NUCLEI AND PARTICLES
PHYS3002
2014/2015

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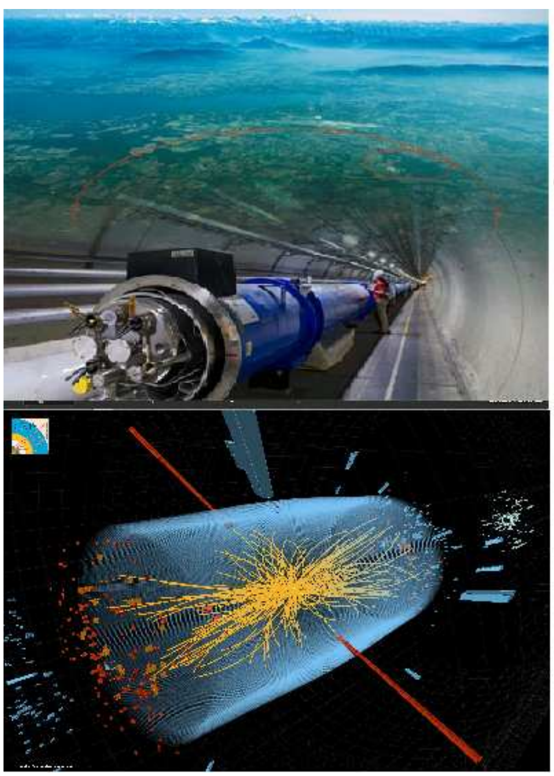
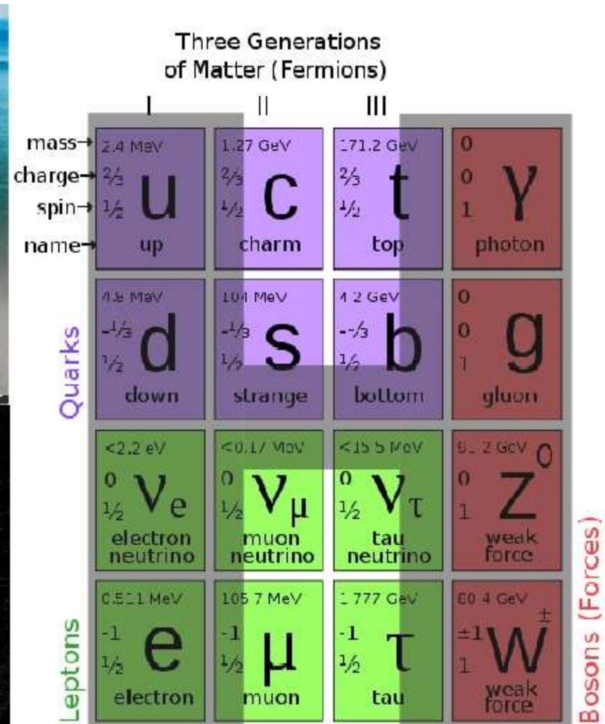
http://www.personal.soton.ac.uk/ab1u06/webpage/phys3002.html

Monday: 11:00 -13:00 02a/2065(L/T)

Friday: 09:00 -10:00 58/1067 (L/T)





Contents

Chapter 1

Introduction

1.1 Module Profile

1.1.1 Teaching and learning Methods

This course provides an introduction to nuclear and particle physics. There are approximately 16 lectures for each section supplemented by directed reading. Lectures delivered using mainly white board/blackboard and with a slight admixture of computer presentation for selected topics. There will be five problem sheets with respective five sessions devoted to the respective problem solutions. Model solutions will be provided after the problem sheets are due to be handed in. The problem sheets also contain non-assessed supplementary questions usually of a descriptive nature designed for deepper understanding of the material.

1.1.2 Learning Outcomes

This course provides a working knowledge of nuclear structure, nuclear decay and certain models for estimating nuclear masses and other properties of nuclei. Alo students will become familiar with the basics of elementary particle physics and particle accelerators. They will have an understanding of building blocks of matter and their interactions via different forces of Nature.

Students will learn about Nuclear Scattering, various properties of Nuclei, the Liquid Drop Model and the Shell Model, radioactive decay, fission and fusion. By the end of the course, the students should be able to classify elementary particles into hadrons and leptons, and understand how hadrons are constructed from quarks. They will also learn about flavour quantum numbers such as isospin, stangeness, etc. and understand which interactions conserve which quantum numbers. They will study the carriers of the fundamental interactions and have a qualitative understanding of QCD as well as the mechanisms of weak and electromagnetic interactions.

1.1.3 Syllabuses

Nuclei
1.Rutherford scattering (classical treatment)
2.and nuclear diffraction.
3.Nuclear properties.
4.Binding energies and Liquid Drop Model.
5.Magic Numbers and the Shell Model.
6.Radioactive decay
7.Fission and fusion
8.Isospin

Particles
1.Accelerators
2.Forces of Nature (strong, weak and electromagnetic interactions and their force carriers)
3.Particle classification
4.The constituent quark model
5.Weak Interactions (W and Z bosons)
6.Electromagnetic interactions
7.Quantum Chromodynamics (interactions of quarks and gluons)
8.Charge conjugation and parity

1.1.4 Non-contact Hours

Students are expected to devote a minimum of 6 hours per week of private study to background reading and problem solving.

1.1.5 Assessment Methods

Assessment is done by written examination at the end of the course. The exam will have a compulsory section A covering the whole course, with 4 - 6 questions and a section B on Nuclei where answers to 1 question out of 2 will be required, and a section C on Particles where answers also to 1 question out of 2 will be required. Each section carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on each.

The problem sheets will contribute 10% to the final mark and only 3 out 5 problems (1^{st},3^{rd} and the 5^{th}) will be marked. It is important to stress that the history of this course clearly shows that only those students who have been attempting to solve *all* problems from the very beginning were the most successful. The completed solutions should be handed in before the deadline indicated on the problem sheet. The problem sheets also contain non-assessed questions which are of a qualitative nature or pure bookwork. The student should work through all of these and ensure that he/she would be able to answer them under examination conditions.

1.1.6 Recommended Books

1. R.A.Dunlap - An Introduction to the Physics of Nuclei and Particles, Thomson, 2004 (- main text)
2.W S C Williams - Nuclear and Particle Physics, Oxford University Press, 1991.
3.D Perkins - Introduction to High Energy Physics, Addison-Wesley, 4th edition.
4.Francis Halzen, Alan D. Martin - Quarks and Leptons: An Introductory Course in Modern Particle Physics, John Wiley, 1984

1.1.7 Other Course Information

The course website
`http://www.hep.phys.soton.ac.uk/~belyaev/webpage/physics_phys3002.html` contains course notes and problem sheets (solutions will be uploaded after their due date) and the selected past exam papers. It also contains revision notes on topics from previous courses, familiarity with which will be assumed during the lectures.

Please note, that all course notes are acessible at the website and not ment to be printed for you.

1.2 History of Particle Physics

Since long ago people were trying to understand the Nature and its fundamental building blocks. We know several ‘theories’ which came from the ancient philosophers. More than two thousand years ago Empedocles (490-430 B.C.) suggested that all matter is made up of four elements: water, earth, air and fire. On the other hand, Democritus developed a theory that the universe consists of empty space and an (almost) infinite number of invisible particles which differ from each other in form, position and arrangement. He called them *atoms*(indivisible in Greek).

Since that time our understanding of fundamental building blocks of Nature has evolved into powerful science called Particle Physics. The main difference between Particle Physics and ancient philosophy is that Particle Physics, as a science, verifies its theoretical predictions by experiment. Theory and Experiment are vital interacting components of Particle Physics and because of these components Particle Physics can be called science. That is exactly the way how Standard Model (SM), which describes our present understanding of fundamental particles and their interactions, has been established. In this course we will briefly discuss the SM elementary particles and their interactions summarized in Fig. 1.1. The last particle

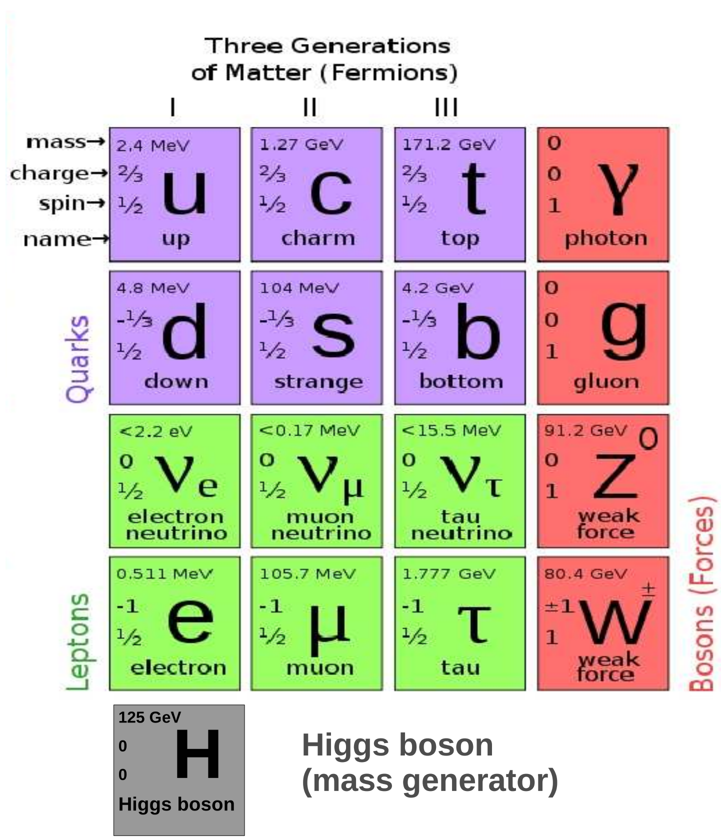


Figure 1.1: A summary of elementary particles of the Standard Model and their interactions.

in the SM, Higgs boson, responsible for the mass generation of other partcles, was discovered on the 4th of July 2012 which was announced by both, ATLAS and CMS collaborations at the Large Hadrom COllider (LHC). This was truly historical event. There could be more particles and theories beyond the SM: presently there are many new promising models beyond the SM which will be tested experimentally in the nearest future.

