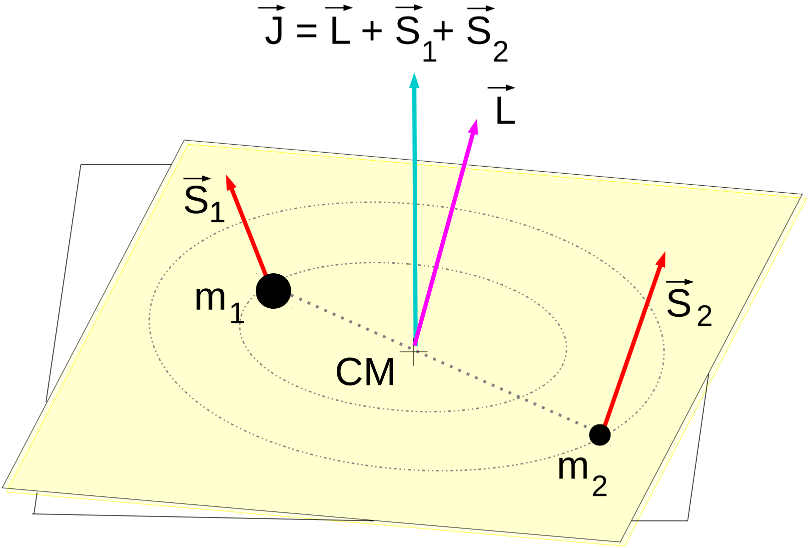
eLISA will be able to constrain the equation of state of dark energy $w = p_\mathsf{DE}/\rho_\mathsf{DE}$ to within a few percent

Extreme mass ratio inspirals (EMRI) for eLISA

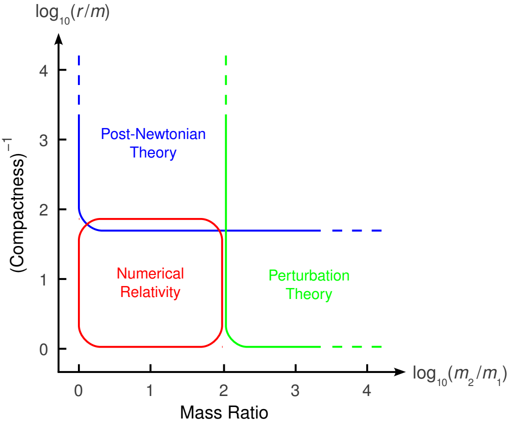


- A neutron star or a stellar black hole follows a highly relativistic orbit around a supermassive black hole. The gravitational waves generated by the orbital motion are computed using black hole perturbation theory
- Observations of EMRIs will permit to test the no-hair theorem for black holes, i.e. to verify that the central black hole is described by the Kerr geometry

Modelling of compact binary inspiral

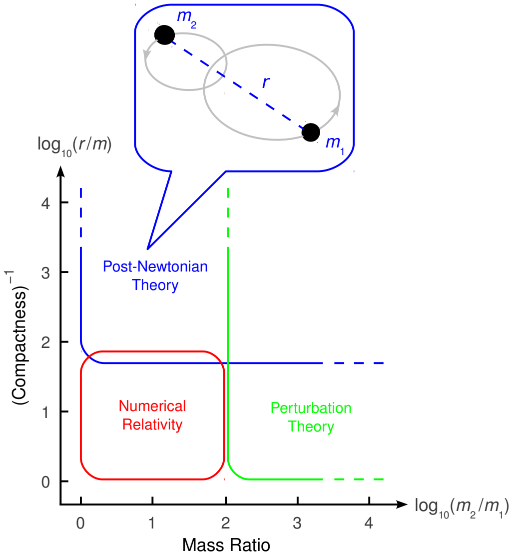


Methods to compute GW templates



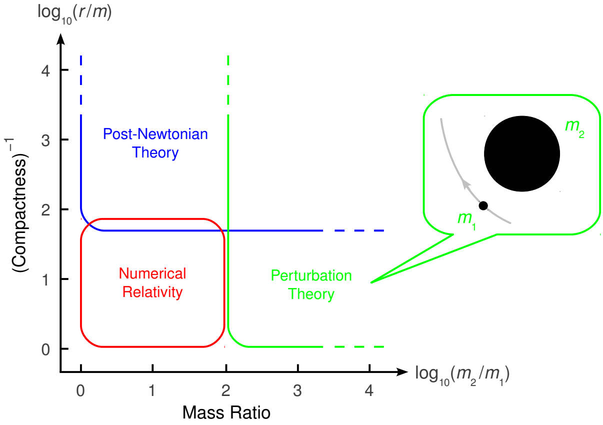
[courtesy Alexandre Le Tiec]

Methods to compute GW templates



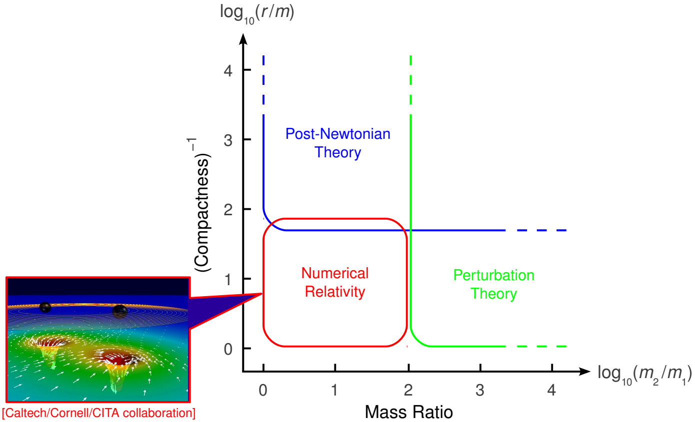
[courtesy Alexandre Le Tiec]

