The Quest for Superstring Scattering Amplitudes

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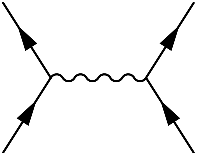
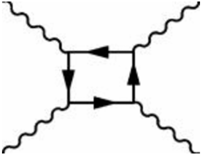
- The computation of scattering amplitudes provides a window into the interactions of a quantum theory. It tell us the probability amplitude of certain outcomes given by experiments
- Standard QFT books teach us how to compute Feynman diagrams of several processes of interest, and to interpret its results
- At tree level one may have to calculate a Feynman diagram like this to understand how electrons interact with each other:



Why strings and why scattering amplitudes

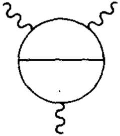
- At one loop, a typical Feynman diagram is the so-called box diagram



- These diagrams become more and more complicated as the loop order increases
- If the quantum theory makes sense, a fundamental condition is that the results must give rise to a probability amplitude. In particular, they must be finite (in the UV regime, at high energies)

Why strings and why scattering amplitudes

- Standard non-supersymmetric gravity described by the Einstein-Hilbert lagrangian gives rise to infinite results. In 4d, the 4-point amplitude at 2 loops diverges (Goroff, Sagnotti 1986) and there is no way to renormalize it



- In technical terms, in $4-\epsilon$ dimensions the Feynman diagram above implies that there is R^3 counterterm in the Lagrangian that diverges as $\epsilon \to 0$,

$$-\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\frac{209}{5760}\sqrt{-g}\,R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\mu\nu}R^{\mu\nu}{}_{\alpha\beta}$$

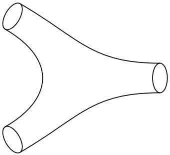
- This is a complete disaster. Standard general relativity fails to be a sensible quantum theory of gravity!

Why strings and why scattering amplitudes

- String theory gives rise to a UV finite theory of quantum gravity. Its scattering amplitudes give rise to finite results at arbitrary loop orders and therefore can be interpreted as probabilities (Mandelstam)
- There are two parameters in the amplitudes: the string coupling constant g_s and the typical size of the string (related to α')
- Unlike QFT where the rules to compute amplitudes are derived from a Lagrangian, in string theory there are prescriptions based on conformal field theory (CFT) techniques to calculate them
- The lagrangian is not a priori known. By computing scattering amplitudes of gravitons one learns more about the theory!
- Quantum gravity doesn't get more "quantum" than computing graviton amplitudes!

Why strings and why scattering amplitudes

- The starting point is the computation of the tree level 3-point amplitude, from which one can reverse engineer the Einstein-Hilbert effective field theory lagrangian
- String theory "diagram"



- Similarly, by computing the tree-level 4-point amplitude and can discover if there are R^4 terms in the Lagrangian. Yes, there are (Gross, Witten)

Why strings and why scattering amplitudes

- Interestingly, the tree-level R^4 interaction is proportional to the Euler zeta value ζ_3, which is one representative of a general class of Multiple Zeta Values (MZV)
- Number theorists have spent centuries studying such numbers, and now their appearance in string theory amplitudes has helped to create a synergy between physicists and mathematicians
- What other MZVs are produced at tree-level? In string theory these numbers are the result of computing disk integrals arising from worldsheet singularities as vertex operators approach each other
- Is there some other mechanism that generates them? Drinfel'd associator (Drummond, Ragoucy 2013; Broedel, Schlotterer, Stieberger, Terasoma 2013)

Why strings and why scattering amplitudes

- String theory does not stop there. The appearance of *multiple zeta values* at tree level is generalized to *elliptic multiple zeta values* (eMZVs) at one loop (Broedel, CM, Schlotterer 2014).
- These eMZVs are associated to a series of functions that lives on a elliptic curve, the Kronecker–Eisenstein series. Beautiful mathematics. There is a lot yet to discover!