Particle Physics as a science has started in the very end of the 19th century. In Table 1.1 a timeline of Particle Physics is presented in a very brief way. More detailed history can be found, for example, at `http://en.wikipedia.org/wiki/Timeline_of_particle_physics` or at `http://web.ihep.su/dbserv/compas/contents.html` in much more detail.

1885	Eugene Goldstein discovered a positively charged sub-atomic particle
1897	J. J. Thomson discovered the electron
1909	Robert Millikan measured the charge and mass of the electron
1911	Ernest Rutherford discovered the nucleus of an atom
1913	Neils Bohr introduced his atomic theory
1919	Ernest Rutherford discovered the proton
1920s	Modern atomic theory developed by Heisenberg, de Broglie and Shroedinger
1932	James Chadwick discovered the Neutron
1964	Up, Down and Strange quarks were discovered
1974	Burton Richter and Samuel Ting discovered the J/ψ particle, demonstrating the existence of Charm quark
1977	Upsilon particle discovered at Fermilab, demonstrating the existence of the bottom quark
1995	Top quark discovered at Fermilab
2000	Tau neutrino proved distinct from other neutrinos at Fermilab
2012	Discovery of the Higgs Boson at the LHC

Table 1.1: A very brief timeline of particle physics

One of the most important milestones in the early history of Particle Physics is the experiment of Ernest Rutherford in 1911 which has proved an existence of the atomic structure with an atomic nucleus. We start this course describing this experiment and physics behind it in Chapter 2. Since that time many exciting discoveries has been made. However, one should stress that the principle behind the Rutherford experiment is one of the main ones being used in the modern collider physics. Rutherford has used the short length of the de Broglie wave of the electrons to probe the internal atomic structure. From well-known formula

$$\lambda = \frac{hc}{E}, \tag{1.1}$$

where λ stands for the wave-length, h is the Plank constant and E is the energy, one can see that the de Broglie wave length of the particle is inversely proportional to its energy. One can use this fact and resolve the structure of the tested object if the wave length if the tester particle is comparable or smaller than the size of the object. So, when the energy of the tester particle is large enough, it will interact with the the object at the respective scale. On the contrary, if the energy of the tester particle is too low, then, due to its large

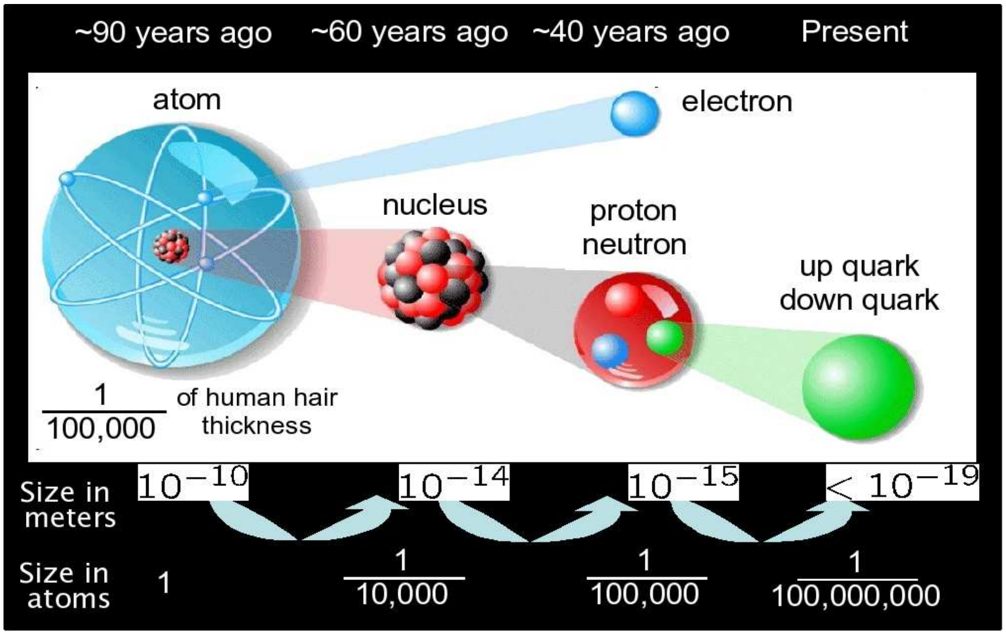


Figure 1.2: Timeline of the scale accessible in Particle Physics

de Broglie wave-length it will just bend around the object under study and no its internal structure will be resolved. The principle behind the Eq.(1.1) which in general relates the scale and the energy, is one of the main foundations of particle physics. Present collider experiments which reached now TeV energy scale (10^{12} electron volt) probe the scale as low as 10^{-19} meters! The timeline of the scale evolution of the distance scale accessible in Particle Physics is presented in Fig. 1.2.

On the other hand, another well known formula

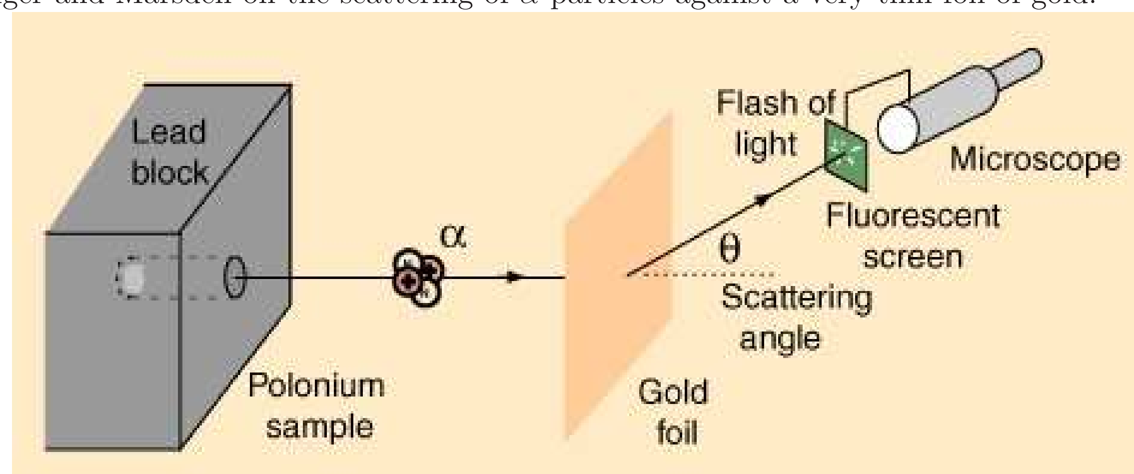
$$E = mc^2 \tag{1.2}$$

relating the energy and the mass tells us that High Energy gives us possibility to produce new heavy particles. This opens another way to explore new theories beyond the SM. The Large Hadron Collider (LHC) is now colliding protons with the highest energy in the world and one can expect that time lime of Particle Physics discoveries will be continued soon.

Chapter 2

Rutherford Scattering

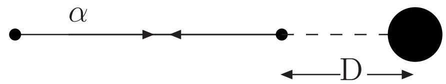
Let us start from the one of the first steps which was done towards understanding the deepest structure of matter. In 1911, Rutherford discovered the nucleus by analysing the data of Geiger and Marsden on the scattering of α-particles against a very thin foil of gold.



The data were explained by making the following assumptions.

- The atom contains a nucleus of charge Ze, where Z is the atomic number of the atom (i.e. the number of electrons in the neutral atom),
- The nucleus can be treated as a point particle,
- The nucleus is sufficently massive compared with the mass of the incident α-particle that the nuclear recoil may be neglected,
- That the laws of classical mechanics and electromagnetism can be applied and that no other forces are present,
- That the collision is elastic.