Methods to compute GW templates



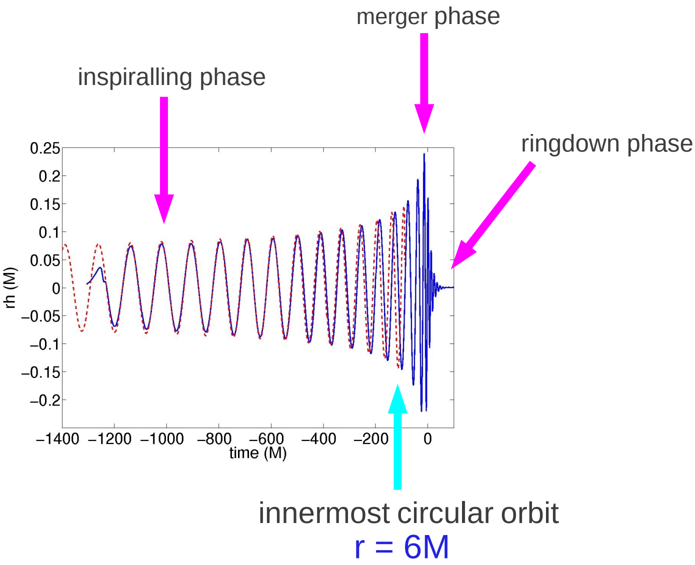
[courtesy Alexandre Le Tiec]

Methods to compute GW templates



[courtesy Alexandre Le Tiec]

The gravitational chirp of compact binaries



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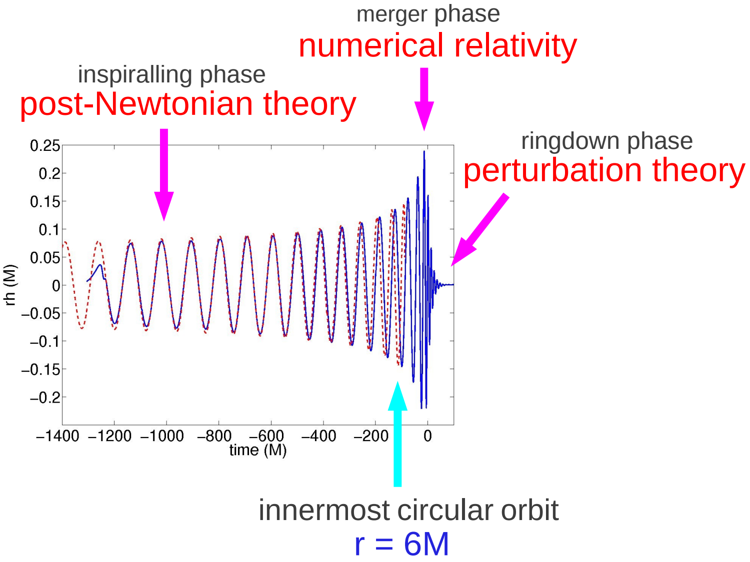
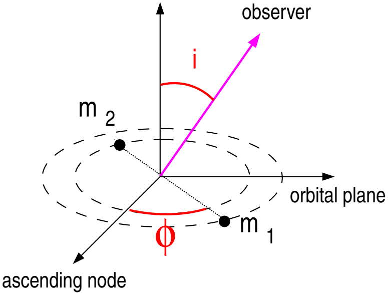


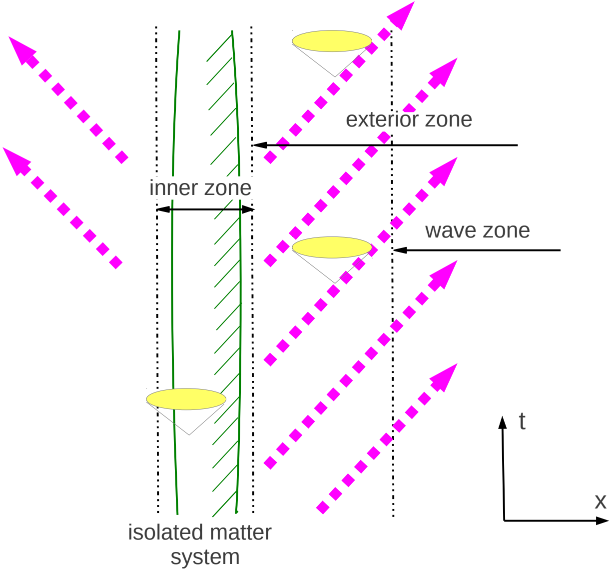
Inspiralling binaries require high-order PN modelling

[Cutler, Flanagan, Poisson & Thorne 1992; Blanchet & Schäfer 1993]

