Non-Relativistic Expansion of General Relativity

Jelle Hartong

University of Edinburgh, School of Mathematics

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In collaboration with:

Niels Obers (Nordita & NBI) and Dennis Hansen (ETH)

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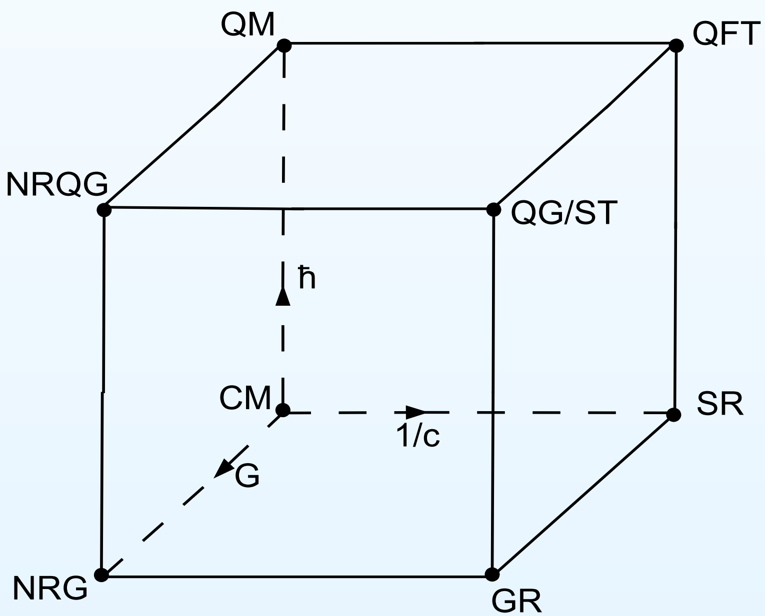
Motivation: Classical GR

- Off shell and covariant approximation of GR.
- Which effects are really relativistic?
- Action for Newtonian gravity.
- Post-Newtonian expansion: weak field and non-relativistic. But there is no need to take a weak field limit. What is strong non-relativistic gravity?
- Universal method to define non-relativistic approximations of any relativistic field theory.

Motivation: Quantum Gravity

- Bronstein cube



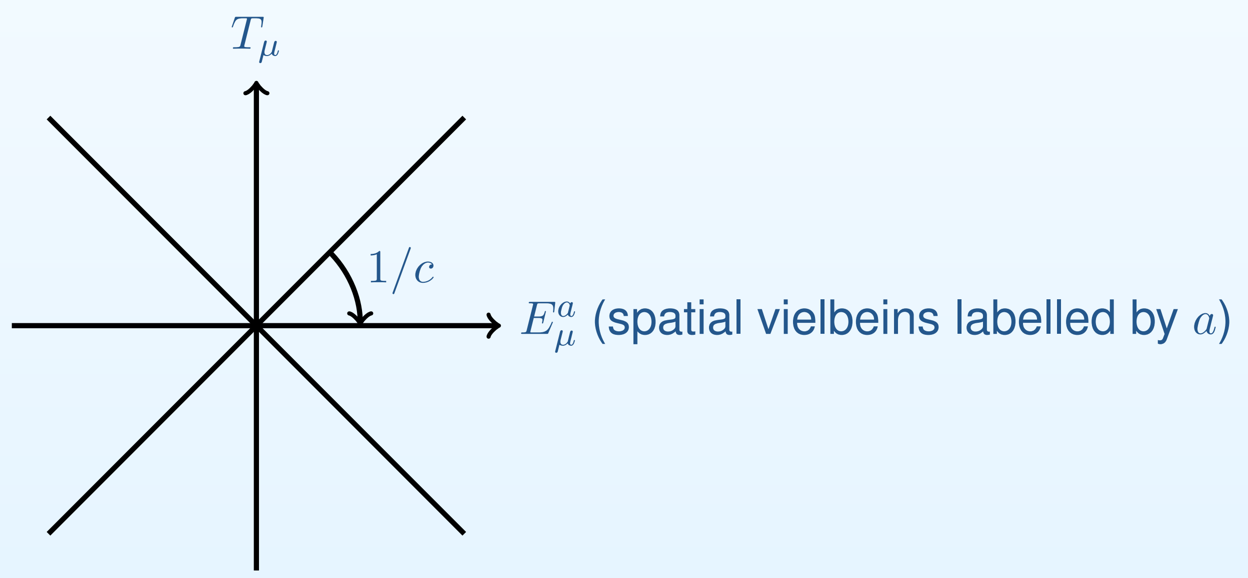
- Is non-relativistic quantum gravity a well-defined limit of string theory/holography?
- Why would NR quantum gravity require a UV completion?
- What does it tell us about holography if it is not restricted to relativistic gravity theories like GR?

