Chapter 6

Radioactivity

Some nuclides have a far higher binding energy than some of its neighbours. When this is the case it is often energetically favourable for a nuclide with a low binding energy ("parent nucleus") to decay into one with a higher binding energy ("daughter nucleus"), giving off either an α-particle, which is the a $^4_2$He (helium) nucleus (α-decay) or an electron (positron) and another very low mass particle called a "antineutrino" ("neutrino"). This is called "β-decay". The difference in the binding energies is equal to the kinetic energy of the decay products

A further source of radioactivity arises when a nucleus in a metastable excited state ("isomer") decays directly or indirectly to its ground state emitting one or more high energy photons (γ-rays).

6.1 Decay Rates

The probability of a parent nucleus decaying in one second is called the "decay constant", (or "decay rate") λ. If we have $N(t)$ nuclei then the number of 'expected' decays per second is $\lambda\, N(t)$. The number of parent nuclei decreases by this amount and so we have

$$\frac{dN(t)}{dt} \;=\; -\lambda N(t). \tag{6.1.1}$$

This differential equation has a simple solution - the number of parent nuclei decays exponentially -

$$N(t) \;=\; N_0 e^{-\lambda t},$$

where N_0 is the initial number of parent nuclei at time $t = 0$.

The time taken for the number of parent nuclei to fall to $1/e$ of its initial value is called the "mean lifetime", τ of the radioactive nucleus, and we can see from eq.(6.1.1) that

$$\tau \;=\; \frac{1}{\lambda}.$$

Quite often one talks about the "half-life", $\tau_{\frac{1}{2}}$ of a radioactive nucleus, which is the time taken for the number of parent nuclei to fall to one-half of its initial value. From eq.(6.1.1) we can also see that

$$\tau_{\frac{1}{2}} \;=\; \frac{\ln 2}{\lambda} \;=\; \ln 2\,\tau.$$

6.2 Random Decay

It was stated above that the "expected" number of decays per second would be $\lambda N(t)$. This does not mean that there will always be precisely this number of decays per second.

Radioactive decay is a random process with a *probability* λ that any one nucleus will decay in one second.

The laws of random distributions tell us that if the expected number of events in a given period of time is ΔN, then the 'error' on this number is $\sqrt{\Delta N}$. More precisely there is a 68% probability that the number of events will be in the range

$$\Delta N \,-\, \sqrt{\Delta N} \;\rightarrow\; \Delta N \,+\, \sqrt{\Delta N}.$$

This means that if we want to measure the decay constant (lifetime, half-life) to within an accuracy of ϵ, we need to collect at least $1/\epsilon^2$ decays.

For example, suppose we have a sample with 10^{12} radioactive nuclei with a mean lifetime of about 10^{10} seconds and we want to measure this lifetime then in 1 second we predict that there will be (with 68% certainty) between

$$\frac{10^{12}}{10^{10}} - \sqrt{\frac{10^{12}}{10^{10}}} \;= 100 - 10 = 90 \text{ and } \frac{10^{12}}{10^{10}} + \sqrt{\frac{10^{12}}{10^{10}}} \;= 100 + 10 = 110,$$

decays per second. So if we want to determine the lifetime to better than 1% we need to observe the decays for 100 secs, for which we expect to have between 9900 and 10100 decays.

One decay per second is a unit of radioactivity known as the Bequerel (Bq) after the person who discovered radioactivity. Radioactivity is more often measured in Curies where one Curie is 3.7×10^{10} decays per second. This is the number of decays per second of one gram of $^{226}_{88}$Ra (radium).

What is the half-life of $^{226}_{88}$Ra?
Neglecting the binding energy the mass of Ra nucleus is

$$M_{Ra} \;=\; 88m_p \,+\, (226-88)m_n \;=\; 3.77 \times 10^{-25}\, kg$$

The number of nuclei in one gram is

$$N_0 \;=\; \frac{10^{-3}}{3.77 \times 10^{-25}} \;=\; 2.67 \times 10^{21}.$$

Therefore of the number of decays per second is 3.7×10^{10} for 2.67×10^{21} nuclei of Ra, we have for the decay constant

$$\lambda \;=\; \frac{3.7 \times 10^{10}}{2.67 \times 10^{21}} \;=\; 1.39 \times 10^{-11}\ s^{-1},$$

which gives us a half-life of

$$\tau_{\frac{1}{2}} \;=\; \frac{\ln 2}{\lambda} \;=\; \frac{0.693}{1.39 \times 10^{-11}} \;=\; 5 \times 10^{10}\ s \quad (1620\ \text{yr})$$

6.3 Carbon Dating

Living organisms absorb the isotope of carbon ${}^{14}_{6}\mathrm{C}$, which is created in the atmosphere by cosmic ray activity. The production of ${}^{14}_{6}\mathrm{C}$ from cosmic ray bombardment exactly cancels the rate at which thatbisotope decays so that the global concentration of ${}^{14}_{6}\mathrm{C}$ remians constant.