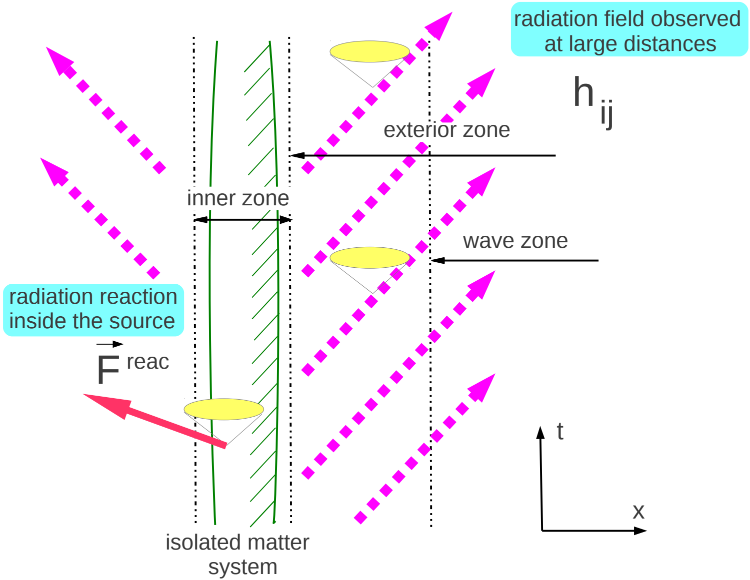


$$\phi(t) = \phi_0 \underbrace{- \frac{M}{\mu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}}_{\text{result of the quadrupole formalism (sufficient for the binary pulsar)}} \Bigl\{ 1 + \underbrace{\frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \cdots + \frac{3\text{PN}}{c^6} + \cdots}_{\text{needs to be computed with 3PN precision at least}} \Bigr\}$$

Isolated matter system in general relativity



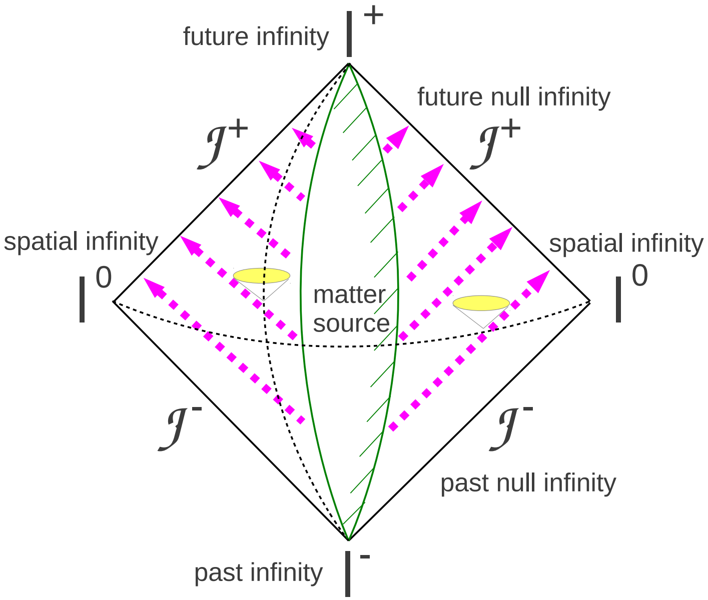
Isolated matter system in general relativity



Isolated matter system in general relativity

1. Generation problem
   - What is the gravitational radiation field generated in a detector at large distances from the source?
2. Propagation problem
   - Solve the propagation effects of gravitational waves from the source to the detector, including non-linear effects
3. Motion problem
   - Obtain the equations of motion of the matter source including all conservative non-linear effects
4. Reaction problem
   - Obtain the dissipative radiation reaction forces inside the source in reaction to the emission of gravitational waves

Conformal picture



Asymptotic structure of space-time

1. What is the struture of space-time far away from an isolated matter system?
2. Does a general radiating space-time satisfy rigourous definitions of asymptotic flatness in general relativity?
3. How to relate the asymptotic structure of space-time [Bondi *et al.* 1962, Sachs 1962] to the matter variable and dynamics of an actual source?
4. How to impose rigourous boundary conditions on the edge of space-time appropriate to an isolated system?

Einstein field equations [Einstein, November 1915!]

They derive from the total gravitational field plus matter action

$$S = \underbrace{\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} + \underbrace{S_{\text{mat}}\big[\Psi, g_{\alpha\beta}\big]}_{\text{matter action}}$$

Varying the metric (with $\delta g_{\alpha\beta} \to 0$ when $|x^\mu| \to \infty$)

$$\boxed{\underbrace{G^{\alpha\beta}[g, \partial g, \partial^2 g]}_{\text{Einstein tensor}} = \frac{8\pi G}{c^4} \underbrace{T^{\alpha\beta}[\Psi, g]}_{\text{matter stress-energy tensor}}}$$

The field equations contain the matter equations

$$\nabla_\mu G^{\alpha\mu} \equiv 0 \Longrightarrow \nabla_\mu T^{\alpha\mu} = 0$$

Gauge-fixed Einstein field equations

$$S_{\text{gauge-fixed}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x \left(\sqrt{-g}\,R \underbrace{-\frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu \mathfrak{g}^{\alpha\mu}\partial_\nu \mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\right) + S_{\text{mat}}$$

where $\mathfrak{g}^{\alpha\beta} = \sqrt{|g|}g^{\alpha\beta}$ is called the ghotic metric

$$\mathfrak{g}^{\mu\nu}\partial_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}}$$

$$\underbrace{\partial_\mu \mathfrak{g}^{\alpha\mu} = 0}_{\text{harmonic-gauge condition}}$$

