(Slowly-varying) Attractors and Bifurcations in Multi-field Inflation

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The cosmological model

What do we know so far? At cosmological scales the Universe is:

- **homogeneous**
- **isotropic**
- mostly composed of **unknown** ingredients

The first two conditions fix the metric to **FLRW**

$$ds^2 = -dt^2 + \frac{a(t)^2}{1 - K_3 r^2}\left(dr^2 + r^2 d\Omega^2\right) \qquad (1)$$

Further observations show compatibility with **zero** spatial curvature.

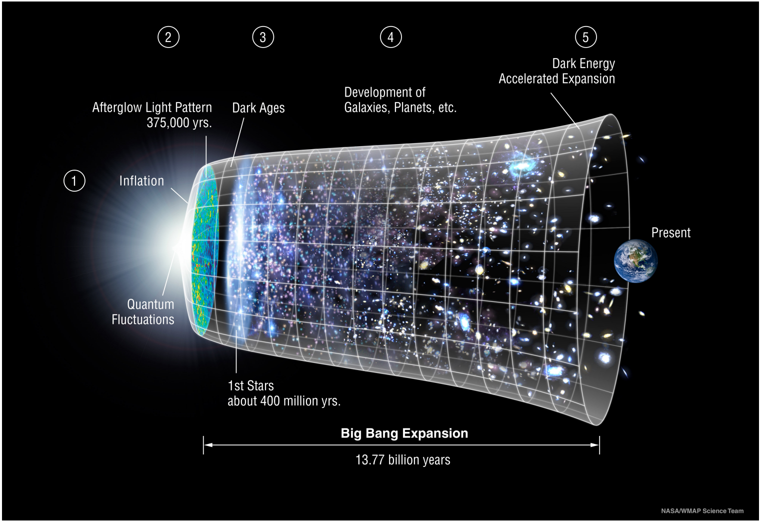


Figure: *The standard cosmological model (NASA WMAP Science Team)*

Inflation: a period with $\ddot{a} > 0$. Was first introduced to tackle some "problems"

- monopoles from GUT [Guth]
- particle horizon (isotropy) [Kazanas] and flatness [Guth]
- other more exotic relics such as domain walls

Later it was realized that it predicted small anisotropies $\Rightarrow$ mechanism for structure formation [Mukhanov]

However, the likelihood of initial conditions, parameter values,..., requires knoweledge of probability densities [Wald, Sloan,..]

- **Early universe cosmology**: Assume

$$g = g_{(0)} + \epsilon g_{(1)} + \cdots \tag{2}$$

$$\phi^J = \phi^J{}_{(0)} + \epsilon \phi^J{}_{(1)} + \cdots \tag{3}$$

and then solve

$$G_{(0)} = T_{(0)}, \qquad \Box \phi^J{}_{(0)} + V^{,J}{}_{(0)} = 0 \tag{4}$$

$$G_{(1)} = T_{(1)}, \qquad \Box \phi^J{}_{(1)} + V^{,J}{}_{(1)} = 0 \tag{5}$$

$$\cdots$$

- First order quantities are connected to observables

Split the metric into scalar, vector and tensor degrees of freedom. After canonical quantization

- **scalar** degrees $R = R(\phi_{(1)}, g_{(1)})$ correspond to

$$\langle R_{\boldsymbol{k}} R_{\boldsymbol{k}'} \rangle = (2\pi)^3 \delta(\boldsymbol{k} + \boldsymbol{k}') P_R(\boldsymbol{k}) \tag{6}$$

- **tensor** $h = h(g_{(1)})$ degrees correspond to

$$\langle h_{\boldsymbol{k}} h_{\boldsymbol{k}'} \rangle = (2\pi)^3 \delta(\boldsymbol{k} + \boldsymbol{k}') P_h(\boldsymbol{k}) \tag{7}$$

- **vector** modes decay for scalar fields

For the rest focus only on background quantities.