A sample of carbon taken from a living organism will have a concentration of one part in $1.3{\times}10^{12}$, and it is being continually rejuvenated, by exchanging carbon with the environment (either by photosynthesis or by eating plants which have undergone photosynthesis or by eating other animals that have eaten such plants.)

On the other hand a sample of carbon from a dead object cannot exchange its carbon with the environment and therefore cannot rejuvenate its concentration of ${}^{14}_{6}\mathrm{C}$.

${}^{14}_{6}\mathrm{C}$ decays radioactively into ${}^{14}_{7}\mathrm{N}$ (nitrogen), via β-decay with a half-life of 5730 years.

Thus by measuring the concentration of the isotope ${}^{14}_{6}\mathrm{C}$ in a fossil sample using techniques of mass spectroscopy, the age of the fossil can be determined.

6.4 Multi-modal Decays

A radioactive nucleus can sometimes decay into more than one channel, each of which has its own decay constant.

An example of this is ${}^{212}_{83}\mathrm{Bi}$ (bismuth) which can either decay as

$${}^{212}_{83}\mathrm{Bi} \;\rightarrow\; {}^{208}_{81}\mathrm{Ti} \;+\; \alpha$$

or

$${}^{212}_{83}\mathrm{Bi} \;\rightarrow\; {}^{212}_{84}\mathrm{Po} \;+\; e^{-} + \bar{\nu}$$

with a total mean lifetime of 536 secs. Ratio of ${}^{208}_{81}\mathrm{Ti}$ (titanium) to ${}^{212}_{84}\mathrm{Po}$ (polonium) from these decays is 9:16 What are the decay constants λ_1 and λ_2 for each of these decay modes? The rate of change of the number of parent nuclei is given by

$$\frac{dN(t)}{dt} \;=\; -\lambda_1 N(t) \;-\; \lambda_2 N(t),$$

with solution

$$N(t) \;=\; N_0 e^{-(\lambda_1+\lambda_2)t}.$$

From the total lifetime we have

$$\lambda_1 \,+\, \lambda_2 \;=\; \frac{1}{536} \;=\; 1.86\times 10^{-3}\, s^{-1}$$

The ratio of the number of decay products is equal to the ratio of the decay constants, i.e.

$$\frac{\lambda_1}{\lambda_2} \;=\; \frac{9}{16}$$

This gives us

$$\lambda_1 \;=\; 6.8\times 10^{-4}\ s^{-1}.$$
$$\lambda_2 \;=\; 11.8\times 10^{-3}\ s^{-1}.$$

6.5 Decay Chains

It is possible that a parent nucleus decays, with decay constant λ_1 into a daughter nucleus, which is itself radioactive and decays (either into a stable nuclide or into another radioactive nuclide) with decay constant λ_2. An example of this is

$$^{210}_{83}\mathrm{Bi} \;\stackrel{\beta}{\rightarrow}\; ^{210}_{84}\mathrm{Po} \;\stackrel{\alpha}{\rightarrow}\; ^{206}_{82}\mathrm{Pb}$$

The mean lifetime for the first stage of decay is 7.2 days and the mean lifetime for the second stage is 200 days.

If at time t we have $N_1(t)$ nuclei of the parent nuclide and $N_2(t)$ nuclei of the daughter nuclide, then for $N_1(t)$ we simply have

$$\frac{dN_1(t)}{dt} \;=\; -\lambda_1 N_1(t) \tag{6.5.2}$$

and therefore,

$$N_1(t) = N_1(0)e^{-\lambda_1 t}, \tag{6.5.3}$$

whereas for N_2 there is a production mechanism which contributes a rate of increase of N_2 equal to the rate of *decrease* of N_1. In addition there is a contribution to the rate of decrease of N_2 from its decay process, so we have

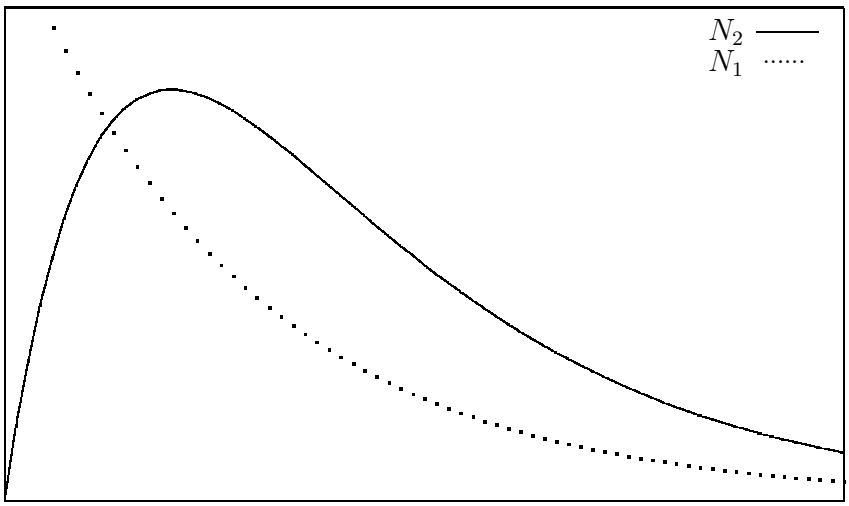
$$\frac{dN_2(t)}{dt} \;=\; \lambda_1 N_1(t) - \lambda_2 N_2(t) \tag{6.5.4}$$