Outline

- How to expand GR in $1/c$: non-relativistic geometry and gauge symmetries and algebras
- Solutions: weak and strong limits of Schwarzschild and AdS/dS spacetime
- $1/c$ expansions of Lagrangians
- Backreaction and matter couplings: Schrödinger–Newton equation
- Comments on holography and string theory

c-dependence of GR

- A convenient way to make the c-dependence of GR manifest is to write $g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}$ and $g^{\mu\nu} = -\frac{1}{c^2} T^\mu T^\nu + \Pi^{\mu\nu}$.
- Signature of $\Pi_{\mu\nu}$ is $(0, 1, \ldots, 1)$.
- Light cones in tangent space have slope $1/c$:



- Write $1/c = \epsilon/\hat{c}$ where $\hat{c}$ is the speed of light. The expansion is then in the dimensionless quantity ϵ. We set $\hat{c} = 1$.

c-dependence of GR

- Goal: write the Einstein–Hilbert action in terms of T_μ and $\Pi_{\mu\nu}$.
- Requires a new choice of connection $C^\rho_{\mu\nu}$ called the 'pre-non-relativistic' connection defined as

$$C^\rho_{\mu\nu} = -T^\rho \partial_\mu T_\nu + \frac{1}{2}\Pi^{\rho\sigma}\left(\partial_\mu \Pi_{\nu\sigma} + \partial_\nu \Pi_{\mu\sigma} - \partial_\sigma \Pi_{\mu\nu}\right)$$

- This connection has torsion proportional to $\partial_\mu T_\nu - \partial_\nu T_\mu$ and satisfies: $\overset{(C)}{\nabla}_\mu T_\nu = 0 = \overset{(C)}{\nabla}_\mu \Pi^{\nu\rho}$
- In terms of T_μ, $\Pi_{\mu\nu}$ the EH Lagrangian is [Hansen, JH, Obers, 2019]

$$\begin{aligned}\mathcal{L}_{\text{EH}} &= \frac{c^6}{16\pi G} E\left[\frac{1}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}\left(\partial_\mu T_\rho - \partial_\rho T_\mu\right)\left(\partial_\nu T_\sigma - \partial_\sigma T_\nu\right) + \frac{1}{c^2}\Pi^{\mu\nu}\overset{(C)}{R}_{\mu\nu}\right.\\ &\left. + \frac{1}{4c^4}\Pi^{\mu\nu}\Pi^{\rho\sigma}\left(\mathcal{L}_T \Pi_{\mu\rho}\mathcal{L}_T \Pi_{\nu\sigma} - \mathcal{L}_T \Pi_{\mu\nu}\mathcal{L}_T \Pi_{\rho\sigma}\right)\right]\end{aligned}$$

$1/c$ expansion

- So far we just reformulated GR in different variables. We will now assume that we can Taylor expand T_μ and $\Pi_{\mu\nu}$ in $1/c$:

$$T_\mu = \tau_\mu + \frac{1}{c^2} m_\mu + \frac{1}{c^4} B_\mu + \mathcal{O}(c^{-6}) \,, \qquad \Pi_{\mu\nu} = h_{\mu\nu} + \frac{1}{c^2}\Phi_{\mu\nu} + \mathcal{O}(c^{-4})$$

- This is what leads to the covariant $1/c$ expansion.
- Note here only even powers. For odd powers see [Ergen, Hamamci, Van den Bleeken, 2020].
- This leads to the metric expansion:

$$g_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + h_{\mu\nu} - 2\tau_{(\mu} m_{\nu)} + c^{-2}\left(\Phi_{\mu\nu} - m_\mu m_\nu - 2\tau_{(\mu} B_{\nu)}\right) + \mathcal{O}(c^{-4})$$

- $1/c$ expansion of the metric was pioneered by [Dautcourt, 1990/97] and generalised in [Van den Bleeken, 2017].