Such system of equations is a well-posed problem ("problème bien posé") in the sense of Hadamard [Choquet-Bruhat 1952]

Post-Minkowskian expansion

[*e.g.* Bertotti & Plebanski 1960; Thorne & Kovàcs 1975]

Weakly self-gravitating isolated matter source

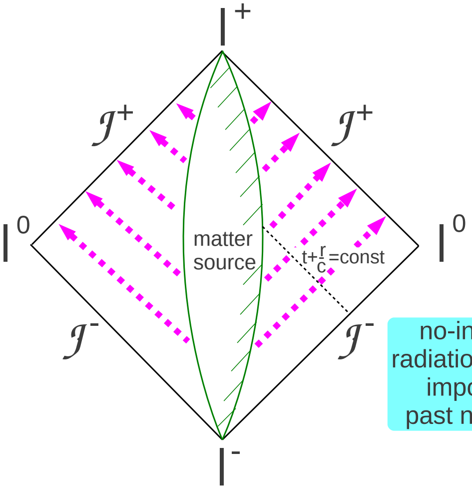
$$\gamma_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \qquad \begin{cases} M \text{ mass of source} \\ a \text{ size of source} \end{cases}$$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)}}_{G \text{ labels the PM expansion}}$$

$$\Box_\eta h^{\alpha\beta}_{(n)} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta}_{(n)} + \overbrace{\Lambda^{\alpha\beta}_{(n)}[h_{(1)}, \cdots, h_{(n-1)}]}^{\text{know from previous iterations}}$$

$$\partial_\mu h^{\alpha\mu}_{(n)} = 0$$

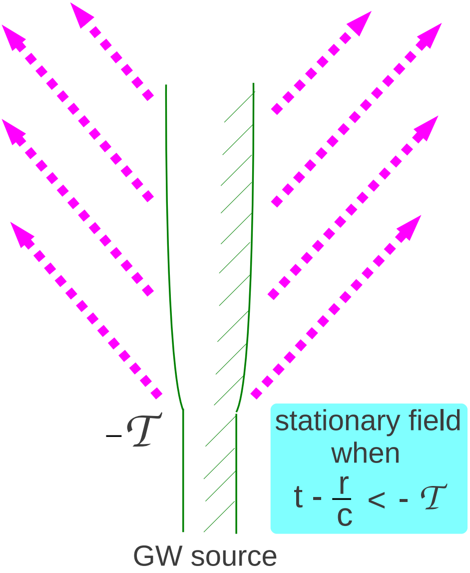
No-incoming radiation condition



no-incoming radiation condition imposed at past null infinity

$$\lim_{\substack{r\to+\infty \\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0$$

Hypothesis of stationarity in the remote past



In practice all GW sources observed in astronomy (*e.g.* a compact binary system) will have been formed and started to emit GWs only from a finite instant in the past $-\mathcal{T}$

The post-Newtonian expansion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1932; Fock 1959; Chandrasekhar 1965]

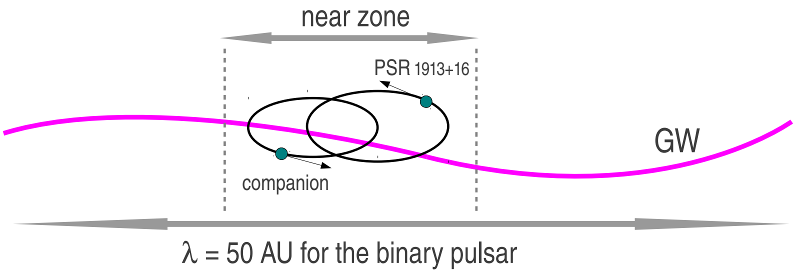
Valid for isolated matter sources that are at once slowly moving, weakly stressed and weakly gravitating (so-called post-Newtonian source) in the sense that

$$\varepsilon_{\mathrm{PN}} \equiv \max\left\{\left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}, \left|\frac{U}{c^2}\right|^{1/2}\right\} \ll 1$$

- $\varepsilon_{\mathrm{PN}}$ plays the role of a slow motion estimate $\varepsilon_{\mathrm{PN}} \sim v/c \ll 1$
- For self-gravitating sources the internal motion is due to gravitational forces (*e.g.* a Newtonian binary system) hence $v^2 \sim GM/a$
- Gravitational wave length $\lambda \sim cP$ where $P \sim a/v$ is the period of motion

$$\frac{a}{\lambda} \sim \frac{v}{c} \sim \varepsilon_{\mathrm{PN}}$$

The post-Newtonian expansion



- Near zone defined by $r \ll \lambda$ covers entirely the post-Newtonian source
- General PN expansion inside the source's near zone

$$h^{\alpha\beta}_{\text{PN}}(\mathbf{x}, t, c) = \sum_{p \geqslant 2} \frac{1}{c^p} h^{\alpha\beta}_p(\mathbf{x}, t, \ln c)$$

Quadrupole moment formalism [Einstein 1916; Landau & Lifchitz 1947]

1. First quadrupole formula

$$h^{\mathrm{TT}}_{ij} = \frac{2G}{c^4 r} P^{\mathrm{TT}}_{ijkl} \left\{ Q^{(2)}_{kl} \left(t - \frac{r}{c}\right) + \mathcal{O}\left(\varepsilon_{\mathrm{PN}}\right) \right\} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

