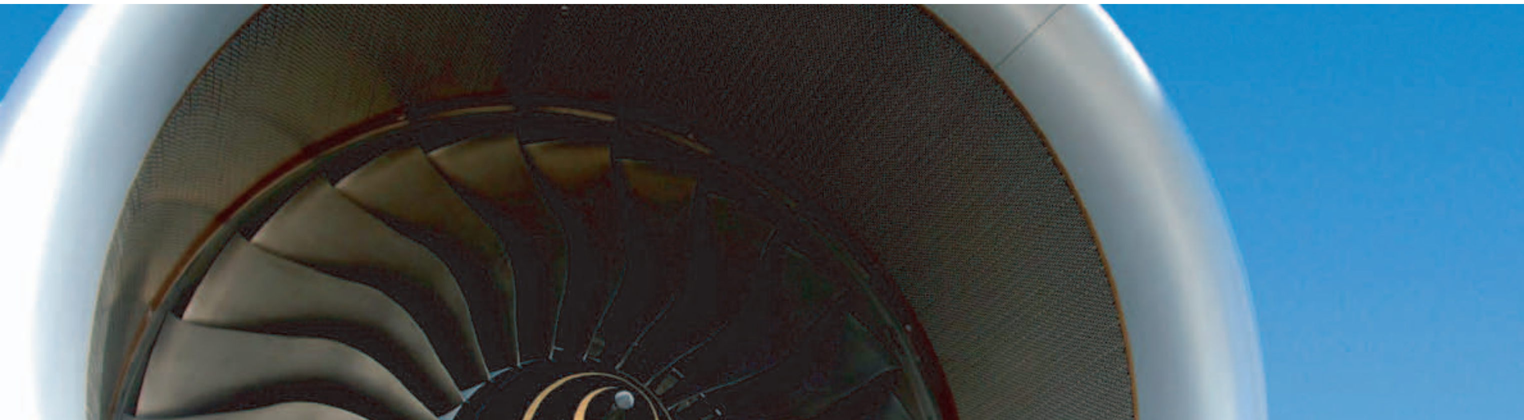

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Stochastic Reduced Basis Methods for PDEs on Random Domains

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Introduction

Geometric uncertainties are ubiquitous in the mathematical characterization of many engineering systems. Such uncertainties may arise due to a number of reasons, including measurement errors, lack of information, manufacturing variability, thermal effects or wear occurring in service. An efficient and reliable way to deal with such problems is to construct probabilistic models of geometrical uncertainties and describe the phenomena of interest by partial differential equations (PDEs) on random domains. However, solving stochastic PDEs on random domains is not as straightforward as compared to the standard case when only the PDE coefficients are uncertain. This motivates the development of a new computational framework to propagate the uncertainty in boundary topology through the governing equations to obtain response statistics.

SRBMs for tackling geometric uncertainty

A novel approach has been developed to tackle *deterministic/stochastic* PDEs defined on random domains which leverages existing *deterministic analysis* tools (e.g. finite element solvers). The approach can handle cases when the uncertain boundary is randomly parameterized or represented by a random field model. The central idea is to work with mesh-based representations of random geometries, where the connectivity information is deterministic but the coordinates of the mesh nodes are random. To ensure that the mesh coordinates vary smoothly as a function of the random geometry, the dependence of each coordinate on the random geometrical features is modelled by a PDE (e.g., the Laplace equation) supplemented with appropriate random Dirichlet boundary conditions.

This setting of the problem enables the element matrices arising from finite element discretization to be readily expressed as a function of the basic random variables characterizing the geometric variability. To create a tractable representation of the random coefficient matrices, we assume they are second-order stochastic processes by admitting Polynomial Chaos (PC) representations. The unknown coefficients in the PC expansion are nonintrusively computed using a multidimensional quadrature scheme which essentially involves repeatedly invoking a deterministic finite element formulation to generate samples of the coefficient matrices for different realizations of the uncertain boundary. Since automatically remeshing various realizations of the domain may lead to differences in the dimensions of the stiffness and force matrices, a mesh morphing/grid perturbation strategy is employed to ensure consistency in the quadrature scheme. The idea is to construct a mesh for the nominal domain of definition and subsequently morph this mesh to any realization of the random domain. In this work, a sequence of Laplace equations with appropriate boundary conditions are solved to achieve mesh morphing. Stochastic Reduced Basis Methods are then applied to the resulting linear random algebraic system of equations to calculate the response statistics.