Weak NR limit of Schwarzschild

- Schwarzschild line element with factors of c reinstated:

$$ds^2 = -c^2\left(1 - \frac{2Gm}{c^2 r}\right)dt^2 + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega_{S^2}$$

- Weak limit: consider m independent of c^2.

$$\begin{aligned} \tau_\mu dx^\mu &= dt\,, \qquad h_{\mu\nu}dx^\mu dx^\nu = dr^2 + r^2 d\Omega_{S^2} \\ m_\mu dx^\mu &= -\frac{Gm}{r}dt\,, \qquad \Phi_{\mu\nu}dx^\mu dx^\nu = \frac{2Gm}{r}dr^2 \end{aligned}$$

- Point mass in flat space with Newtonian potential $\Phi = -\frac{Gm}{r}$.
- Absolute time t: τ is exact.

Strong NR limit of Schwarzschild

- Strong limit: $m = c^2 M$; M independent of c^2 [Van den Bleeken, 2017].

$$\tau_\mu dx^\mu = \sqrt{1 - \frac{2GM}{r}} dt \,, \qquad h_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_{S^2}$$

$$m_\mu dx^\mu = 0 = \Phi_{\mu\nu} dx^\mu dx^\nu$$

- This strong gravity expansion of the Schwarzschild metric is not captured by Newtonian gravity, but is still described as a Newton–Cartan geometry.

- This provides us with a different approximation of GR as compared to the post-Newtonian expansion.

- τ is no longer exact but $\tau \wedge d\tau = 0$ (hypersurface orthogonality). Strong limit captures gravitational time dilation: clocks tick slower/faster depending on position on a constant time slice.

Gauge structure

- Consider Einstein–Cartan formalism.
- Poincaré algebra generators: H, P_a, B_a, J_{ab} ($\mathfrak{so}(d)$)

$$[H\,, B_a] = P_a\,, \qquad [P_a\,, B_b] = \frac{1}{c^2} H\delta_{ab}\,, \qquad [B_a\,, B_b] = -\frac{1}{c^2} J_{ab}$$

- Cartan connection: $\mathcal{A}_\mu = HT_\mu + P_a E^a_\mu + B_a {\omega_\mu}^a + \frac{1}{2} J_{ab} {\omega_\mu}^{ab}$
  $({\omega_\mu}^a\,, {\omega_\mu}^{ab})$ make up the spin connection.
- Transformation: $\delta\mathcal{A}_\mu = \mathcal{L}_\Xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu, \Sigma]$
  $\Sigma = B_a\lambda^a + \frac{1}{2} J_{ab}\lambda^{ab}$
- No GR torsion: $\mathcal{F}_{\mu\nu}|_H = 0 = \mathcal{F}_{\mu\nu}|_{P_a}$. Solved by writing spin connection in terms of vielbeins (and derivatives).
- Expanding vielbeins T_μ and E^a_μ in $1/c^2$ leads to an algebra expansion where the new generators are $T^{(m)} = T \otimes c^{-2m}$

Algebra expansion

- This leads to the infinite dimensional algebra:

$$\left[H^{(m)}, B_a^{(n)}\right] = P_a^{(m+n)}, \quad \left[P_a^{(m)}, B_b^{(n)}\right] = \delta_{ab} H^{(m+n+1)}$$

$$\left[B_a^{(m)}, B_b^{(n)}\right] = -J_{ab}^{(m+n+1)}$$

  n counts the order in the c^{-2} expansion.