If the collision between the incident particle whose kinetic energy is T and electric charge ze ($z = 2$ for an α-particle), and the nucleus were head on,



the distance of closest approach D is obtained by equating the initial kinetic energy to the Coulomb energy at closest approach, i.e.

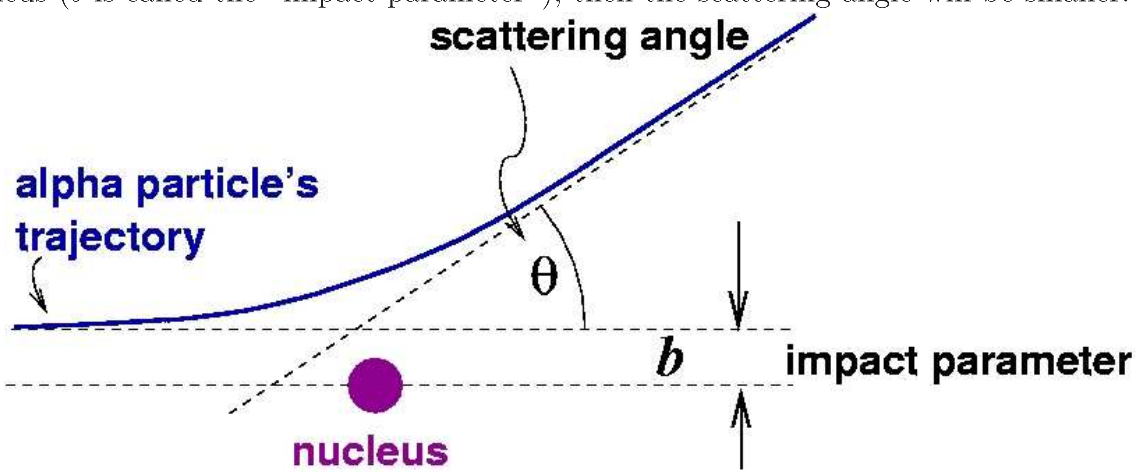
$$T = \frac{z\,Z\,e^2}{4\pi\epsilon_0 D},$$

or

$$D = \frac{z\,Z\,e^2}{4\pi\epsilon_0 T}$$

at which point the α-particle would reverse direction, i.e. the scattering angle θ would equal π.

On the other hand, if the line of incidence of the α-particle is a distance b, from the nucleus (b is called the "impact parameter"), then the scattering angle will be smaller.

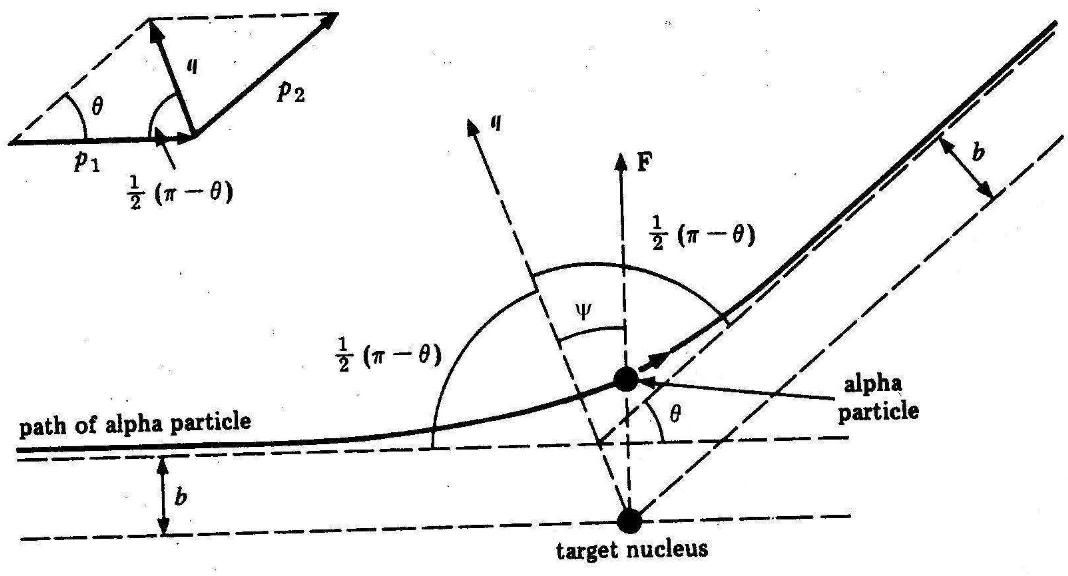


2.1 Relation between scattering angle and an impact parameter

The relation between b and θ is given by

$$\tan\left(\frac{\theta}{2}\right) = \frac{D}{2b} \tag{2.1.1}$$

This relation is derived using Newton's Second Law of Motion, Coulomb's law for the force between the α-particle and and nucleus, and conservation of angular momentum. The derivation is given in this section. Here we note that $\theta = \pi$ when $b = 0$ as stated above and that as b increases the α-particle 'glances' the nucleus so that the scattering angle decreases.



Geometrical relationships in Rutherford scattering.

The initial and final momenta, p_1, p_2 are equal in magnitude (p) (recall, that, elastic scattering is assumed), so that together with the momentum change $\mathbf{q}$ they form an isosceles triangle with angle θ between the initial and final momenta, as shown above.

Using the sine rule we have

$$\frac{q}{p} = \frac{\sin\theta}{\sin\left(\frac{1}{2}(\pi-\theta)\right)} \;=\; 2\sin\left(\frac{\theta}{2}\right). \tag{2.1.2}$$

The direction of the vector $\mathbf{q}$ is along the line joining the nucleus to the point of closest approach of the α-particle.

We assume that the nucleus is much heavier than the α-particle so we can neglect its recoil. We also neglect any relativistic effects.

The position of the α-particle is given in terms of two-dimensional polar coordinates r, ψ with the nucleus as the origin and $\psi = 0$ chosen to be the point of closest approach.

By Newton's second law, the rate of change of momentum in the direction of $\mathbf{q}$ is the component of the force acting on the α-particle due to the electric charge of the nucleus. By Coulomb's law the magnitude of the force is

$$F \;=\; \frac{zZe^2}{4\pi\epsilon_0 r^2},$$

where $Z\,e$ is the electric charge of the nucleus, and $z\,e$ is the electric charge of the incident particle (for an α-particle $z = 2$). Using $T = \frac{zZe^2}{4\pi\epsilon_0 D}$ expression relating kinetic energy and the closest approach for head-on collision, one finds

$$F \;= \frac{TD}{r^2}$$

. The component of this force in the direction of $\mathbf{q}$ is

$$F_{\mathbf{q}}(t) \; = \frac{TD}{r^2}\cos\psi(t)$$

and, therefore, the change of momentum ($F_{\mathbf{q}}(t) = \frac{d\mathbf{q}}{dt}$) is given by

$$q \;=\; \int \frac{zZe^2}{4\pi\epsilon_0 r^2}\cos\psi\,dt. \tag{2.1.3}$$

We can replace integration over time by integration over the angle ψ using

$$dt \;=\; \frac{d\psi}{\dot\psi},$$

where $\dot\psi$ can be obtained form conservation of angular momentum,

$$L \;=\; m_\alpha r^2\dot\psi.$$

The initial angular momentum is given by

$$L \;=\; bp,$$

so we have

$$\dot\psi \;=\; \frac{b\,p}{m_\alpha\,r^2},$$

so that eq.(2.1.3) becomes

$$q \;=\; \int \frac{TD\,m_\alpha\,r^2}{r^2\,b\,p}\cos\psi\,d\psi \;=\; \int \frac{D\,p}{2b}\cos\psi\,d\psi, \tag{2.1.4}$$

where kinetic energy of α-particle $T = p^2/(2m_\alpha)$ related its momenta and its mass was substituted at the last step. Note that r^2 has cancelled.

From the diagram we see that the limits on ψ are

$$\psi \;=\; \pm\frac{1}{2}(\pi-\theta),$$

so that we get

$$q \;=\; \frac{Dp}{2b}2sin\left(\frac{1}{2}(\pi-\theta)\right)$$

Now using eq.(2.1.2) we get

$$2p\sin\left(\frac{\theta}{2}\right) \;=\; \frac{Dp}{2b}2sin\left(\frac{1}{2}(\pi-\theta)\right)$$

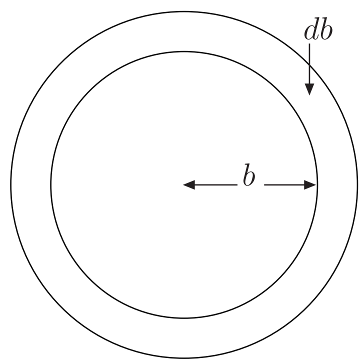
from where it follows that

$$\tan(\theta/2) = \frac{D}{2b}$$

2.2 Flux and cross-section

The "flux", F of incident particles is defined as the number of incident particles arriving per unit area per second at the target.