Single-field inflation

Assume simple Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V\right) \tag{8}$$

Substitute $\boldsymbol{g} \to \boldsymbol{g}_{FRW}$, $K_3 = 0$ and $\phi(t, x^i) \to \phi(t) \Rightarrow$ minisuperspace Lagrangian

$$L_{ms} = a^3\left[\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2\right] + a^3\left(\frac{1}{2}\dot{\phi}^2 - V\right) \tag{9}$$

EOM (with $H \equiv \dot{a}/a$)

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V\,, \qquad \dot{H} = -\frac{1}{2}\dot{\phi}^2 \tag{10}$$

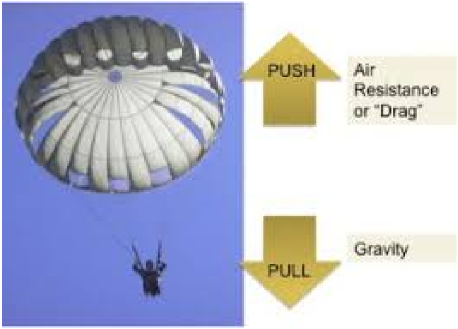
$$\underbrace{\ddot{\phi}}_{\text{acceleration}} + \underbrace{3H\dot{\phi}}_{\text{Hubble friction}} + \underbrace{V_{,\phi}}_{\text{potential gradient}} = 0 \tag{11}$$

- Similarities with parachute fall

$$\ddot{x} + b\dot{x} + mg = 0 \tag{12}$$

terminal velocity: $\ddot{x} = 0 \Leftrightarrow \dot{x} = -mg/b$

- For inflation it corresponds to the slow-roll velocity. Hubble friction balances gradient $\Rightarrow$ **slowly varying** motion



Phase space plot

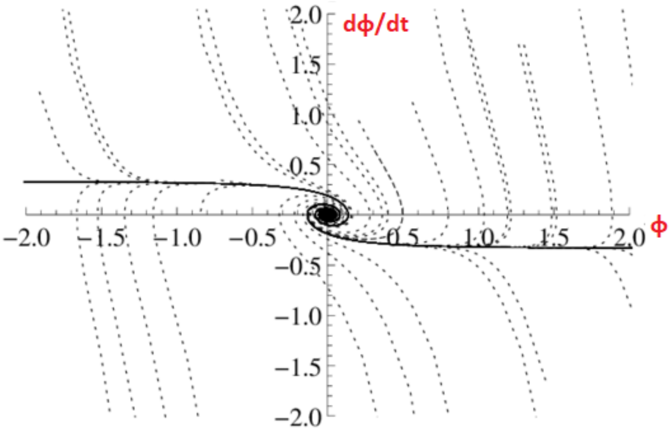


Figure: Numerical solution for a massive quadratic field (*arXiv:1309.2611*)

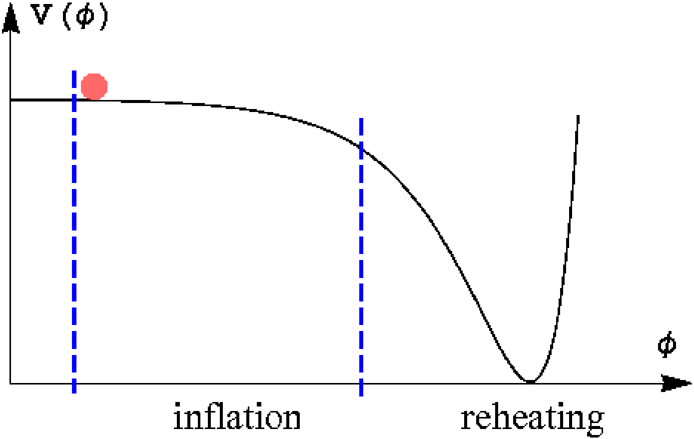


Figure: *Starobinsky model (most favorable)*

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Klein-Gordon is

1. non-linear
2. second order in time

$\Rightarrow$ no general analytical solutions. However, **autonomous** so can apply reduction of order

- Transform as first order system

$$y = \dot{\phi} \tag{13}$$

$$\dot{y} = -3Hy - V_{,\phi} \tag{14}$$

$$\dot{H} = -\frac{1}{2}y^2 \tag{15}$$

- Time reparameterization $t \to \phi$ ($\dot{\phi} \neq 0$): $d/dt \to y d/d\phi$

$$yy_{,\phi} = -3Hy - V_{,\phi} \tag{16}$$

$$H_{,\phi} = -\frac{1}{2}y \tag{17}$$

with the Friedman constraint $3H^2 = \frac{1}{2}y^2 + V$

- Klein-Gordon first order but **non-autonomous** $\Rightarrow$ not an improvement
- Solve the **inverse problem**: given a solution y_{sol} which V satisfies the ODE

- Eliminating y gives V for some H

$$V = 3H^2 - 2H_{,\phi}^2 \tag{18}$$

Known as the superpotential method [Salopek, Bond]

- H can be eliminated by defining $u = y/H$. Used in dark energy models and dynamical systems