- We can quotient this algebra by setting to zero all generators with level $n > L$ for some L [Khasanov, Kuperstein, 2011].

- At level $n = 0$ the algebra is isomorphic to the Galilean algebra which is the Inönü–Wigner contraction of the Poincaré algebra. We will consider the algebra up to level $n = 1$, i.e. by truncating all level two and higher generators.

Level one algebra

- $H \equiv H^{(0)}, P_a \equiv P_a^{(0)}, G_a \equiv B_a^{(0)}, J_{ab} \equiv J_{ab}^{(0)}$, ($G_a$ Galilean boost) and $N \equiv H^{(1)}$, $T_a \equiv P_a^{(1)}$, $B_a \equiv B_a^{(1)}$, $S_{ab} \equiv J_{ab}^{(1)}$.

$$[H\,,G_a] = P_a\,, \qquad [P_a\,,G_b] = N\delta_{ab}$$
$$[N\,,G_a] = T_a\,, \qquad [H\,,B_a] = T_a\,, \qquad [G_a\,,G_b] = -S_{ab}$$
$$[S_{ab}\,,P_c] = \delta_{ac}T_b - \delta_{bc}T_a\,, \qquad [S_{ab}\,,G_c] = \delta_{ac}B_b - \delta_{bc}B_a$$

  Left out commutators with J_{ab} [Hansen, JH, Obers, 2018/19].
- Modding out T_a, B_a and S_{ab} gives Bargmann algebra
- Only modding out T_a, B_a in 3D gives extended Bargmann used in Chern–Simons theories [Papageorgiou, Schroers, 2009]; [Bergshoeff, Rosseel, 2016]; [JH, Lei, Obers, 2016] with $S_{ab} = S\epsilon_{ab}$.
- Cartan formalism: T_a related to $\Phi_{\mu\nu}$. Quotienting to Bargmann is only possible when $\Phi_{\mu\nu}$ decouples ($d\tau = 0$).

Geometry

- Consider again the metric expansion:

$$g_{\mu\nu} = -c^2\tau_\mu\tau_\nu + h_{\mu\nu} - 2\tau_{(\mu}m_{\nu)} + c^{-2}\left(\Phi_{\mu\nu} - m_\mu m_\nu - 2\tau_{(\mu}B_{\nu)}\right) + \mathcal{O}(c^{-4})$$

- Inverse metric expands as: $g^{\mu\nu} = h^{\mu\nu} + \mathcal{O}(c^{-2})$ with $\tau_\mu h^{\mu\nu} = 0$.
- We can view the $1/c$ expansion as an expansion around a geometry with degenerate metrics $\tau_\mu\tau_\nu$ and $h^{\mu\nu}$ where all the higher order fields m_μ and $\Phi_{\mu\nu}$ are like gauge connections.
- Expanding the generator of infinitesimal diffeos: $\Xi^\mu = \xi^\mu + \frac{1}{c^2}\zeta^\mu + O(c^{-4})$ leads to gauge transformations for the subleading fields m_μ and $\Phi_{\mu\nu}$ wrt subleading diffeos ζ^μ.
- $h_{\mu\nu}$ is not a 'metric' because it transforms under 'local Galilean boosts': shifts of $h_{\mu\nu}$ and m_μ that leave $h_{\mu\nu} - 2\tau_{(\mu}m_{\nu)}$ invariant.

NR limits of AdS/dS spacetimes

- global coord.: $ds^2 = -c^2 \cosh^2 \rho dt^2 + l^2 \left(d\rho^2 + \sinh^2 \rho d\Omega^2_{d-1}\right)$
- Corresponding type II NC geometry ($m_\mu = 0 = \Phi_{\mu\nu}$):

$$\tau_\mu dx^\mu = \cosh \rho dt \,, \qquad h_{\mu\nu} dx^\mu dx^\nu = l^2 \left(d\rho^2 + \sinh^2 \rho d\Omega^2_{d-1}\right)$$