The number of particles, $dN(b)$, with impact parameter between b and $b + db$ is this flux multiplied by the area between two concentric circles of radius b and $b + db$



$$dN(b) \;=\; F\, 2\pi b\, db \tag{2.2.5}$$

Differentiating eq.(2.1.1) gives us

$$db = -\frac{D}{4\sin^2(\theta/2)} d\theta \tag{2.2.6}$$

which allows us to write an expression for the number of α-particles scattered through an angle between θ and $\theta + d\theta$ after substitution Eq.(2.2.6) and Eq.(2.1.1) into Eq.(2.2.5):

$$dN(\theta) \;= F\pi \frac{D^2}{4} \frac{\cos(\theta/2)}{\sin^3(\theta/2)} d\theta. \tag{2.2.7}$$

(the minus sign has been dropped as it merely indicates that as b increases, the scattering angle θ decreases - $N(\theta)$ must be positive).

The "differential cross-section", $d\sigma/d\theta$, with respect to the scattering angle is the number of scatterings between θ and $\theta + d\theta$ per unit flux, per unit range of angle, i.e.

$$\frac{d\sigma}{d\theta} \;= \frac{dN(\theta)}{F d\theta} \;=\; \pi \frac{D^2}{4} \frac{\cos(\theta/2)}{\sin^3(\theta/2)}.$$

It is more usual to quote the differential cross-section with respect to a given solid angle Ω, which is related to the scattering angle θ and the azimuthal angle ϕ by

$$d\Omega \;=\; \sin\theta d\theta d\phi \;=\; 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta d\phi.$$

The relation between the number of events, the flux, differential solid angle $d\Omega$ and differential cross section is given by

$$\frac{dN}{d\Omega} = F\frac{d\sigma}{d\Omega}$$

. in analogy to the relation for differential angle $d\theta$.

The integrastion over the azimuthal angle just gives a factor of 2π so we may write

$$\frac{d\sigma}{d\theta} = 2\pi\frac{d^2\sigma}{d\theta d\phi}$$

so that

$$\frac{d^2\sigma}{d\theta d\phi} = \frac{D^2}{8}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}.$$

and substitute $d\theta d\phi$ by $d\Omega$ (using the above relation) to obtain

$$\frac{d\sigma}{d\Omega} = \frac{D^2}{8}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\frac{1}{2\sin(\theta/2)\cos(\theta/2)} = \frac{D^2}{16\sin^4(\theta/2)}.$$

Differential cross-sections have the dimension of an area. These are usually quoted in terms of "barns". I barn is defined to be $10^{-28}\,m^2$, so that, for example, 1 millibarn (mb) is an area of $10^{-31}\,m^2$.

The unit of length that is often used in nuclear physics is the "fermi" (fm) which is defined to be 10^{-15} m and energies are usually quoted in electron volts (Kev, MeV, or GeV). A cross-section of 1 fm^2 corresponds ot 10 mb. For the purposes of numerical calculations, it is worth noting that

$$\hbar c = 197.3\,\mathrm{MeV\,fm},$$

so that

$$\frac{e^2}{4\pi\epsilon_0} = \alpha\,\hbar c = \frac{1}{137}\times 197.3\,\mathrm{MeV\,fm}$$

For example, the distance of closest approach is therefore given by

$$D = \frac{197.3}{137}\frac{zZ}{T}\,\mathrm{fm},$$

where the kinetic energy T is given in MeV.

2.3 Results and interpretation of the Rutherford experiment

Although the differential cross-section falls rapidly with the scattering angle, the cross-section at large angles is still much larger than would have been obtained from Thomson's 'current cake' model of the atom in which electrons are embedded in a 'dough' of positive charge -

so that as the α-particle moves through the atom it suffers a large number of small-angle scatterings in random directions.

We notice that the differential cross-section diverges as the scattering angle goes to zero. However we note from eq.(2.1.1) that small angle scattering implies a large impact parameter. The distance of the incident particle from any nucleus can only grow to about half of the distance between the nuclei in the gold foil. In fact, the total number of particles scattered into a given solid angle is the differential cross-section multiplied by the flux, multiplied by the number of nuclei in the foil - or more precisely in the part of the foil that is 'illuminated' by the incident α-particles. We assume that the foil is sufficiently thin so that multiple scatterings are very unlikely and we can make the approximation that all the nuclei lie in a single plane. The mass of a nucleus with atomic mass number A is given to a very good approximation by Am_p, total number of nuclei per unit area of foil is given by

$$\rho\delta\frac{1}{Am_p}$$

where ρ is the density, δ is the thickness of the foil, A is the atomic mass. This means that the fraction of α particles scattered into a small interval of solid angle $d\Omega$ is given by

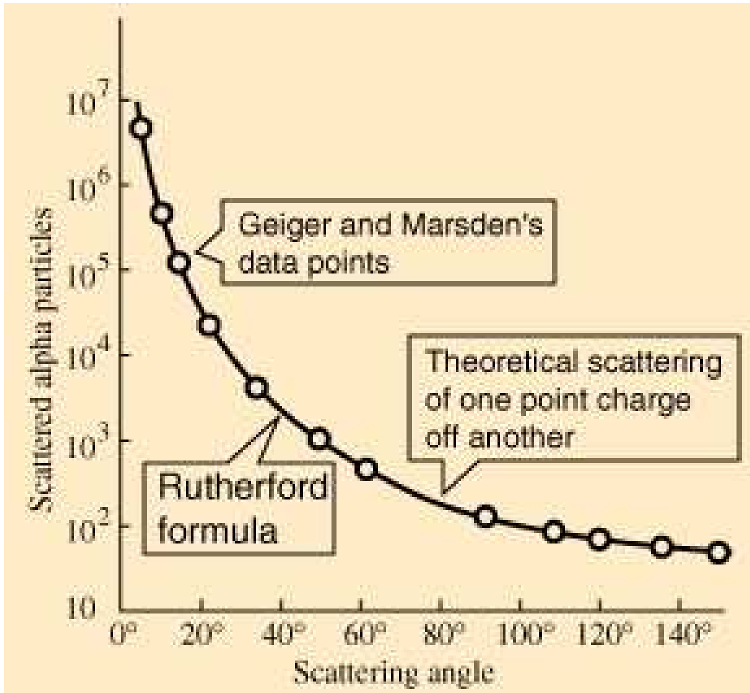
$$\frac{\delta n}{n} = \rho\delta\frac{1}{Am_p}\frac{d\sigma}{d\Omega}d\Omega \qquad (2.3.8)$$

Solid angle is defined such that an area element dA at a distance r from the scattering centre subtends a solid angle

$$d\Omega = \frac{dA}{r^2},$$

so that if we place a detector with an acceptance area dA at a distance r from the foil and at an angle θ to the direction of the incident α-particles then the fraction of incident α-particles enter the detector is given by replacing $d\Omega$ by dA/r^2 in eq.(2.3.8)

This theoretical result compares very well with the data taken by Geiger and Marsden.



Geiger and Marsden's data points
Theoretical scattering of one point charge off another
Rutherford formula
Scattered alpha particles
10^7
10^6
10^5
10^4
10^3
10^2
10
0° 20° 40° 60° 80° 100° 120° 140°
Scattering angle

Chapter 3

Nuclear Size and Shape

The unit of nuclear length is called the "fermi", (fm)

$$1\,fm \;=\; 10^{-15}\; m.$$

There are deviations from the Rutherford scattering formula when the energy of the incident α-particle becomes too large, so that the distance of closest approach is of order a few fermi's.

The reason for this is that the Rutherford scattering formula was derived assuming that the nucleus was a point particle. In reality it has a finite size with a radius R of order 10^{-15} m.

The nucleus therefore has a charge distribution, $\rho(\mathbf{r})$. In terms of quantum mechanics we have

$$\rho(\mathbf{r}) \;=\; Ze|\Psi(\mathbf{r})|^2,$$

where Z is the atomic number and is equal to the number of protons in the nucleus, and Ψ is the wave-function for one of these protons. ($|\Psi(\mathbf{r})|^2$ is therefore the probability density for one proton). Nuclear 'radius' is not really a very precise term - it is the extent over which the electric charge distribution of the proton, and therefore its wavefunction, is not too small, although in principle the wave-function extends throughout all space.

It is difficult to produce α-particles with sufficient energy to probe the charge distribution of the nucleus, so we use high energy electrons instead.