2. Einstein quadrupole formula

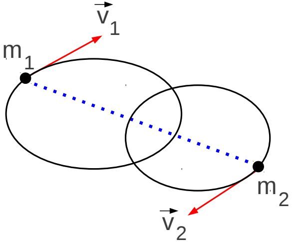
$$\mathcal{F}^{\mathrm{GW}} \equiv \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathrm{GW}} = \frac{G}{5c^5} \left\{ Q^{(3)}_{ij} Q^{(3)}_{ij} + \mathcal{O}\left(\varepsilon^2_{\mathrm{PN}}\right) \right\}$$

3. Radiation reaction quadrupole formula [Burke & Thorne 1970]

$$F^{\mathrm{reac}}_i = -\frac{2G}{5c^5} \rho_{\mathrm{N}}\, x^j\, Q^{(5)}_{ij} + \mathcal{O}\left(\varepsilon^7_{\mathrm{PN}}\right)$$

Gravitational radiation is a small 2.5PN effect $\sim \varepsilon^5_{\mathrm{PN}}$ when seen in the near zone

Application to compact binaries [Peters & Mathews 1963]



$$\begin{cases} a & \text{semi-major axis of relative orbit} \\ e & \text{eccentricity of relative orbit} \\ \omega = \frac{2\pi}{P} & \text{orbital frequency} \end{cases}$$

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{M}$$

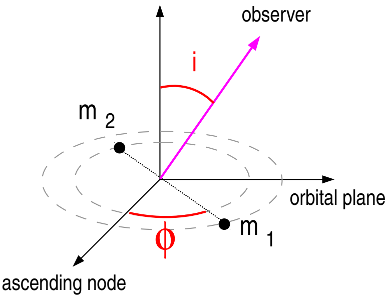
$$\boxed{\nu = \frac{\mu}{M} \qquad 0 < \nu \leqslant \frac{1}{4}}$$

$$\boxed{\langle \mathcal{F}^{\text{GW}} \rangle = \frac{32}{5}\frac{c^5}{G}\nu^2 \left(\frac{GM}{ac^2}\right)^5 \underbrace{\frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}}_{\text{“enhancement” factor } f(e)}}$$

Energy balance argument $\frac{\mathrm{d}E}{\mathrm{d}t} = -\langle \mathcal{F}^{\text{GW}} \rangle$ together with Kepler's law $GM = a^3\omega^2$

$$\boxed{\dot{P} = -\frac{192\pi}{5c^5}\left(\frac{2\pi GM}{P}\right)^{5/3} \nu\, f(e)}$$

Waveform of inspiralling compact binaries



$$h_+ = \frac{2G\mu}{c^2 D_\mathsf{L}} \left(\frac{GM\omega}{c^3}\right)^{2/3} \left(1+\cos^2 i\right) \cos\left(2\phi\right)$$

$$h_\times = \frac{2G\mu}{c^2 D_\mathsf{L}} \left(\frac{GM\omega}{c^3}\right)^{2/3} \left(2\cos i\right) \sin\left(2\phi\right)$$

The distance of the source $r = D_\mathsf{L}$ is measurable from the GW signal [Schutz 1986]

Orbital phase of inspiralling compact binaries

for quasi circular orbits
$$\begin{cases} E = -\dfrac{Mc^2}{2}\nu\, x \\ \mathcal{F}^{\mathsf{GW}} = \dfrac{32}{5}\dfrac{c^5}{G}\nu^2 x^5 \end{cases}$$

where $x = \left(\dfrac{GM\omega}{c^3}\right)^{2/3}$ = PN parameter = $\mathcal{O}(\varepsilon^2_{\mathsf{PN}})$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathsf{GW}} \iff \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{64}{5}\frac{c^3\nu}{GM}\,x^5 \iff \frac{\dot{\omega}}{\omega^2} = \frac{96\nu}{5}\nu\left(\frac{GM\omega}{c^3}\right)^{5/3}$$

$$\boxed{\begin{aligned} a(t) &= \left(\frac{256}{5}\frac{G^3M^3\nu}{c^5}(t_c - t)\right)^{1/4} \\ \phi(t) &= \phi_c - \frac{1}{32\nu}\left(\frac{256}{5}\frac{c^3\nu}{GM}(t_c - t)\right)^{5/8} \end{aligned}}$$

Multipolar-post-Minkowskian expansion

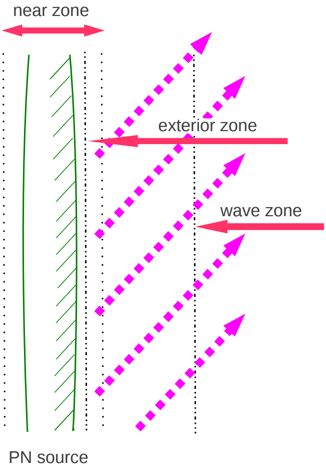
[Blanchet & Damour 1986; Blanchet 1987]

- Starts with the solution of the linearized equations outside an isolated source in the form of multipole expansions [Thorne 1980]
- An explicit MPM algorithm is constructed out of it by induction at any order n in the post-Minkowskian expansion
- A finite-part (FP) regularization based on analytic continuation is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

1. The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system
2. It is asymptotically simple at future null infinity in the sense of Penrose [1963, 1965] and recovers there the Bondi-Sachs [1962] formalism