- Important quantity $\epsilon \equiv -\dot{H}/H^2 = 3K/(K+V)$: $\ddot{a} > 0 \Leftrightarrow \epsilon < 1$. Study evolution of this variable
- Time and field redefinition

$$t \to N = \ln a \,, \qquad u = \frac{\dot{\phi}}{H} = \phi' \tag{19}$$

and Klein-Gordon becomes

$$\phi' = u \tag{20}$$

$$\underbrace{u' - \frac{1}{2}u^3}_{\text{acceleration } \ddot{\phi}} + \underbrace{3u}_{\text{Hubble friction } 3H\dot{\phi}} + \underbrace{\left(3 - \frac{1}{2}u^2\right)(\ln V)_{,\phi}}_{\text{gradient } V_{,\phi}} = 0 \tag{21}$$

- What do we gain? Note that $\epsilon = 1/2u^2$ and so

$$\phi' = s\sqrt{2\epsilon} \tag{22}$$

$$\epsilon' = -(3-\epsilon)\left[2\epsilon + s\sqrt{2\epsilon}(\ln V)_{,\phi}\right] \tag{23}$$

with $s = \text{sign}(\dot{\phi})$

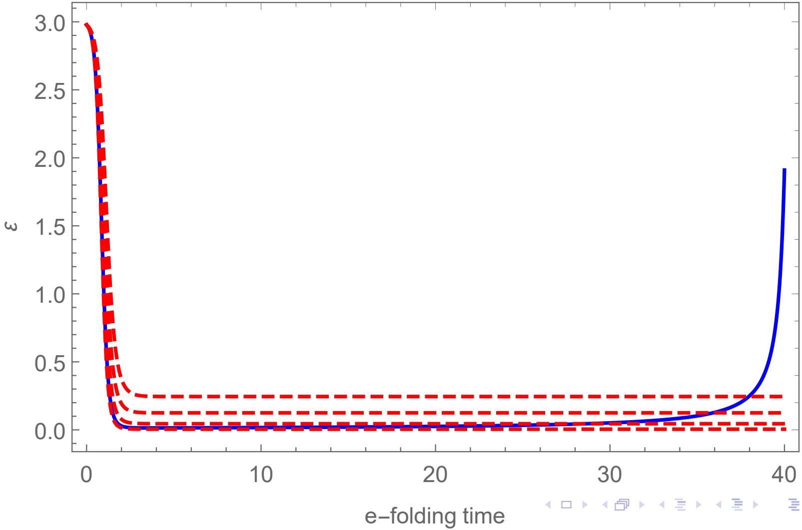
- If $(\ln V)_{,\phi}$ =const then 3 **critical points**:
  1. $\epsilon = \frac{1}{2}(\ln V)_{,\phi} \equiv \epsilon_V$ (scaling solution)
  2. $\epsilon = 3$ (kinetic domination)
- scaling stable for $(\ln V)_{,\phi} < 6 \Leftrightarrow \epsilon_V < 3$
- kinetic stable for $(\ln V)_{,\phi} > 6 \Leftrightarrow \epsilon_V > 3$
- *Side note*: **separable** ODE $\Rightarrow$ general analytical solution

- define $p = (\ln V)_{,\phi}$. For $p_1 < p_2 \Rightarrow \epsilon_1' < \epsilon_2' \Rightarrow \epsilon_1 < \epsilon_2$. For **field-dependent** p appropriate exponentials bound evolution
- rate of growth for inflation can be estimated using exponentials. Specifically for slowly varying p

$$p' \ll 1 \Leftrightarrow \eta_V - \epsilon_V \ll 1 \tag{24}$$

  i.e. the slow-roll conditions are equivalent to an exponential with a slowly-varying exponent. The slow-roll solution (late-time) is close to a scaling solution, which slowly varies with time
- Slow-roll models **imitate** solutions which have proper attractors $\Rightarrow$ quasi-attractors

Phase space plot revisited



Key points so far

- Exact solutions can be constructed via reduction of order
- Formulated in dynamical systems terms $p = (\ln V)_{,\phi}$ controls evolution of ϵ
- With differential inequalities an estimate for growth of ϵ can be found
- Slow-roll models are small deformations of scaling solutions

- Multiple scalar fields with minimal derivative couplings. Minisuperspace matter Lagrangian

$$\mathcal{L}_m = a^3 \left(\frac{1}{2} \mathcal{G}_{IJ} \dot{\phi}^I \dot{\phi}^J - V \right) \tag{25}$$

where $\mathcal{G}_{IJ}$ behaves as a **metric**

- **Non-minimal** models with $L_{\mathrm{gr}} = \sqrt{-g} f(\phi) R$ can be brought in previous form via a conformal transformation $g \to \Omega(\phi) g$ Jordan frame $\to$ Einstein frame [Kaiser, Sfakianakis]