For electrons the projectile charge z is replaced by 1 in the Rutherford scattering formula. There is one further change which is due to the fact that these electrons are moving relativistically with a velocity v close to c. This correction was first calculated by Mott and we have

$$\frac{d\sigma}{d\Omega}_{|\mathrm{Mott}} \;=\; \frac{d\sigma}{d\Omega}_{|\mathrm{Rutherford}} \left(1 - \frac{v^2}{c^2}\sin^2\left(\frac{\theta}{2}\right)\right)$$

We account for the charge distribution of the nucleus by writing the differential cross-section as

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{|\mathrm{Mott}} |F(q^2)|^2. \tag{3.0.1}$$

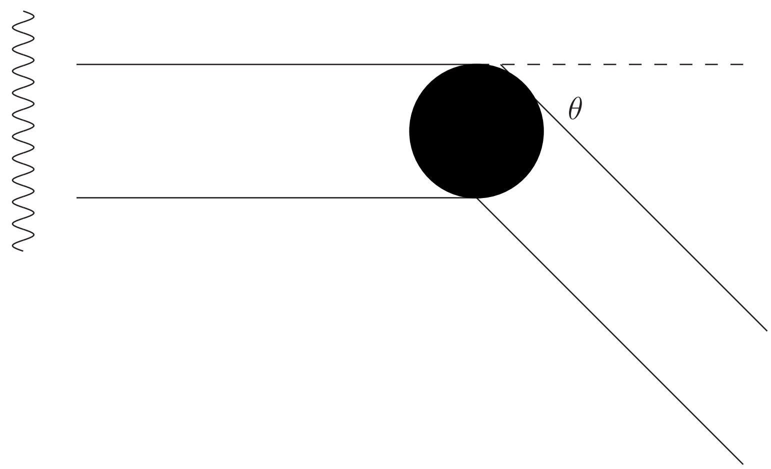
The correction factor $F(q^2)$ is called the "electric form-factor" and $\mathbf{q}$ is the momentum transfered by the electron in the scattering and its magnitude is related to the scattering angle by (see previous chapter)

$$q = 2\,p\,\sin\left(\frac{\theta}{2}\right),$$

where p is the momentum of the incident electron.

To understand the structure of the electric form-factor we need to recall that the electron has a de Broglie wavelength $\lambda = h/p$, and when this wavelength is of the order of the nuclear 'radius' we get a diffraction pattern.

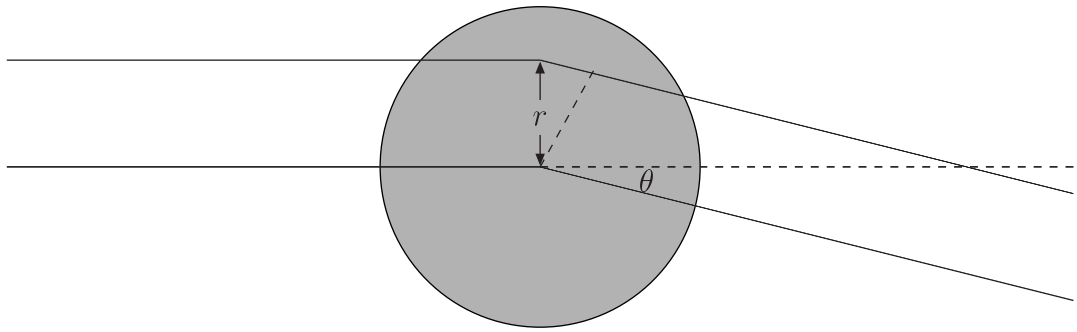
As a simple example suppose that the nucleus were a solid sphere of radius R with an infinite potential inside the sphere and zero potential outside, so that the electron cannot penetrate the sphere.



The wave that passes over the nucleus travels a distance $2R\sin\theta$ further than the wave that passes below the nucleus. If this difference is equal to $\lambda/2$, $3\lambda/2$ $\cdots$ then we get destructive interference. At these angles the differential cross-section vanishes.

The real case is a little more complicated than that. A proper quantum mechanical treatment (which is exactly analogous to diffraction in optics) shows that the electric form-factor is actually the Fourier transform of the charge distribution. For a spherically symmetric charge distribution this leads to

$$F(q^2) = \frac{4\,\pi\,\hbar}{Z\,e\,q} \int r\,\rho(r) \sin\left(\frac{qr}{\hbar}\right)\,dr. \tag{3.0.2}$$



Qualitatively, the reason for this is that the part of the wavefront that passes through the nucleus at a distance r from the centre and is scattered through an angle θ travels a further distance than the part of the wave that passes through the centre, by an amount proportional to r and therefore suffers a phase change (relative to the part of the wave passing through the centre). This phase change also depends on the scattering angle θ and is equal to $qr/\hbar$. This means that different parts of the wavefront suffer a different phase change (just as in optical diffraction) - these different amplitudes are summed to get the total amplitude at some scattering angle θ and this gives rise to the diffraction pattern. The contribution to the amplitude from the part of the wavefront which passes at a distance r from the centre of the nucleus is proportional to the charge density, $\rho(r)$, at r. The total scattering amplitude is therefore the sum of the amplitudes from all these different parts, which is what the integral in eq.(3.0.2) means.

Thus we see that a study of the diffractive scattering of electrons from a nucleus can give us information about the charge distribution inside the nucleus.

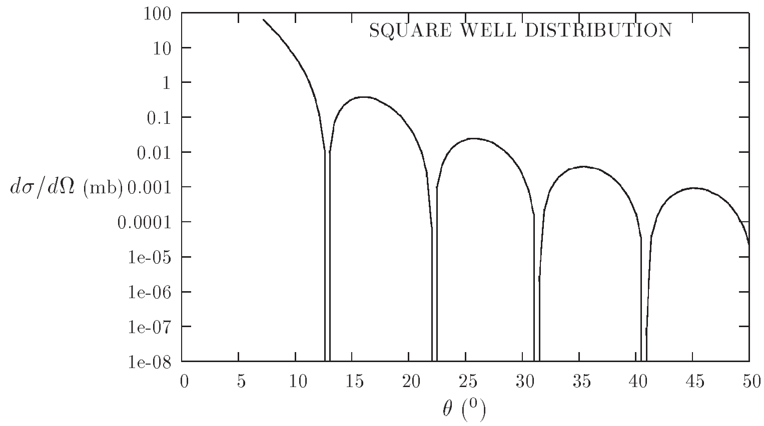
For example, if we assume that the charge distribution is a constant for $r < a$ and zero outside

$$\begin{aligned}\rho(r) &= \frac{3Ze}{4\pi R^3}, \quad r < R \\ &= 0 \quad r > R,\end{aligned}$$

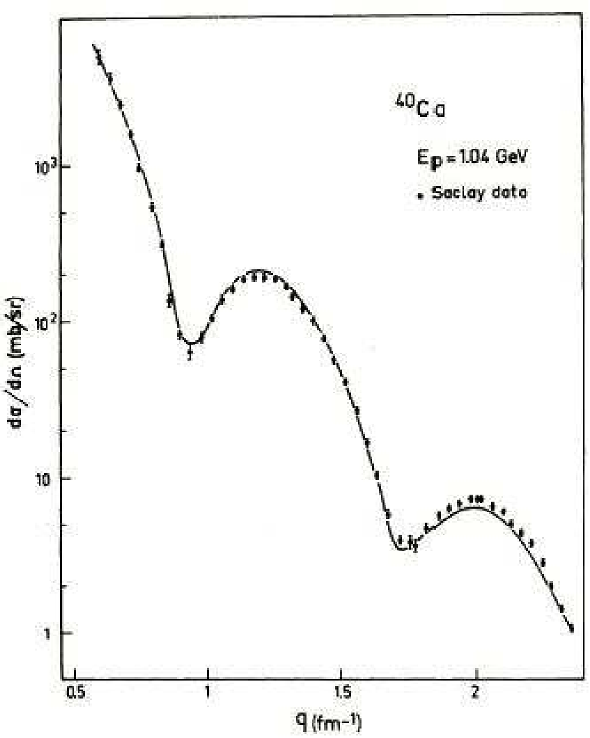
the integral in the Fourier transform eq.(3.0.2) can be done analytically via integrating by parts to give

$$F(q^2) = 3\left(\frac{\hbar}{qR}\right)^3 \left(\sin(qR/\hbar) - \frac{qR}{\hbar}\cos(qR/\hbar)\right).$$

Feeding this back into eq.(3.0.1) for the diffractive differential cross-section we get



This is not quite what is observed in experiment which is more like this example of scattering of electrons of energy 1.04 GeV against a Ca nucleus



We see that although there are oscillations in the differential cross-section, it never actually vanishes. The reason for the discrepancy is that the square-well model for the charge distribution is unrealistic. The charge distribution rapidly becomes small as r exceeds a few fermi's, but never goes to zero.