- AdS(+)/dS(-): $ds^2 = -\left(c^2 \pm H^2 r^2\right) dt^2 + \frac{dr^2}{1 \pm \frac{H^2 r^2}{c^2}} + r^2 d\Omega^2_{d-1}$
  radius $l = \frac{c}{H}$ with H independent of c.
- The resulting NC geometry is the Newton–Hooke space:

$$\tau = dt \,, \qquad h_{\mu\nu} dx^\mu dx^\nu = d\vec{x} \cdot d\vec{x} \,, \qquad m = \pm \frac{1}{2} H^2 \vec{x}^2 dt$$

- Using NC gauge transformations this is also:

AdS : $\tau = dt' \,, \qquad h'_{\mu\nu} dx^\mu dx^\nu = \cos^2(Ht) d\vec{x}' \cdot d\vec{x}' \,, \qquad m' = 0$

dS : $\tau = dt' \,, \qquad h'_{\mu\nu} dx^\mu dx^\nu = e^{2Ht} d\vec{x}' \cdot d\vec{x}' \,, \qquad m' = 0$

Type I vs Type II Newton–Cartan geometry

- The NC geometry just described is called type II.
- Type I is obtained by removing $\Phi_{\mu\nu}$ and by changing the gauge transformation of m_μ to $\delta m_\mu = \partial_\mu \Lambda$. This is what is ordinarily called *the* NC geometry and it can be obtained by applying the Cartan formalism to the Bargmann algebra [Andringa, Bergshoeff, de Roo, Panda, 2011].
- On shell Newtonian gravity [Trautman, 1963]:

$$\bar{R}_{\mu\nu} = 8\pi G \frac{d-2}{d-1} \rho \tau_\mu \tau_\nu \,, \qquad d\tau = 0$$

  we used a type I, NC metric compatible connection $\bar{\Gamma}^{\rho}_{\mu\nu}$.
- This equation has been known for a long time and is invariant under type I NC gauge transformations, so it seemed reasonable to look for a type I invariant action.

Type I vs Type II Newton–Cartan geometry

- The algebra expansion does not lead to the Bargmann algebra and so the off shell gauge structure is not type I.
- It is not consistent to set $d\tau = 0$ off shell.
- Type II becomes type I if and only if we are on shell and $d\tau = 0$ as then $\Phi_{\mu\nu}$ decouples and the gauge transformations for m_μ coincide.
- To find an action formalism for NR gravity we need to expand the EH Lagrangian.

Lagrangian expansions

- Expanding Lagrangians: $\mathcal{L}(c, \phi, \partial_\mu \phi)$ where $\phi = \phi_{(0)} + c^{-2}\phi_{(2)} + \cdots$
- Assuming the overall power of the Lagrangian is c^N we define $\tilde{\mathcal{L}}(\sigma) = c^{-N}\mathcal{L}(c, \phi, \partial_\mu \phi)$ where $\sigma = c^{-2}$
- Taylor expand $\tilde{\mathcal{L}}(\sigma)$ around $\sigma = 0$, i.e.

$$\tilde{\mathcal{L}}(\sigma) = \tilde{\mathcal{L}}(0) + \sigma \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \sigma}|_{\sigma=0} + \phi_{(2)} \left[\frac{\partial \tilde{\mathcal{L}}(0)}{\partial \phi_{(0)}} - \partial_\mu \left(\frac{\partial \tilde{\mathcal{L}}(0)}{\partial \partial_\mu \phi_{(0)}} \right) \right] \right) + \cdots$$

- The eom of the NLO field of the NLO Lagrangian is the eom of the LO field of the LO Lagrangian.