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- EOM

$$D_t\dot{\phi}^J + 3H\dot{\phi}^J + V^{,J} = 0 \tag{26}$$

$$\dot{H} = -\frac{1}{2}\mathcal{G}_{IJ}\dot{\phi}^I\dot{\phi}^J \tag{27}$$

$$3H^2 = \frac{1}{2}\mathcal{G}_{IJ}\dot{\phi}^I\dot{\phi}^J + V \tag{28}$$

where D_t is the covariant time derivative associated with $\mathcal{G}$

- Solutions with $\ddot{\phi}^I \approx 0$? Based on previous discussion can look for scaling two-field solutions

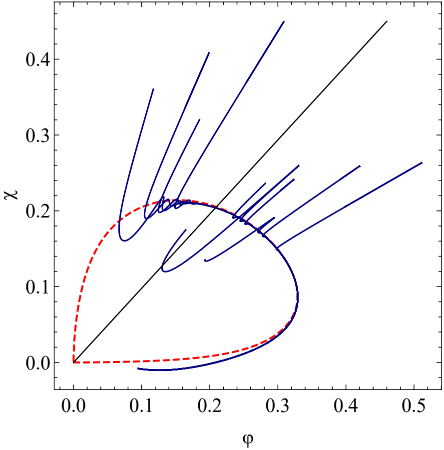


Figure: *The one-parameter "attractor" solution of* **angular** *inflation* where $V = \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}m_\phi^2\phi^2$ and $\mathcal{G}_{IJ} = \frac{\alpha}{(1-\chi^2-\phi^2)^2}\delta_{IJ}$.

- If an **one-parameter** (approximate) solution exists then align coordinates such that $\dot{\chi} = 0$, while ϕ is evolving

$$\ddot{\phi} + \Gamma^{\phi}_{\phi\phi}\dot{\phi}^2 + \cdots + 3H\dot{\phi} + V^{,\phi} = 0 \quad (29)$$

$$\ddot{\chi} + \Gamma^{\chi}_{\phi\phi}\dot{\phi}^2 + \cdots + 3H\dot{\chi} + V^{,\chi} = 0 \quad (30)$$

- Solution requires $\ddot{\chi} = 0$ and $\Gamma^{\chi}_{\phi\phi}\dot{\phi}^2 + V^{,\chi} \equiv V^{,\chi}_{\mathrm{eff}} = 0$
- With more complicate argument: $\ddot{\phi} \approx 0 \Rightarrow D_t\dot{\phi} \approx 0$
- **Inflaton** is subject to vanishing covariant acceleration, while the "**heavy**" field is stabilized at a **critical point** of its **effective potential**

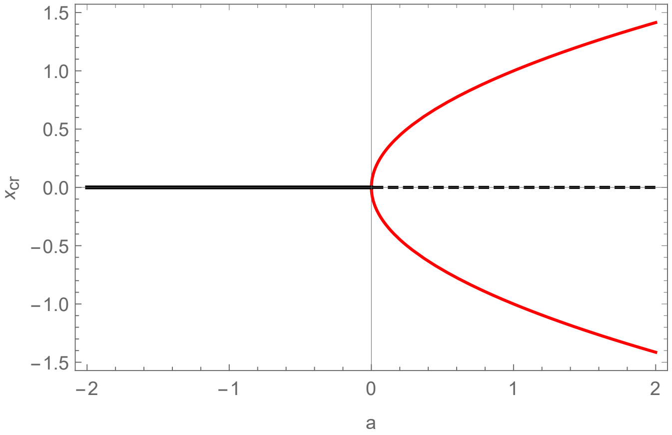
- Bifurcations: **alteration in stability** of critical points
- Prototypical example:

$$x' = -x(x^2 - a) \qquad \frac{\partial x'}{\partial x} = a - 3x^2 \tag{31}$$

1. $a < 0$ then $x = 0$ only critical point (stable)
2. $a > 0$ two more critical points at $x = \pm\sqrt{a}$ (stable), and $x = 0$ (unstable)

- Eq. (31): normal form of a **pitchfork bifurcation**

$$x' = -x(x^2 - a)$$



initial critical point becomes unstable. #stable − #unstable remains the same ⇒ 2 new stable CP

- If the effective gradient has more critical points then stability may depend on several model parameters (length of curvature, masses,..)
- An appropriate choice of parameters guarantees pitchfork bifurcations
- This was known as geometrical destabilization [Renaux-Petel, Turzinsky]; a geodesic solution becomes unstable and two others may appear. If not account properly can lead to wrong predictions