Why strings and why scattering amplitudes

- Computing other amplitudes at different loop orders and different number of points we can find many other corrections to supergravity as predicted by string theory (e.g. $D^p R^q$ terms in the effective action) that depend on α'.
- These corrections can be used to test string dualities, non-renormalization theorems etc (Green, Gutperle, Vanhove et. al.)
- When $\alpha' \to 0$ one recovers the results that would have been obtained by standard QFT methods with Feynman diagrams

Why strings and why scattering amplitudes

- String theory has a set of rules (e.g. based on conformal field theory) that *in theory* allows us to compute scattering amplitudes and obtain these corrections

Yogi Berra

In theory there is no difference between theory and practice.
In practice there is.

- My work so far has been dedicated to computing string scattering amplitudes in practice
- The pure spinor formalism provides a convenient framework to extend the known limits considerably compared with the standard RNS and GS formulations

Why Pure Spinors?

Issue with RNS

- Spacetime supersymmetry is not manifest

Issue with GS

- Covariant quantization is not possible

Advantages of PS

- Manifest spacetime supersymmetry (10D superfields)
- Covariant quantization (BRST methods, cohomology)

Tree-level Amplitudes

Tree-level N-point

$$\mathcal{A}_N = \langle V_1(z_1) V_2(z_2) V_3(z_3) \int dz_4 U_4(z_4) \ldots \int dz_N U_N(z_N) \rangle$$

- V_i and U_i are vertex operators containing information about the particles (strings) being scattered
- Usual CFT methods: OPE's integrate out conformal weight 1 variables, then integrate out zero-modes
- **Naively**, higher-point amplitudes generate too many terms and become huge very quickly
- But they give rise to pure spinor superspace expressions . . .

Tree level amplitudes

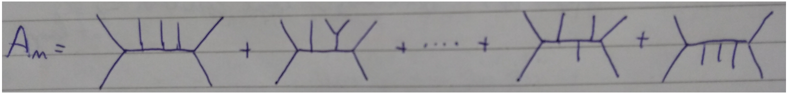
- However the general n-point amplitude was found in 2011! (CM, Schlotterer, Stieberger)
- It is important to simplify known formulas and to find tricks and shortcuts when going forward
- I am a huge fan of recursions. Their rules are in general simple and yet they can generate huge expressions (that would look intractable at first sight)
- A bit of analogy first . . .

Simplicity in Recursions

- Familiar story of young Gauss. Teacher wanted to punish the class and ordered them to sum all integers from 1 to 100
- The straightforward way

$$1+2=3,\ 3+3=6,\ 6+4=10,\ 10+5=11,\ \ldots,$$

  is laborious and takes a lot of time.
- This actually resembles summing over all Feynman diagrams one by one



Simplicity in Recursions

- Gauss noticed a recursive pattern, summing the endpoints:

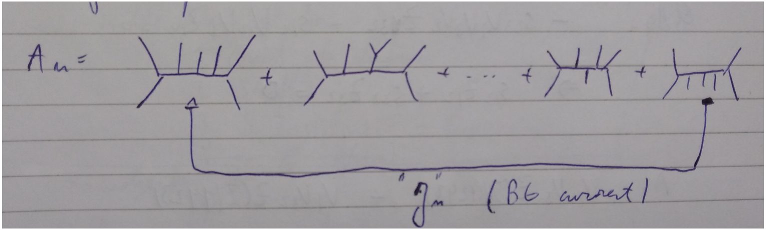
$$\overbrace{1+\overbrace{2+3\cdots+98+99}^{2+99=101}+100}_{}$$

$$\underbrace{1+2+3\cdots+98+99+100}_{1+100=101}$$

- Repeating it 50 times he got the answer: $101 \times 50 = 5050$
- Lesson: regrouping terms can lead to tremendous simplications!