EH Lagrangian

$$\mathcal{L}_{\text{EH}} = \frac{c^6}{16\pi G} E \left[\frac{1}{4} \Pi^{\mu\nu} \Pi^{\rho\sigma} \left(\partial_\mu T_\rho - \partial_\rho T_\mu \right) \left(\partial_\nu T_\sigma - \partial_\sigma T_\nu \right) + \frac{1}{c^2} \Pi^{\mu\nu} \overset{(C)}{R}_{\mu\nu} + \frac{1}{4c^4} \Pi^{\mu\nu} \Pi^{\rho\sigma} \left(\mathcal{L}_T \Pi_{\mu\rho} \mathcal{L}_T \Pi_{\nu\sigma} - \mathcal{L}_T \Pi_{\mu\nu} \mathcal{L}_T \Pi_{\rho\sigma} \right) \right]$$

- Connection: $C^\rho_{\mu\nu} = -T^\rho \partial_\mu T_\nu + \frac{1}{2} \Pi^{\rho\sigma} \left(\partial_\mu \Pi_{\nu\sigma} + \partial_\nu \Pi_{\mu\sigma} - \partial_\sigma \Pi_{\mu\nu} \right)$
- Expanding: $\mathcal{L}_{\text{EH}} = \frac{c^6}{16\pi G} \left[\mathcal{L}_{\text{LO}} + \sigma \mathcal{L}_{\text{NLO}} + \sigma^2 \mathcal{L}_{\text{N}^2\text{LO}} + O(\sigma^3) \right]$

$$\mathcal{L}_{\text{LO}} = \frac{e}{4} h^{\mu\nu} h^{\rho\sigma} \left(\partial_\mu \tau_\rho - \partial_\rho \tau_\mu \right) \left(\partial_\nu \tau_\sigma - \partial_\sigma \tau_\nu \right)$$

$$\mathcal{L}_{\text{NLO}} = e h^{\mu\nu} \check{R}_{\mu\nu} + \frac{\delta \mathcal{L}_{\text{LO}}}{\delta \tau_\mu} m_\mu + \frac{\delta \mathcal{L}_{\text{LO}}}{\delta h_{\mu\nu}} \Phi_{\mu\nu}$$

where the connection is $\check{\Gamma}^\rho_{\mu\nu} = C^\rho_{\mu\nu}|_{\sigma=0}$.

Non-relativistic gravity

- For the eom of the N^{2}LO Lagrangian involving only NLO fields we can use $\tau\wedge d\tau=0$ off shell: $\mathcal{L}_{\mathrm{NRG}}=\mathcal{L}_{\mathsf{N^2LO}}|_{\tau\wedge d\tau=0}+\mathcal{L}_{\mathsf{LM}}=$

$$\frac{e}{16\pi G}\Big[h^{\mu\rho}h^{\nu\sigma}K_{\mu\nu}K_{\rho\sigma}-(h^{\mu\nu}K_{\mu\nu})^2-2m_\nu\left(h^{\mu\rho}h^{\nu\sigma}-h^{\mu\nu}h^{\rho\sigma}\right)\check{\nabla}_\mu K_{\rho\sigma}$$

$$+\Phi h^{\mu\nu}\check{R}_{\mu\nu}+\frac{1}{4}h^{\mu\rho}h^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}+\frac{1}{2}\zeta_{\rho\sigma}h^{\mu\rho}h^{\nu\sigma}(\partial_\mu\tau_\nu-\partial_\nu\tau_\mu)$$

$$-\Phi_{\rho\sigma}h^{\mu\rho}h^{\nu\sigma}\left(\check{R}_{\mu\nu}-\check{\nabla}_\mu a_\nu-a_\mu a_\nu-\frac{1}{2}h_{\mu\nu}h^{\kappa\lambda}\check{R}_{\kappa\lambda}+h_{\mu\nu}e^{-1}\partial_\kappa\left(eh^{\kappa\lambda}a_\lambda\right)\right)\Big]$$

[Hansen, JH, Obers, 2020]

- $K_{\mu\nu}$ is the extrinsic curvature, $F=dm-a\wedge m$ and a is essentially the derivative of the lapse function N in $\tau=NdT$.
- In 3D, if we force $d\tau=0$ with a LM then this becomes a Chern–Simons theory for the extended Bargmann algebra [Papageorgiou, Schroers, 2009]; [Bergshoeff, Rosseel, 2016]; [JH, Lei, Obers, 2016].