The MPM-PN formalism

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



The MPM-PN formalism

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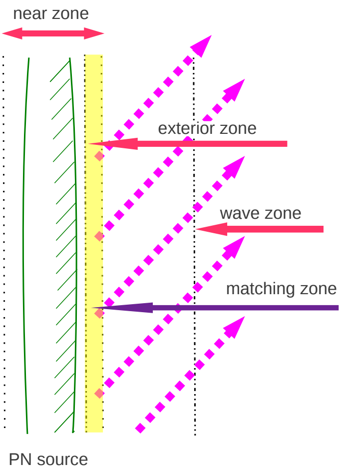


The matching equation

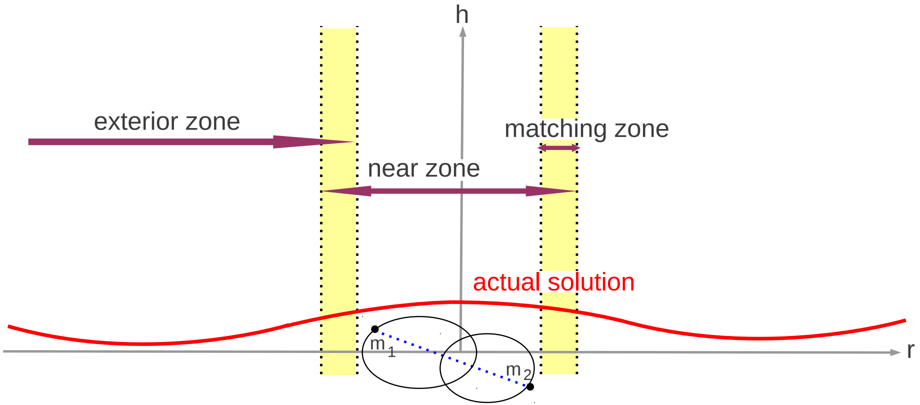
- This is a variant of the theory of matched asymptotic expansions
  [*e.g.* Lagerström *et al.* 1967; Kates 1980; Anderson *et al.* 1982]

$$\text{match} \quad \left\{ \begin{array}{c} \text{the multipole expansion} \quad \mathcal{M}(h^{\alpha\beta}) \equiv h^{\alpha\beta}_{\text{MPM}} \\ \text{with} \\ \text{the PN expansion} \quad \bar{h}^{\alpha\beta} \equiv h^{\alpha\beta}_{\text{PN}} \end{array} \right.$$

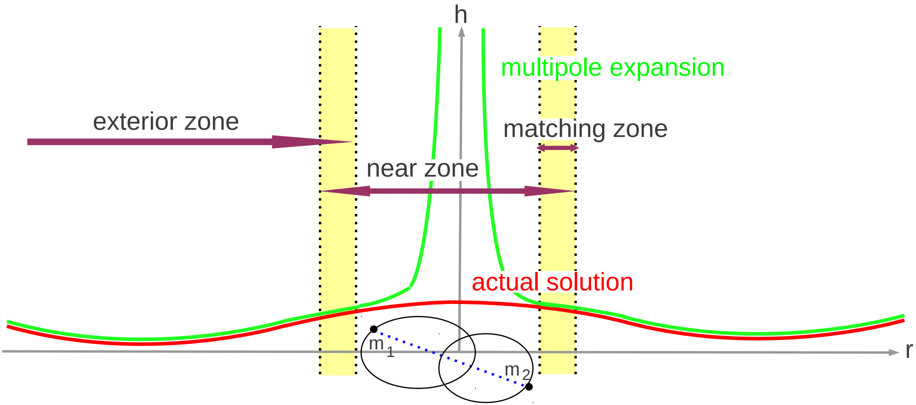
$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

  - Left side is the NZ expansion ($r \to 0$) of the exterior MPM field
  - Right side is the FZ expansion ($r \to \infty$) of the inner PN field
- The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field
- It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source

The matching equation



The matching equation



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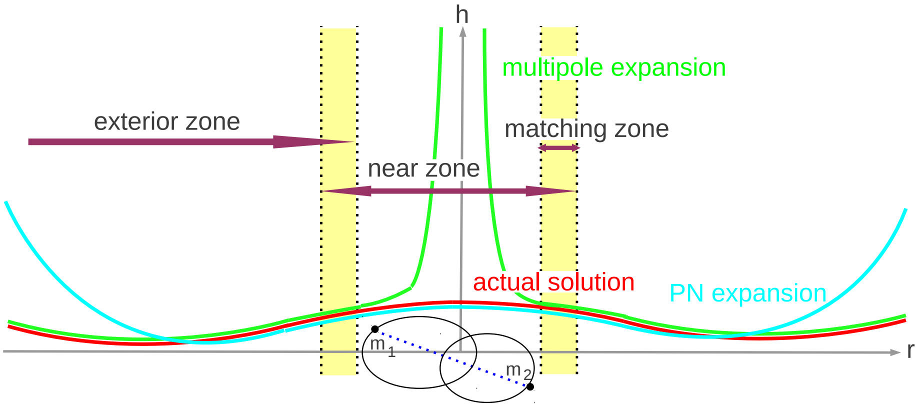
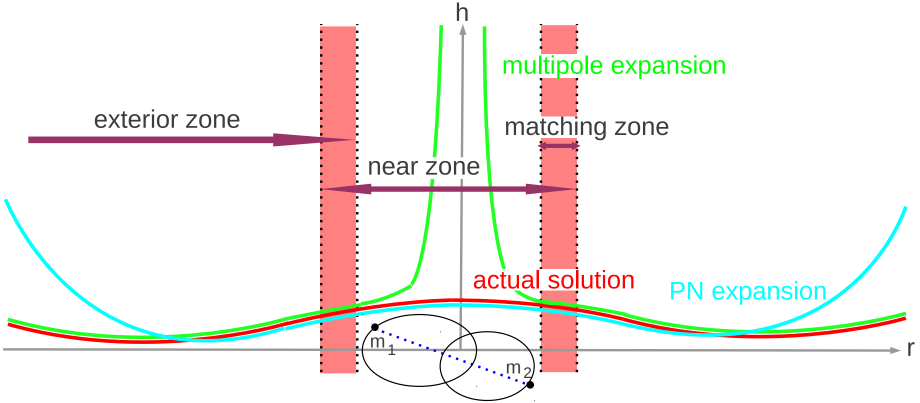