A rough (to with about 30%) estimate of the nuclear radius R can be obtained from the first minimum of the diffraction pattern and assuming that this occurs when

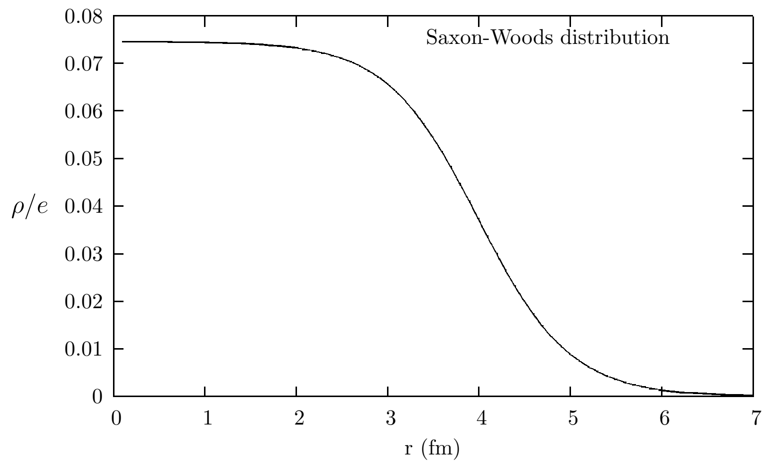
$$\frac{qR}{\hbar} \approx \pi$$

In the above case this occurs at $q/\hbar \approx 1\text{fm}^{-1}$ (the x-scale is given in fm^{-1} which means it is really $q/\hbar$), giving an approximate nuclear radius of about 3 fm.

A more realistic charge distribution is the Saxon-Woods model for which

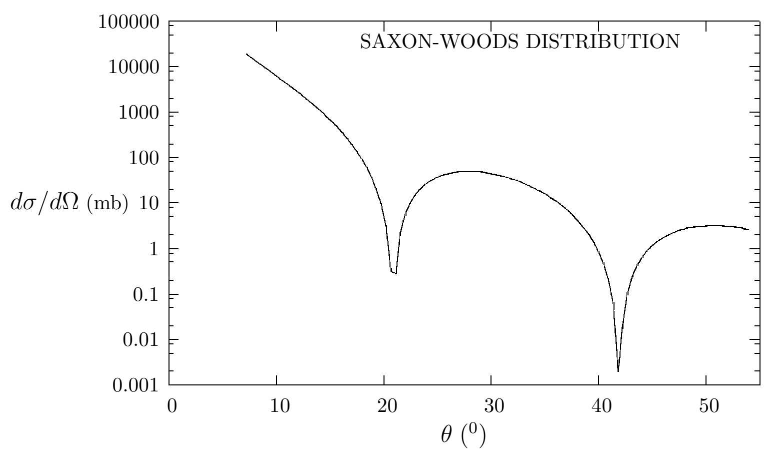
$$\rho(r) \propto \frac{1}{1+\exp((r-R)/\delta)},$$

which looks like



We can interpret R as the nuclear 'radius' and δ as the 'surface depth' - it measures the range in r over which the charge distribution changes from the order of its value at the centre to much smaller than this value.

This leads to a differential cross-section which looks like



This has dips but no zeros and is much more similar in shape to the experimental results.

In fact, the Saxon-Woods model fits data from most nuclei rather well with empirical values for R and δ depending on the atomic mass number, A (the total number of protons and neutrons in the nucleus):

$$R \;=\; (1.18A^{1/3} - 0.48)\,\mathrm{fm}$$

$$\delta \;=\; 0.4 \;-\; 0.5\;\;\mathrm{fm}\;\;\text{for A} \;>\; 40$$

The first term in the expression for R is easily understandable as one would expect the volume occupied by a nucleus to be proportional to A, so that the radius is proportional to $A^{1/3}$.

3.1 Electric Quadrupole Moments

So far, we have assumed that the charge distribution is spherically symmetric. If that were the case we would have

$$< x^2 > = < y^2 > = < z^2 > = \frac{1}{3} < r^2 >,$$

where

$$< x^2 > = \frac{1}{Ze}\int x^2 \rho(\mathbf{r}) d^3\mathbf{r}$$

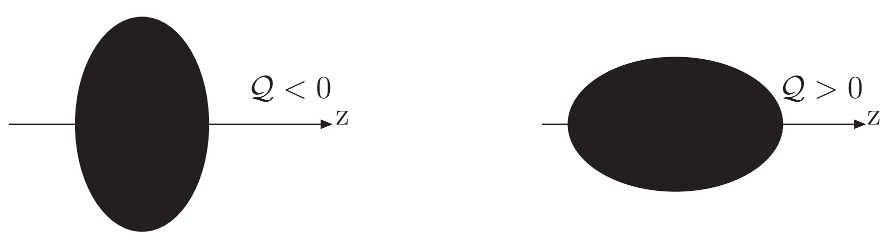
etc.

However, for many nuclei this is not the case and they possess an "electric quadrupole moment" defined (with respect to an axis z) as

$$\mathcal{Q} \;=\; \int (3z^2 - r^2)\rho(\mathbf{r}) d^3\mathbf{r}$$

The $\mathcal{Q}/e$ has dimensions of area and is therefore usually quoted in barnes.

Nuclei that possess and electric quadrupole moment have a shape which is an oblate spheroid for $\mathcal{Q} < 0$ and a prolate spheroid for $\mathcal{Q} > 0$.



On the other hand, the electric dipole moment, which is a vector defined by

$$\mathbf{d} \;=\; \int \mathbf{r}\rho(\mathbf{r}) d^3\mathbf{r},$$

is almost zero. The reason for this that to a very good approximation, the wavefunction of a proton in a nucleus is a parity eigenstate, i.e.

$$\Psi(\mathbf{r}) \;=\; \pm\Psi(-\mathbf{r})$$

which implies

$$\rho(\mathbf{r}) \;=\; \rho(-\mathbf{r}),$$

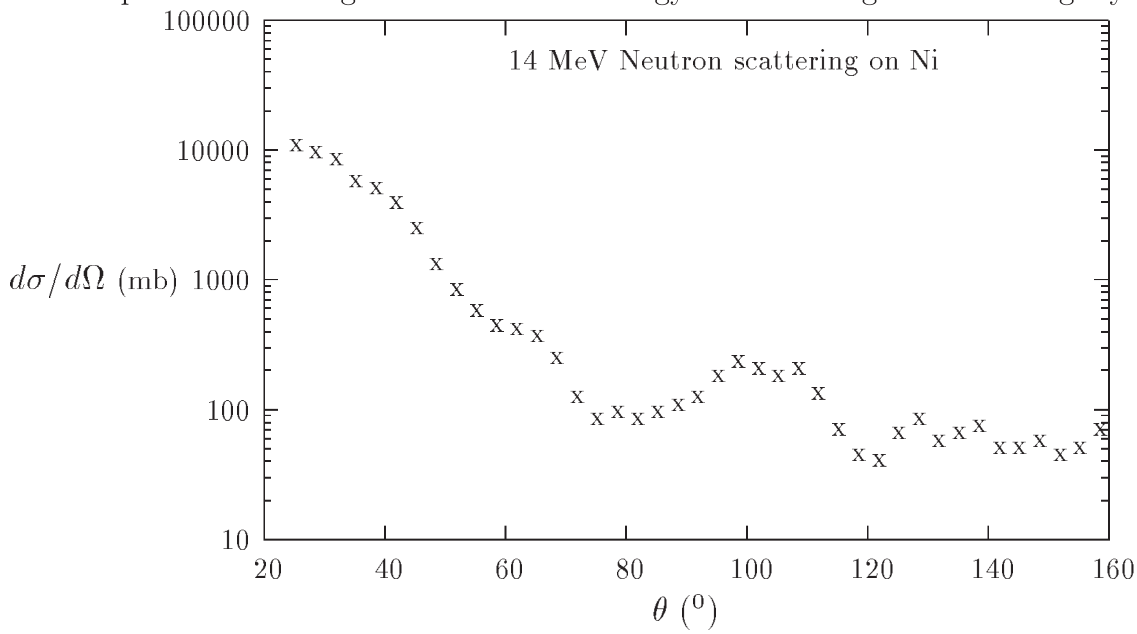
so that the electric dipole moment vanishes by symmetric integration.

3.2 Strong Force Distribution

The protons and neutrons inside a nucleus are held together by a strong nuclear force. This has to be strong enough to overcome the Coulomb repulsion between the protons, but unlike the Coulomb force, it extends only over a short range of a few fermi's.

Electron diffractive scattering is used to examine the distribution of electric charge (i.e. the protons) within the nucleus. Similar experiments are performed using high energy neutrons in order to probe the distribution of the strong force, i.e the distribution of all "nucleons" (neutrons and protons). In this case the form factor $F(q)$ is not the electric form-factor but the form-factor associated with the strong force.

For example the scattering of neutrons with energy of 14 MeV against a Ni target yields:



The Saxon-Woods model is also useful for the analyses of these data and yields a nuclear radius (for large A), given by

$$R \;=\; 1.2A^{1/3}\,\text{fm}$$

and

$$\delta \;=\; 0.75\,\text{fm}.$$

We see that the strong force extends over approximately the same region as the nuclear charge, and that the 'volume' of the nucleus is proportional to the number of nucleons.