Point Particles

- The proper time particle Lagrangian is

$$\mathcal{L} = -mc\left(-g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}\right)^{1/2}$$

- Expand the metric and scalars $X^{\mu} = x^{\mu} + \frac{1}{c^2}y^{\mu} + O(c^{-4})$
- The action of a particle on type II TNC geometry

$$\mathcal{L} = -mc^2\tau_{\mu}\dot{x}^{\mu} + m\left(\left(\partial_{\nu}\tau_{\mu} - \partial_{\mu}\tau_{\nu}\right)\dot{x}^{\nu}y^{\mu} + \frac{1}{2}\frac{\bar{h}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{\tau_{\rho}\dot{x}^{\rho}}\right) + O(c^{-2})$$

- The y^{μ} EOM forces $d\tau = 0$. Coupling to NRG gives Newtonian gravity and geodesics obeying Newton's second law:

$$\ddot{x}^{\mu} + \bar{\Gamma}^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = 0\,, \qquad \tau_{\mu}\dot{x}^{\mu} = 1$$

Coupling to matter

- The NRG Lagrangian appears at order c^2 and so couples to the c^2 order of the expansions of the matter Lagrangians.
- If we add $\mathcal{L} = -m\int d\lambda \tau_\mu \dot{x}^\mu \delta(x - x(\lambda))$ (which is the c^2 and LO term in the expansion of the massive point particle Lagrangian) then we recover Newtonian gravity coupled to a point particle.
- The x^μ eom forces $d\tau = 0$ and in this case $\Phi_{\mu\nu}$ decouples on shell from the other NC fields.
- The EOM for the type I NC fields can be summarised as

$$\bar{R}_{\mu\nu} = 8\pi G \frac{d-2}{d-1}\rho\tau_\mu\tau_\nu\,, \qquad \rho = m\int d\lambda\,\frac{\delta(x - x(\lambda))}{e}\,, \qquad d\tau = 0$$

Point ‘particles’ for strong gravity

- Geodesics when $\tau \wedge d\tau = 0$. Different expansion:

$$X^\mu = x^\mu + \frac{1}{c} y^\mu + O(c^{-2}) \,, \qquad \tau_\mu \dot{x}^\mu = 0$$

- In this case the action is ($\tau_\mu y^\mu$ is a Lagrange multiplier)

$$S = \int d\lambda \mathcal{L} = -mc \int d\lambda \left[\left(\frac{d}{d\lambda} (\tau_\mu y^\mu) - \dot{x}^\nu a_\nu \tau_\mu y^\mu \right)^2 - h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{1/2}$$

- Fixing worldline reparametrisation symmetries, the EOM involving x^μ are, using $\tau = N dT$ (time function T):

$$\ddot{x}^\mu + \check{\Gamma}^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = \frac{1}{2} h^{\mu\sigma} \partial_\sigma N^{-2} \,, \qquad \frac{1}{2} h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{1}{2} N^{-2} = \mathcal{E}$$

- This is identical to what we know from GR and from it we can derive the GR contribution to perihelion precession.

Scalar Fields

- Free massive complex scalar field $\phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}$

$$\mathcal{L} = -\frac{1}{2c}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \varphi^2\left(g^{\mu\nu}\partial_\mu\theta\partial_\nu\theta + m^2c^2\right)\right]$$

- We expand the modulus and phase of ϕ as

$$\varphi = \varphi_{(0)}+c^{-2}\varphi_{(2)}+O(c^{-4})\,, \qquad \theta = c^2\left(\theta_{(0)} + c^{-2}\theta_{(2)} + O(c^{-4})\right)$$

- Expanding the Lagrangian we find that on shell we must have $\partial_\mu\theta_{(0)} = -m\tau_\mu$ so that $d\tau = 0$.