The matching equation



General solution for the multipolar field [Blanchet 1995, 1998]

$$\mathcal{M}(h^{\mu\nu}) = \mathrm{FP}\Box^{-1}_{\text{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{\infty}\partial_L\left\{\frac{M_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where $$M_L^{\mu\nu}(t) = \mathrm{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\,\underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem

General solution for the inner PN field

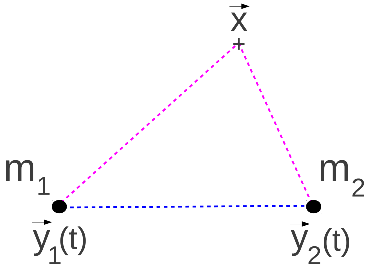
[Poujade & Blanchet 2002, Blanchet, Faye & Nissanke 2004]

$$\bar{h}^{\mu\nu} = \mathsf{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{\infty}\partial_L\left\{\frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r}\right\}}_{\text{homogeneous antisymmetric solution}}$$

where $$R_L^{\mu\nu}(t) = \mathsf{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_1^{\infty}\mathrm{d}z\,\gamma_\ell(z)\,\underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x}, t - zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$$

- The radiation reaction effects starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects associated with tails are contained in the second term and start at 4PN order

Problem of point particles



$$U(\mathbf{x},t) = \frac{Gm_1}{|\mathbf{x}-\mathbf{y}_1(t)|} + \frac{Gm_2}{|\mathbf{x}-\mathbf{y}_2(t)|}$$

$$\frac{\mathrm{d}^2\mathbf{y}_1}{\mathrm{d}t^2} = (\boldsymbol{\nabla} U)(\mathbf{y}_1(t),t) \overset{?}{=} -Gm_2\frac{\mathbf{y}_1-\mathbf{y}_2}{|\mathbf{y}_1-\mathbf{y}_2|^3}$$

- For extended bodies the self-acceleration of the body cancels out by Newton's action-reaction law
- For point particles one needs a self-field regularization to remove the infinite self-field of the particle

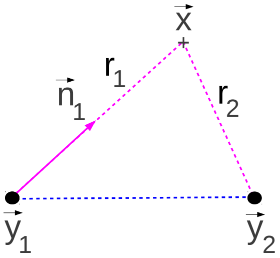
Problem of the self-field regularization

Let $F(\mathbf{x})$ be singular at source points $\mathbf{y}_1$ and $\mathbf{y}_2$, *e.g.* when $r_1 = |\mathbf{x} - \mathbf{y}_1| \to 0$

$$F(\mathbf{x}) = \sum_{a_{\text{min}} \leqslant a \leqslant N} r_1^a \underset{1}{f}{}_a(\mathbf{n}_1, \mathbf{y}_2) + o(r_1^N)$$

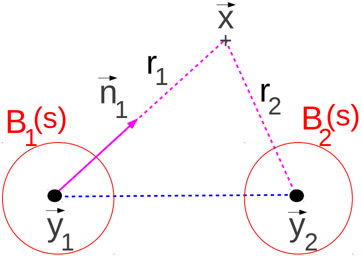
1. How to define $F(\mathbf{y}_1)$?
2. What is the meaning of $F(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}_1)$?
3. What is the meaning of $\int \mathrm{d}^3\mathbf{x}\, F(\mathbf{x})$?
4. How to differentiate singular functions, *e.g.* $\partial_i \partial_j F$?

Hadamard self-field regularization [Hadamard 1932; Schwartz 1978]



$$F(\mathbf{y}_1) \equiv \langle \underset{1}{f}{}_0 \rangle = \int \frac{\mathrm{d}\Omega_1}{4\pi} \underset{1}{f}{}_0(\mathbf{n}_1)$$

Hadamard self-field regularization [Hadamard 1932; Schwartz 1978]



$$\mathrm{Pf}\int d^3\mathbf{x}\,F = \lim_{s\to 0}\left\{\int_{\mathbb{R}^3\setminus B_1\cup B_2} d^3\mathbf{x}\,F + 4\pi\sum_{a+3<0}\frac{s^{a+3}}{a+3}\langle \underset{1}{f}{}_a\rangle + 4\pi\ln\left(\frac{s}{s_1}\right)\langle \underset{1}{f}{}_{-3}\rangle + 1\leftrightarrow 2\right\}$$