Inserting the solution of eq.(6.5.2) into eq.(6.5.4) gives

$$\frac{dN_2(t)}{dt} \;=\; \lambda_1 N_1(0)e^{-\lambda_1 t} - \lambda_2 N_2(t).$$

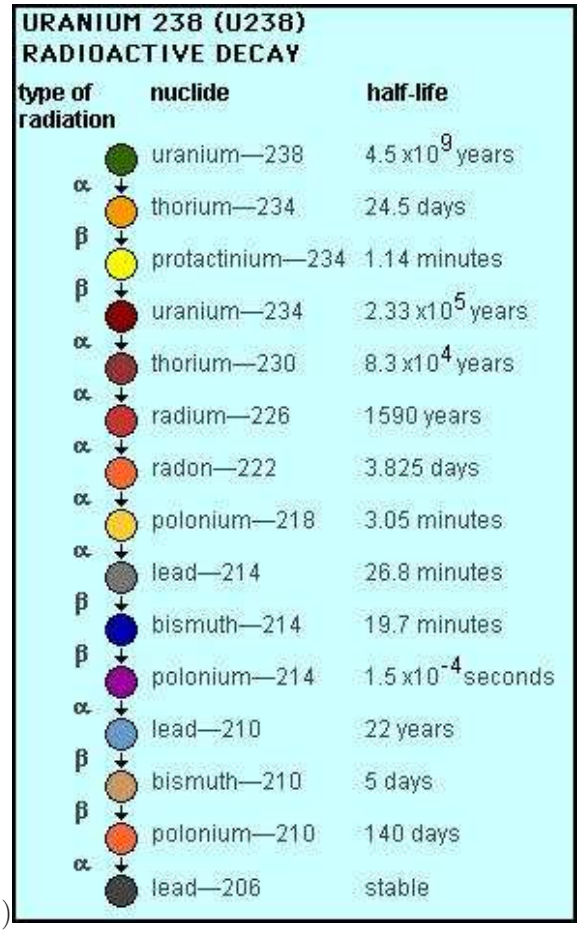
This is an inhomogeneous differential equation whose solution with $N_2(0) = 0$ is given by

$$N_2(t) \;=\; N_1(0)\frac{\lambda_1}{(\lambda_2 - \lambda_1)}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$



What is happening is that initially as the parent decays the quantity of the daughter nuclide grows faster than it decays. But after some time the available quantity of the parent nuclide is depleted so the production rate decreases and the decay rate of the daughter nuclide begins to dominate so that the quantity of the daughter nuclide also decreases.

Some heavy nuclides have a very long decay chain, decaying at each stage to another unstable nuclide before eventually reaching a stable nulcide. An example of this is $^{238}_{92}\mathbf{U}$, which decays in no fewer than 14 stages - eight by α-decay and six by β-decay before reaching a stable isotope of Pb. The lifetimes for the individual stages vary from around 10^{-4} s. to 10^9 years.



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In such cases, if the first parent is very long-lived, so that the number of parent nuclei does not decrease much, it is possible to reach what is known as "secular equilibrium", in which the quantities of various daughter nuclei remains unchanged. This happens when the numbers of nuclei in the chain N_A, $N_B\, N_C \cdots$ are in the ratio

$$\lambda_A N_A \;=\; \lambda_B N_B, \quad \text{etc.},$$

where λ_A, $\lambda_B \cdots$ are the decay rates for these nuclides. What is happening here is that the rate of production of daughter B, is the rate of decay of A, which is $\lambda_A\, N_A$ and this is equal to $\lambda_B\, N_B$, the rate of decay of B, so the quantity of B nuclei remains unchanged.

6.6 Induced Radioactivity

It is possible to convert a nuclide which is not radioactive into a radioactive one by bombarding it with neutrons or other particles. The stable nuclide (sometimes) absorbs the projectile in order to become an unstable, radioactive nucleus.

For example bombarding ${}^{23}_{11}$Na (sodium) with neutrons can convert the nuclide to ${}^{24}_{11}$Na, which is radioactive and decays via β-decay to ${}^{24}_{12}$Mg (magnesium).

In this case if we assume that the rate at which the radioactive nuclide (with decay constant λ) is being generated is R, then the number of such nuclei is given by the differential equation

$$\frac{dN(t)}{dt} \;=\; R \;-\lambda N$$

If at time $t = 0$ the number of these nuclei is zero (i.e. we start the bombardment at $t = 0$) then the solution to this differential equation is

$$N(t) \;=\; \frac{R}{\lambda}\left(1 - e^{-\lambda t}\right)$$

This starts at zeros and then grows so that asymptotically

$$R \;=\; \lambda N,$$

which is the equilibrium state in which the production rate R is equal to the decay rate λN.