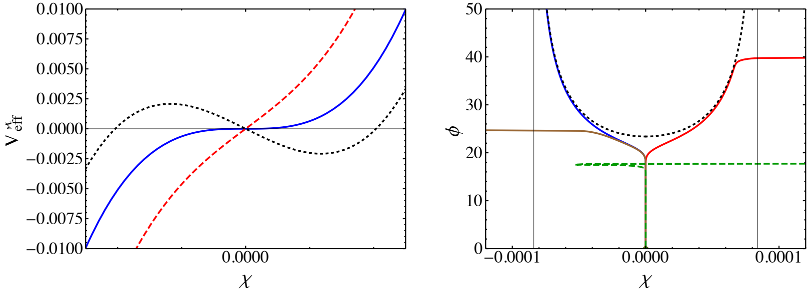


Figure: **Sidetracked** *model:* $ds^2 = d\chi^2 + \left(1 + \frac{\chi^2}{L^2}\right) d\phi^2$ *and* $V = \frac{1}{2} m_\chi^2 \chi^2 + \frac{1}{2} m_\phi^2 \phi^2$. *Left: Effective gradient. Right: Evolution on the* $\phi - \chi$ *plane*

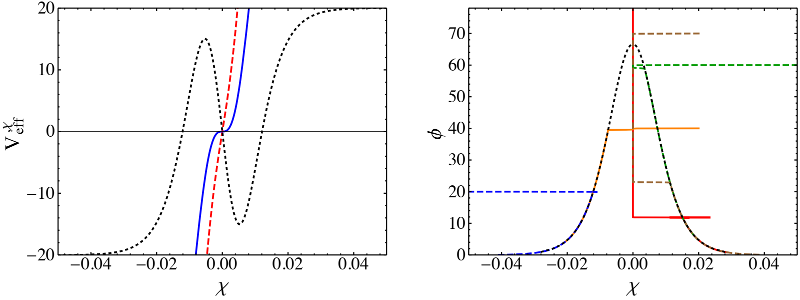


Figure: **Hyperinflation** *model:* $ds^2 = d\chi^2 + \cosh^2\left(\frac{\chi}{L}\right) d\phi^2$ *and* $V = \frac{1}{2}m^2\phi^2 + \frac{1}{2}m^2\chi^2\frac{\phi}{L}$. *Left: Effective gradient. Right: Evolution on the* $\phi - \chi$ *plane*

- Dynamical system and linearization:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \qquad \dot{\delta \boldsymbol{x}} = \boldsymbol{J} \cdot \delta \boldsymbol{x} \tag{32}$$

  Eigenvalues of $\boldsymbol{J}$ determine local behaviour around a solution.

- If $Re(\lambda_i) < 0 \Rightarrow$ system asymptotically stable. If one zero, special treatment

- Note that eigenvalues of $\dot{\delta \boldsymbol{y}} = \tilde{A}\delta \boldsymbol{y}$, where $\delta y_{(i)} = f_{(i)}\delta x_{(i)}$ provide no information about (32)

$$ds^2 = g^2(\phi)d\chi^2 + f^2(\chi)d\phi^2 \,, \tag{33}$$

that includes the commonly used of a metric with an isometry.
Linearizing Klein-Gordon

- $(V_{\mathrm{eff}}{}^{,\chi})_{,\chi} > 0$: defines a mass (M) which coincides with the effective mass of isocurvature perturbations on super-Hubble scales (μ_s^2) **only** when $g = 1$, that is for problems with **isometry**.
- $(3 - \epsilon) > -(\ln g)'$: defines a critical value for ϵ beyond which motion becomes unstable.

Thus, background stability is not always the same as stability of cosmological perturbations

- Solutions of Einstein's equations with a spacelike killing vector

$$ds^2 = dz^2 + e^{A(z)} ds^2_{n-1} \tag{34}$$

  If subspace Minkowski and $A(z) \to 0$ at the boundary $\Rightarrow$ AdS

- Domain walls $\Leftrightarrow$ cosmology [Skenderis et al.]
- (Approximate) solutions mentioned earlier will have a domain walls analogue. RG flow $\Leftrightarrow$ slow-roll parameter ϵ

- We presented exact solutions and dynamical systems analysis for the Klein-Gordon equation
- We demonstrated the resemblance between the slow-roll approximation and scaling solutions
- We presented two-field solutions for non-trivial field manifolds
- We proposed a unification scheme of different (viable) inflationary models based on their attractors and bifurcations. This can be extended to domain wall solutions

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