- The Schrödinger field $\psi = \sqrt{m}\varphi_{(0)}e^{i\theta_{(2)}}$ propagates on a type I NC background sourcing NC gravity with mass $\rho = m\psi\psi^*$ leading to the Schrödinger–Newton equation:

$$i\partial_t\psi(t,x) = \left(-\frac{1}{2m}\vec{\partial}^2 - m^2G\int d^3x'\,\frac{\psi(t,x')\psi^\star(t,x')}{|\vec{x}-\vec{x}'|}\right)\psi(t,x)$$

Structure of the EOM

- For $\tau \wedge d\tau = 0$ we obtain something more general: strong NRG ($\Phi_{\mu\nu}$ does not decouple on shell).
- Natural gauge choice:

$$\tau = N dt\,, \qquad h_{\mu\nu} dx^\mu dx^\nu = \gamma_{ij} dx^i dx^j\,, \qquad m = \Phi dt - N^{-1} \gamma_{ij} N^i dx^j$$

with N the lapse function, γ_{ij} an invertible Riemannian metric, Φ Newton's potential and N^i the shift vector.

1). EOM I: spatial derivatives of N and γ_{ij}. Time dependence through integration constants.

2). EOM II: time dependence of N and γ_{ij} coupled to spatial derivatives of the NLO fields: Φ, N^i and Φ_{ij}.

- See also [Van den Bleeken, 2019].

Comments

- FLRW also solves the eom of NRG coupled to a fluid.
- The Tolman–Oppenheimer–Volkoff solution for a fluid star is a solution of NRG coupled to a fluid.
- It passes the three classical GR tests: gravitational redshift, perihelion and bending of light.
- For Maxwell there are two expansions that agree with the known on shell electric and magnetic limits.
- Kerr geometry and odd powers of $1/c$.
- What is a controlled way of doing this at any order in $1/c$: maybe better in a first order formalism?
- Hamiltonian analysis and asymptotic symmetries.
- Applications to astrophysics?

Strings

- Can we construct a string theory whose worldsheet beta functions correspond to NR gravity?
- NR strings go back to [Gomis, Ooguri, 2000] and [Danielsson, A. Guijosa and M. Kruczenski, 2000] for flat target spaces.
- These strings are obtained from $1/c^2$ expansion of the form $g_{\mu\nu} = c^2\left(-T^0_\mu T^0_\nu + T^1_\mu T^1_\nu\right) + \Pi^\perp_{\mu\nu}$ in non-trivial Kalb–Ramond backgrounds. The worldsheet is still a CFT.
- In general curved backgrounds [Bergshoeff, Gomis, Yan, 2018]; [Harmark, JH, Menculini, Obers, Oling, 2019]. These target spaces are described by type I NC geometry with an additional circle that strings must wind.
- Beta functions: [Yan, Yu, 2019] and [Gallegos, Gürsoy, Zinnato, 2019]
- So far no string theory is known for type II NC backgrounds.

Holography

- Limits of Chern–Simons theories [JH, Lei, Obers, Oling, 2017]
- Near BPS limits of strings on AdS$_5 \times S^5$ and spin-matrix limits of $\mathcal{N} = 4$ SYM [Harmark, Orselli, 2014].
- Duality between NR strings and quantum mechanical limits of $\mathcal{N} = 4$ SYM [Harmark, JH, Menculini, Obers, Yan, 2018].
- The worldsheet theories are non-relativistic, e.g.

$$\mathcal{L}_{LL} = \frac{Q}{4\pi}\left[\cos\theta\dot{\phi} - \frac{1}{4}\left(\theta'^2 + \sin^2\theta\phi'^2\right)\right]$$

  Landau–Lifshitz sigma model for near BPS limit in $SU(2)$ sector. Spin chain momentum is zero $\int_0^{2\pi} d\sigma \sin\theta\phi' = 0$.
- String moves on a NC-like target space $\mathbb{R} \times S^2$ and is pure winding along an additional S^1.

Outlook

- Is there a well-defined corner called NRQG? Does it have a string theory description and if so why would it need one? Can it be holographic?
- Can we use the $1/c$ expansion to learn more about quantum mechanics in gravitational backgrounds? [Pikovski, Zych, Costa, Brukner, 2015]
- Can we systematically compute post-Newtonian corrections to gravity coupled to generic matter systems using the $1/c$ expansion?
- Is strong NR gravity a useful starting point for certain astrophysical problems? Can we study the 2-body problem in that regime?
- What can we say about asymptotic symmetries in NR gravity and first law type relations?