In the present numerical framework, we adopt a non-intrusive approach which enables existing deterministic tools to be leveraged for stochastic analysis. In contrast to the current state-of-the-art methods, the proposed method neither needs the (possibly) cumbersome Jacobian calculations nor results in complicated SPDEs. In addition advances made in the field of mesh morphing can be readily leveraged.

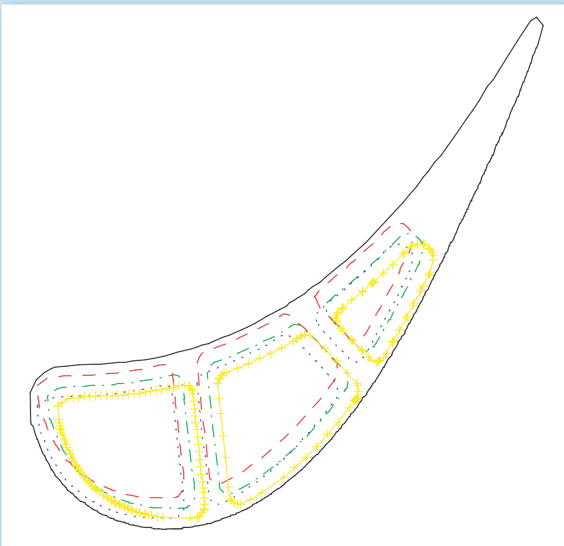

Fig.1 Turbine blade geometry with possible core positions.

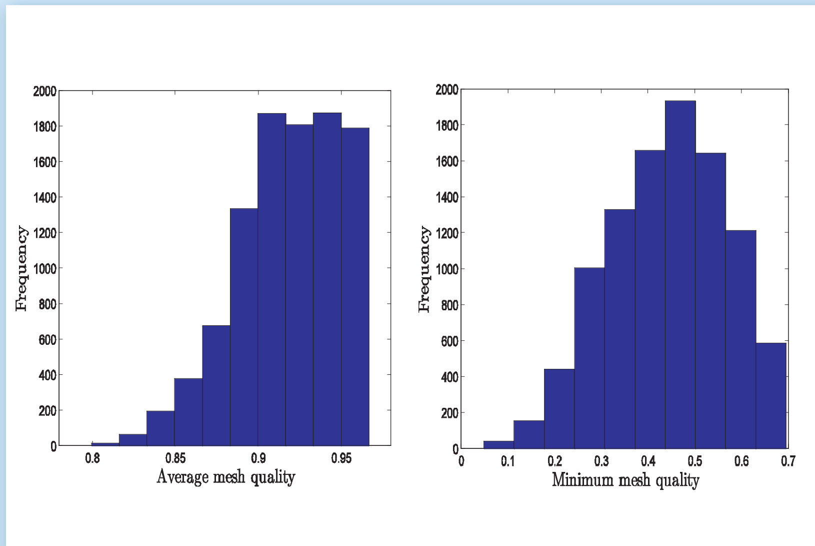

Fig.2 Histograms of average and minimum mesh qualities using a Laplace equation based mesh morphing strategy on the turbine blade.

Numerical Studies & Results

We now present a numerical study involving heat transfer in a core cooled gas turbine blade. In such blades relatively cold air is circulated through the cores resulting in cooling of the blades. However, as a result of manufacturing variability, the core locations may vary from blade to blade leading to an uncertainty problem. We present a simplistic 2D model of a blade where the statistical moments of the temperature profile are the subject of interest. A typical turbine blade profile with a few possible core positions is shown in **Fig. 1** (the cores are varied independently and uniformly along each spatial direction with the limits being ±4% of the nominal core location).

A benchmark Monte-Carlo simulation (MCS) with 10,000 samples has been performed to compute the mean and standard deviation of the temperature profile. **Fig.2** shows the average and minimum mesh quality metrics obtained by the Laplace equation based mesh morpher for an ensemble of random geometries. The spatial distribution of the mean and standard deviation of the temperature profile can be seen in **Fig.3**. **Fig.4** shows the convergence trends of the absolute error in mean and the standard deviation profiles of the temperature for the SRBM scheme (compared to the benchmark MCS) on the outer edge of the turbine blade. Finally the absolute error in the first two statistical moments of the temperature profile are mapped onto the nominal domain and shown in **Fig.5**.