Chapter 4

The Liquid Drop Model

4.1 Some Nuclear Nomenclature

- **Nucleon:** A proton or neutron.
- **Atomic Number, Z:** The number of protons in a nucleus.
- **Atomic Mass number, A:** The number of nucleons in a nucleus.
- **Nuclide:** A nucleus with a specified value of A and Z. This is usually written as ${}^{A}_{Z}\{Ch\}$ where Ch is the Chemical symbol. e.g. ${}^{56}_{28}$Ni means Nickel with 28 protons and a further 28 neutrons.
- **Isotope:** Nucleus with a given atomic number but different atomic mass number, i.e. different number of neutrons. Isotopes have very similar atomic and chemical behaviour but may have very different nuclear properties.
- **Isotone:** Nulceus with a given number of neutrons but a different number of protons (fixed (A-Z)).
- **Isobar:** Nucleus with a given A but a different Z.
- **Mirror Nuclei:** Two nuclei with odd A in which the number of protons in one nucleus is equal to the number of neutrons in the other and vice versa.

4.2 Binding Energy

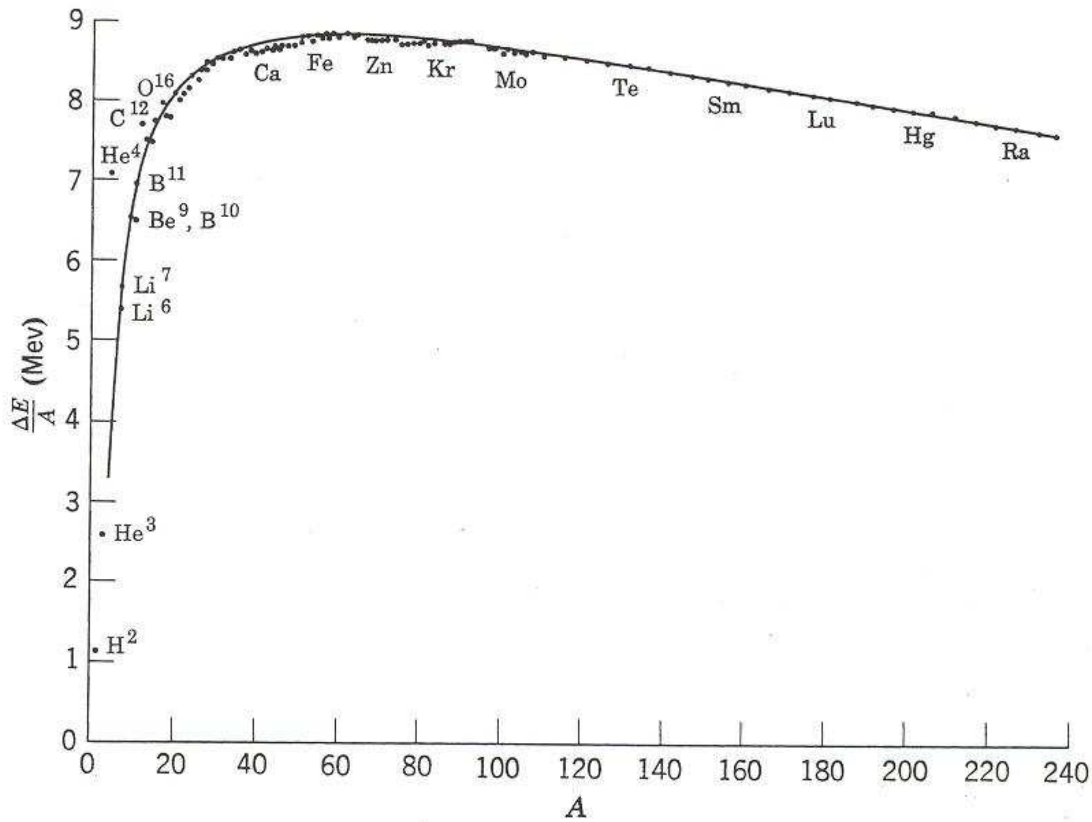
The mass of a nuclide is given by

$$m_N \;=\; Z\,m_p + (A-Z)\,m_n \;-\; B(A,Z)/c^2,$$

where $B(A,Z)$ is the binding energy of the nucleons and depends on both Z and A. The binding energy is due to the strong short-range nuclear forces that bind the nucleons together.

Unlike Coulomb binding these cannot even in principle be calculated analytically as the strong forces are much less well understood than electromagnetism.

Binding energies per nucleon increase sharply as A increases, peaking at iron (Fe) and then decreasing slowly for the more massive nuclei.



The binding energy divided by c^2 is sometimes known as the "mass defect".

4.3 Semi-Empirical Mass Formula

For most nuclei (nuclides) with $A > 20$ the binding energy is well reproduced by a semi-empirical formula based on the idea the the nucleus can be thought of as a liquid drop.

1. **Volume term:** Each nucleon has a binding energy which binds it to the nucleus. Therefore we get a term proportional to the volume i.e. proportional to A.

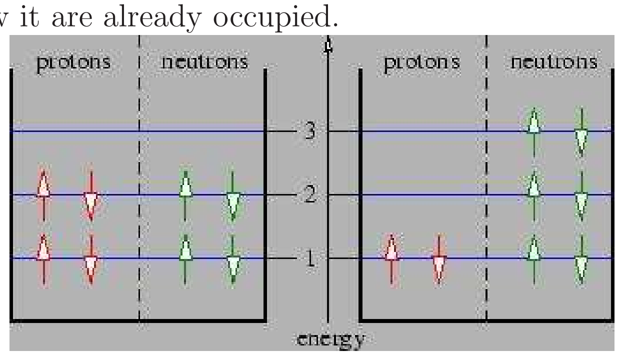
$$a_V \, A$$

This term reflects the short-range nature of the strong forces. If a nucleon interacted with *all* other nucleons we would expect an energy term of proportional to $A(A-1)$, but the fact that it turns out to be proportional to A indicates that a nucleon only interact with its nearest neighbours.

2. **Surface term:** The nucleons at the surface of the 'liquid drop' only interact with other nucleons inside the nucleus, so that their binding energy is reduced. This leads to a reduction of the binding energy proportional to the surface area of the drop, i.e. proportional to $A^{2/3}$
$$-a_S\, A^{2/3}.$$

3. **Coulomb term:** Although the binding energy is mainly due to the strong nuclear force, the binding energy is reduced owing to the Coulomb repulsion between the protons. We expect this to be proportional to the square of the nuclear charge, Z, (the electromagnetic force is long-range so each proton interact with all the others), and by Coulomb's law it is expected to be inversely proportional to the nuclear radius, (the Coulomb energy of a charged sphere of radius R and charge Q is $3Q^2/(20\pi\epsilon_0 R)$) The Coulomb term is therefore proportional to $1/A^{1/3}$
$$-a_C \frac{Z^2}{A^{1/3}}$$

4. **Asymmetry term:** This is a quantum effect arising from the Pauli exclusion principle which only allows two protons or two neutrons (with opposite spin direction) in each energy state. If a nucleus contains the same number of protons and neutrons then for each type the protons and neutrons fill to the same maximum energy level (the 'fermi level'). If, on the other hand, we exchange one of the neutrons by a proton then that proton would be required by the exclusion principle to occupy a higher energy state, since all the ones below it are already occupied.



The upshot of this is that nuclides with $Z = N = (A-Z)$ have a higher binding energy, whereas for nuclei with different numbers of protons and neutrons (for fixed A) the binding energy decreases as the square of the number difference. The spacing between energy levels is inversely proportional to the volume of the nucleus - this can be seen by treating the nucleus as a three-dimensional potential well- and therefore inversely proportional to A. Thus we get a term
$$-a_A \frac{(Z-N)^2}{A}$$

5. **Pairing term:** It is found experimentally that two protons or two neutrons bind more strongly than one proton and one neutron.
In order to account for this experimentally observed phenomenon we add a term to the binding energy if number of protons and number of neutrons are *both* even, we subtract

the same term if these are *both* odd, and do nothing if one is odd and the other is even. Bohr and Mottelson showed that this term was inversely proportional to the square root of the atomic mass number.
We therefore have a term

$$\frac{\left((-1)^Z+(-1)^N\right)}{2}\frac{a_P}{A^{1/2}}.$$

The complete formula is, therefore

$$B(A,Z) \;=\; a_V\,A\;-a_S\,A^{2/3}\;-a_C\frac{Z^2}{A^{1/3}}\;-a_A\frac{(Z-N)^2}{A}\;+\frac{\left((-1)^Z+(-1)^N\right)}{2}\frac{a_P}{A^{1/2}}$$

From fitting to the measured nuclear binding energies, the values of the parameters a_V, a_S, a_C, a_A, a_P are

$$\begin{aligned}
a_V &= 15.56 \text{ MeV}\\
a_S &= 17.23 \text{ MeV}\\
a_C &= 0.697 \text{ MeV}\\
a_A &= 23.285 \text{ MeV}\\
a_P &= 12.0 \text{ MeV}
\end{aligned}$$

For most nuclei with $A > 20$ this simple formula does a very good job of determining the binding energies - usually better than 0.5%.