Simplicity in Recursions

- In the 80s Berends and Giele discovered a recursion within the problem of computing tree-level amplitudes
- Instead of summing diagrams one by one, group them in batches into so-called *currents* $J^m_{12\ldots p}$ to get an efficient recursive formula!



$$A(1,2,\ldots,n) = s_{12\ldots n-1} J^m_{12\ldots n-1} J^m_n$$

- Berends–Giele recursive method is (still) one of the most efficient ways to compute tree-level amplitudes

Berends Giele recursion in pure spinor superspace

- The idea is to treat many superfields together in packages with definite BRST properties, the building blocks $V_{123\ldots n}$
- Defined from iterated computation of OPEs among vertex operators

Recursive building blocks from OPEs

$$V^1(z_1)U^2(z_2) \to \frac{V_{12}}{z_{21}}, \quad V_{123\ldots(p-1)}(z_1)U^p(z_p) \to \frac{V_{123\ldots p}}{z_{p1}}$$

Recursive PS cohomology method for FT amplitudes

(C.M., Schlotterer, Stieberger, Tsimpis, '10)

N-point color-ordered SYM tree amplitudes

$$A_n(1,2,\ldots,n) = \langle E_{123\ldots(n-1)} V_n \rangle$$

- Recursive cohomology problem in pure spinor superspace

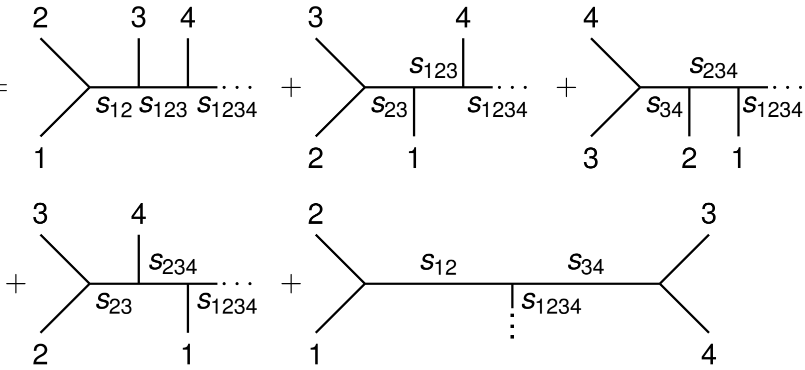
$$E_{123\ldots p} \equiv \sum_{j=1}^{p-1} M_{12\ldots j} M_{j+1\ldots p}$$

$$QM_{123\ldots p} \equiv E_{123\ldots p},$$

where $M_{12\ldots}$ are Berends–Giele supercurrents built from $V_{12\ldots}$ and propagators

- Diagramatic method with cubic graphs

$M_{1234} =$



$$= \frac{1}{s_{1234}}\left(\frac{V_{1234}}{s_{12}s_{123}} + \frac{V_{3214}}{s_{23}s_{123}} + \frac{V_{3421}}{s_{34}s_{234}} + \frac{V_{3241}}{s_{23}s_{234}} + \frac{V_{1234} - V_{1243}}{s_{12}s_{34}}\right)$$

Tree-level superstring amplitudes

- These FT recursions were the backbone of the method to tackle the combinatorial growth of terms in the string tree amplitudes

String amplitude as $(N-2)!$ building blocks

$$\mathcal{A} = \int KN \sum_{p=1}^{N-2} \frac{V_{12\ldots p} V_{N-1,\ldots,p+1} V_N}{(z_{12} z_{23} \cdots z_{p-1,p})(z_{N-1,N-2} \cdots z_{p+2,p+1})} + \mathcal{P}(2,\ldots,N-2)$$

String amplitude as $(N-3)!$ FT amplitudes

$$\mathcal{A} = \int KN \Big[\prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \mathcal{A}_{YM}(1,2,\ldots,N) + \mathcal{P}(2,\ldots,N-2) \Big]$$

Graviton amplitudes

- Schematically, closed string states are related to squares of open string: closed $=$ open $\otimes$ open
- This structure is reflected in the KLT relations between graviton amplitudes (M_n) and gluon amplitudes ($\mathcal{A}_n$) (Kawai, Lewellen, Tye 1986)

$$M_n = \mathcal{A}_n^t S \mathcal{A}_n$$

  where S is the KLT matrix
- Expanding the string disk integrals in powers of α' leads to a plethora of stringy corrections to the Einstein-Hilbert lagrangian

$$\mathcal{L}_{\rm tree} \sim R + \alpha'^3 \zeta_3 R^4 + \alpha'^5 \zeta_5 (D^4 R^4 + D^2 R^5) + \cdots$$