It can be noted from these results that the present approach provides highly accurate results as the approximation order is increased. Further, the computational cost of the proposed approach is two orders of magnitude lower than standard MCS.

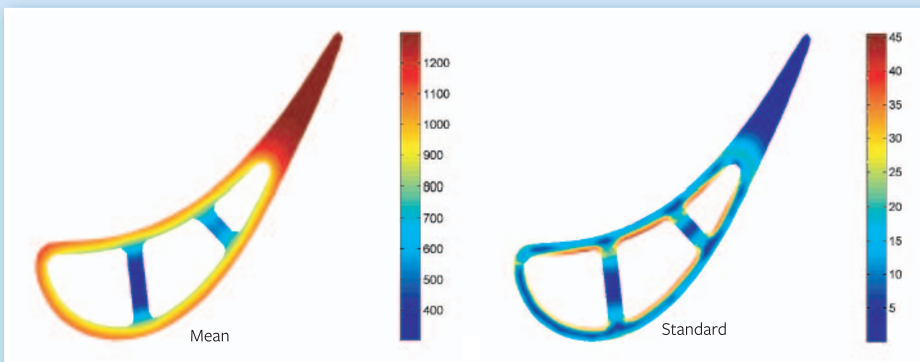

Fig.3 Mean and standard deviation of the temperature profile mapped onto the nominal geometry.

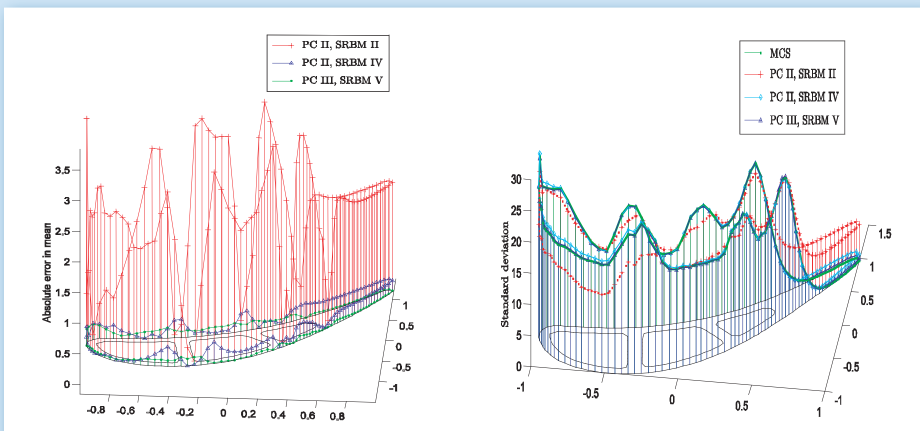

Fig.4 Absolute error in mean compared to MCS mean and the standard deviation profiles of the temperature on the outer edge of the blade.

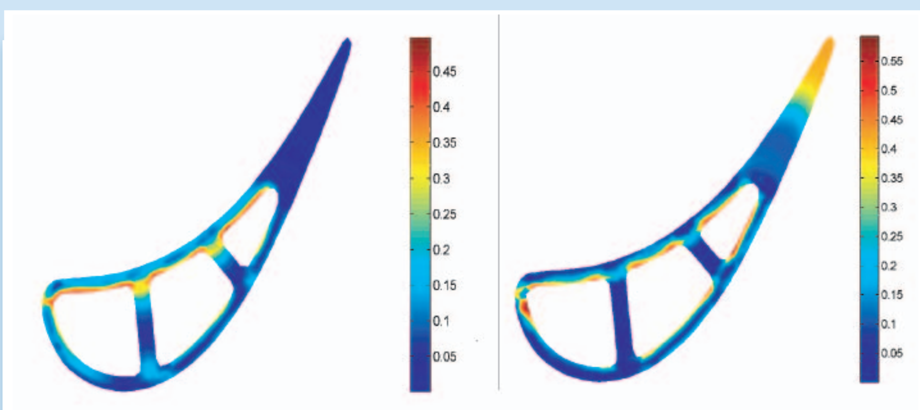

Fig.5 Absolute error in mean and standard deviation of the temperature profile mapped onto the nominal geometry.

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