Dimensional regularization [t'Hooft & Veltman 1972; Bollini & Giambiagi 1972]

- Einstein's field equations are solved in d spatial dimensions (with $d \in \mathbb{C}$) with distributional sources. In Newtonian approximation

$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

- For two point-particles $\rho = m_1 \delta(\mathbf{x} - \mathbf{y}_1) + m_2 \delta(\mathbf{x} - \mathbf{y}_2)$ where δ is the d-dimensional Dirac function we get

$$\boxed{U(\mathbf{x}, t) = \frac{2(d-2)k}{d-1} \left(\frac{G m_1}{|\mathbf{x} - \mathbf{y}_1|^{d-2}} + \frac{G m_2}{|\mathbf{x} - \mathbf{y}_2|^{d-2}} \right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}}$$

- Computations are performed when $\Re(d)$ is a large negative complex number so as to kill all self-terms, and the result is analytically continued for any $d \in \mathbb{C}$ except for poles occuring at integer values of d
- The poles are absorbed into a renormalization of the trajectories of the particles so the physical result is finite

4PN equations of motion of compact binaries

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}}$$

$$+\underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}}+\underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}}+\underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}}+\underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}}+\underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}}+\mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN
- [Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001] — ADM Hamiltonian
- [Blanchet & Faye 2000; de Andrade, Blanchet & Faye 2001] — Harmonic equations of motion
- [Itoh, Futamase & Asada 2001; Itoh & Futamase 2003] — Surface integral method
- [Foffa & Sturani 2011] — Effective field theory

4PN
- [Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014] — ADM Hamiltonian
- [Bernard, Blanchet, Bohé, Faye & Marsat 2015] — Fokker Lagrangian

3.5PN energy flux of compact binaries

[Blanchet, Faye, Iyer & Joguet 2002]

$$\mathcal{F}^{\mathsf{GW}} = -\frac{32c^5}{5G}\nu^2 x^5 \Big\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \underbrace{[\cdots]\, x^{5/2}}_{\text{2.5PN tail}} + \underbrace{[\cdots]\, x^3}_{\substack{\text{3PN} \\ \text{includes a tail-of-tail}}} + \underbrace{[\cdots]\, x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(x^4\right) \Big\}$$

The orbital frequency and phase for quasi-circular orbits are deduced from an energy balance argument

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathsf{GW}}$$

3.5PN dominant gravitational wave modes

[Faye, Marsat, Blanchet & Iyer 2012; Faye, Blanchet & Iyer 2014]

$$h_{22} = \frac{2G\,m\,\nu\,x}{R\,c^2}\sqrt{\frac{16\pi}{5}}\,e^{-2\mathrm{i}\psi}\left\{1 + x\left(-\frac{107}{42} + \frac{55\nu}{42}\right) + 2\pi x^{3/2} + x^2\left(-\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512}\right) + \underbrace{[\cdots]\;x^{5/2}}_{\text{2.5PN}} + \underbrace{[\cdots]\;x^{3}}_{\text{3PN}} + \underbrace{[\cdots]\;x^{7/2}}_{\text{3.5PN}} + \mathcal{O}\left(x^4\right)\right\}$$

$$h_{33} = \cdots$$

$$h_{31} = \cdots$$

Tail contributions in this expression are factorized out in the phase variable

$$\psi = \phi - \frac{2GM\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right)$$

4PN spin-orbit effects in the orbital frequency

[Marsat, Bohé, Faye, Blanchet & Buonanno 2013]

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\,\nu\, x^{5/2}\Big\{\overbrace{1 + x\,[\cdots] + x^{3/2}\,[\cdots] + x^2\,[\cdots] + x^{5/2}\,[\cdots] + x^3\,[\cdots]}^{\text{non-spin terms}}$$
$$+ \underbrace{[\cdots]\,x^{3/2}}_{\text{1.5PN SO}} + \underbrace{[\cdots]\,x^2}_{\text{2PN SS}} + \underbrace{[\cdots]\,x^{5/2}}_{\text{2.5PN SO}} + \underbrace{[\cdots]\,x^3}_{\text{3PN SO}_{\text{tail}}\ \&\ \text{SS}}$$
$$+ \underbrace{[\cdots]\,x^{7/2}}_{\text{3.5PN SO}} + \underbrace{[\cdots]\,x^4}_{\text{4PN S0}_{\text{tail}}\ \&\ \text{SS}} + \mathcal{O}\left(x^4\right)\Big\}$$

- Leading SO and SS terms due to [Kidder, Will & Wiseman 1993; Kidder 1995]
- Many next-to-leading (NL) SS terms mostly in the EOM computed within the ADM Hamiltonian and the Effective Field Theory

Summary of current PN results

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN)	4PN non-spin 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (L) SSS	3.5PN non-spin 4PN (NNL) SO 3PN (NL) SS 3.5PN (L) SSS	3PN non-spin 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian [Jaranowski, Schäfer, Damour, Steinhoff]	4PN non-spin 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (L) SSS		
Effective Field Theory (EFT) [Porto, Rothstein, Foffa, Sturani, Levi, Ross]	3PN non-spin 2.5PN (NL) SO 4PN (NNL) SS	2PN non-spin 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE) [Will, Wiseman, Kidder, Pati]	2.5PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS
Surface Integral [Itho, Futamase, Asada]	3PN non-spin		