Amplitudes and higher-derivative corrections

- Reverse engineer the higher-derivative string effective action from scattering amplitudes
- Use string prescription to compute amplitudes and then write an action which reproduces them

$$S = \int d^{10}x\, e^{-2\phi}(R + R^4 + \cdots) + R^4 + e^{2\phi} D^4 R^4 + e^{4\phi} D^6 R^4 + \cdots$$

- $e^{-2\phi} R^4$: 4-point tree-level amplitude (Gross, Witten '86)
- R^4: 4-point one-loop amplitude (Green, Schwarz)
- $e^{2\phi} D^4 R^4$: 4-point 2-loop amplitude (D'Hoker, Phong; Berkovits '05)
- $e^{4\phi} D^6 R^4$: 4-point 3-loop amplitude (CM, H.Gomez '13)

S-duality and higher-derivative corrections

- For type IIB, use S-duality to guess interactions (Green, Gutperle, Vanhove et al.)

$$S = \int d^{10}x\sqrt{g}\big[e^{-1/2\phi}\zeta_3 E_{3/2}R^4 + e^{1/2\phi}\zeta_5 E_{5/2}D^4R^4 + e^{\phi}\mathcal{E}D^6R^4 +$$

- Coefficients given by modular forms (Eisenstein series etc): non-renormalization theorems, relative coefficients for interactions among different loop orders

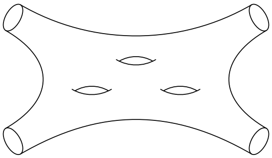
$$2\zeta_3 E_{3/2} = 2\zeta_3 e^{-3/2\phi} + \frac{2\pi^2}{3}e^{1/2\phi} + \ldots$$

$$2\zeta_5 E_{5/2} = 2\zeta_5 e^{-5/2\phi} + \frac{4\pi^4}{135}e^{3/2\phi} + \ldots$$

$$\mathcal{E} = 4\zeta_3^2 e^{-3\phi} + 8\zeta_2\zeta_3 e^{-\phi} + \frac{48}{5}\zeta_2^2 e^{\phi} + \frac{16\zeta_4\pi^2}{189}e^{3\phi}$$

- Scattering amplitudes & S-duality arguments should agree

The 3-loop amplitude



Using the prescription

$$\mathcal{A}_3 = \kappa^4 e^{4\lambda} \int_{\mathcal{M}_3} \prod_{j=1}^{6} d^2\tau_j \int_{\Sigma_4} \left| \langle \mathcal{N}(b,\mu_j)\, U^1(z_1)\dots U^4(z_4)\rangle \right|^2$$

and several tricks to simplify calculations one gets (CM, Gomez 2013)

$$\mathcal{A}_3 = (2\pi)^{10}\delta^{(10)}(k)\,\kappa^4 e^{4\lambda}\,\frac{\pi\,\zeta_6}{3^3}\left(\frac{\alpha'}{2}\right)^6 (s_{12}^3 + s_{13}^3 + s_{14}^3)\, K\overline{K}$$

which agrees with the S-duality prediction of Green and Vanhove from 2005

$$S^{\alpha'^6} = C_3 \int d^{10}x\sqrt{-g}\, D^6\mathcal{R}^4 (4\zeta_3^2 e^{-2\phi} + 8\zeta_2\zeta_3 + \frac{48}{5}\zeta_2^2 e^{2\phi} + \frac{8}{9}\zeta_6 e^{4\phi})$$

Conclusions

- Computing string scattering amplitudes is important for many reasons
- Being able to compute them requires a mindset of actively trying to simplify old formulas as well as looking at the problems from new perspectives. There are no guidelines for what can and cannot be done
- The pure spinor formalism provides a great starting tool to do such computations
- With the computations come a lot of new identities, patterns, and connections with the mathematical literature
- I have just sketched a small subset of recent developments in this area
- Many things left to do and discover!