For example we estimate the binding energy per nucleon of ${}^{80}_{35}$Br (Bromine), for which Z=35, A=80 ($N = 80 - 35 = 45$) and insert into the above formulae to get

$$\begin{aligned}
\text{Volume term:}\quad & (15.56\times 80) \;=\; 1244.8\,\text{MeV}\\
\text{Surface term:}\quad & (-17.23\times(80)^{2/3}) \;=\; -319.9\,\text{MeV}\\
\text{Coulomb term:}\quad & \left(\frac{0.697\times 35^2}{(80)^{1/3}}\right) \;=\; -198.2\,\text{MeV}\\
\text{Asymmetry term:}\quad & \left(\frac{23.285\times(45-35)^2}{80}\right) \;=\; -29.1\,\text{MeV}\\
\text{Pairing term:}\quad & \left(\frac{-12.0}{(80)^{1/2}}\right) \;=\; -1.3\,\text{MeV}
\end{aligned}$$

Note that we *subtract* the pairing term since both (A-Z) and Z are odd. This gives a total binding energy of 696.3 MeV. The measured value is 694.2 MeV.

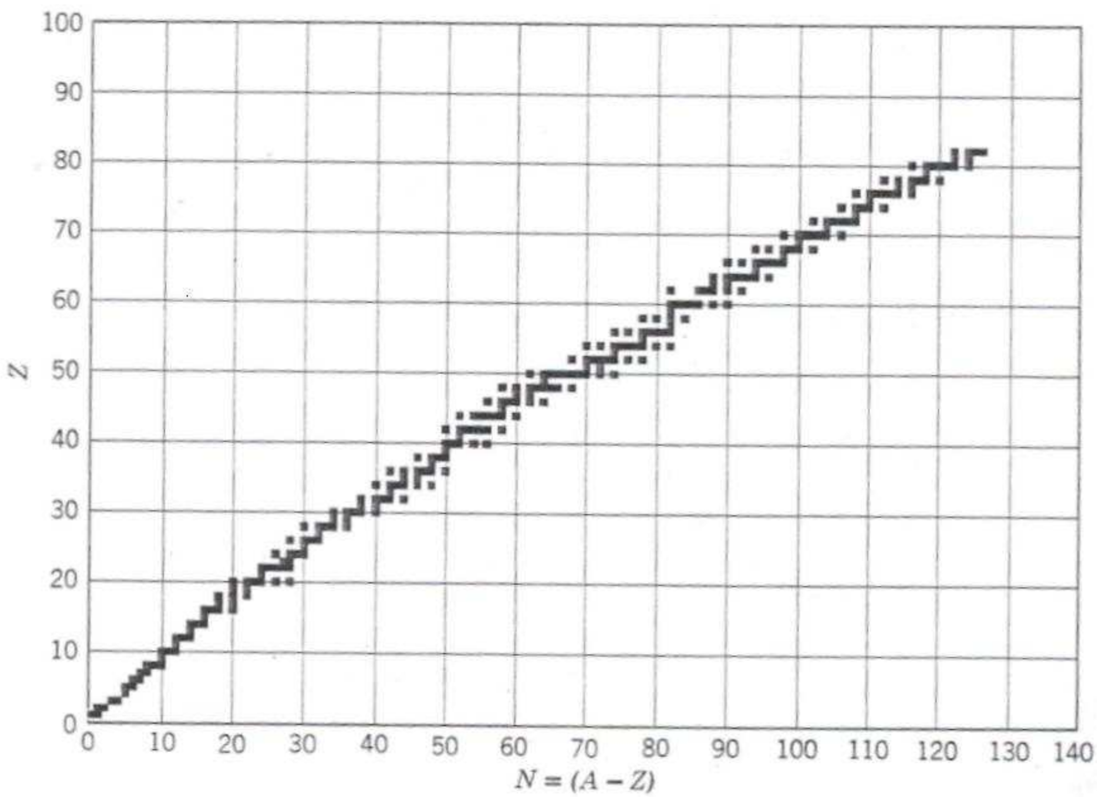
In order to calculate the mass of the nucleus we *subtract* this binding energy (divided by c^2) from the total mass of the protons and neutrons ($m_p = 938.4MeV/c^2, \quad m_n = 939.6MeV/c^2$)

$$m_{Br} \;=\; 35m_p + 45m_n - 696.1MeV/c^2 \;=\; 74417 \text{ Mev/c}^2.$$

Nuclear masses are nowadays usually quoted in MeV/c^2 but are still sometimes quoted in atomic mass units, defined to be 1/12 of the *atomic* mass of ${}^{12}_{6}$C (Carbon). The conversion factor is

$$1\,a.u. \;=\; 931.5 \text{ MeV/c}^2$$

Since different isotopes have different atomic mass numbers they will have different binding energies and some isotopes will be more stable than others. It turns out (and can be seen by looking for the most stable isotopes using the semi-empirical mass formula) that for the lighter nuclei the stable isotopes have approximately the same number of neutrons as protons, but above A $\sim$ 20 the number of neutrons required for stability increases up to about one and a half times the number of protons for the heaviest nuclei.



Qualitatively, the reason for this arises from the Coulomb term. Protons bind less tightly than neutrons because they have to overcome the Coulomb repulsion between them. It is therefore energetically favourable to have more neutrons than protons. Up to a certain limit this Coulomb effect beats the asymmetry effect which favours equal numbers of protons and neutrons.

Chapter 5

Nuclear Shell Model

5.1 Magic Numbers

The binding energies predicted by the Liquid Drop Model underestimate the actual binding energies of "magic nuclei" for which either the number of neutrons $N = (A - Z)$ or the number of protons, Z is equal to one of the following "magic numbers"

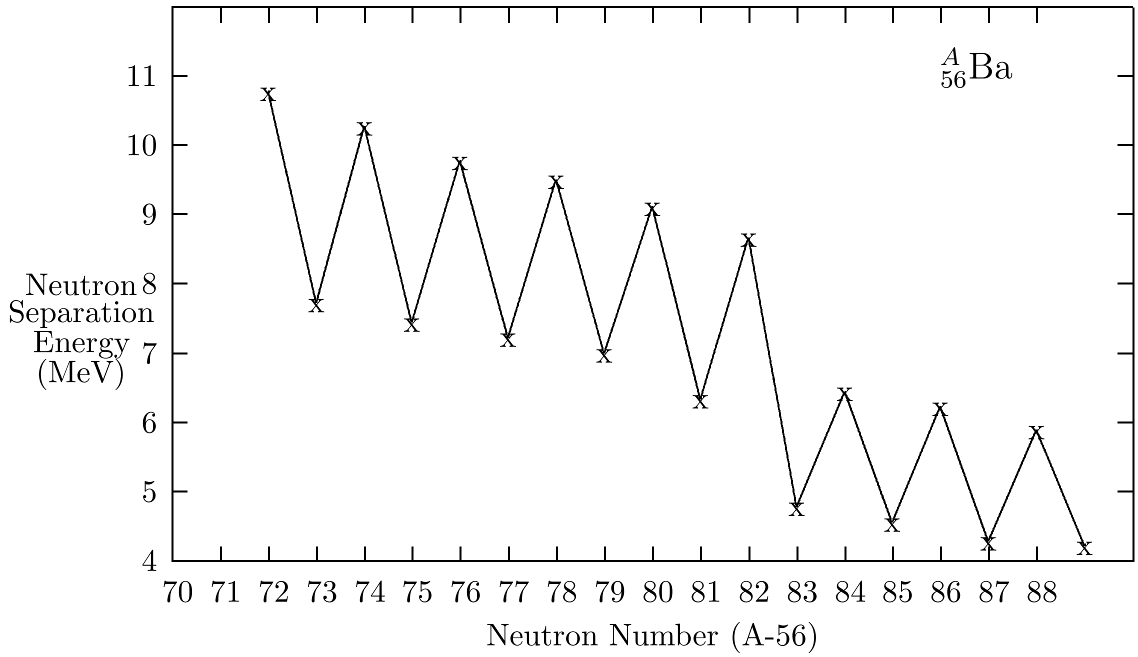
$$2,\ 8,\ 20,\ 28, 50, 82, 126.$$

This is particularly the case for "doubly magic" nuclei in which *both* the number of neutrons and the number of protons are equal to magic numbers.

For example for $^{56}_{28}$Ni (nickel) the Liquid Drop Model predicts a binding energy of 477.7 MeV, whereas the measured value is 484.0 MeV. Likewise for $^{132}_{50}$Sn (tin) the Liquid Drop model predicts a binding energy of 1084 MeV, whereas the measured value is 1110 MeV.

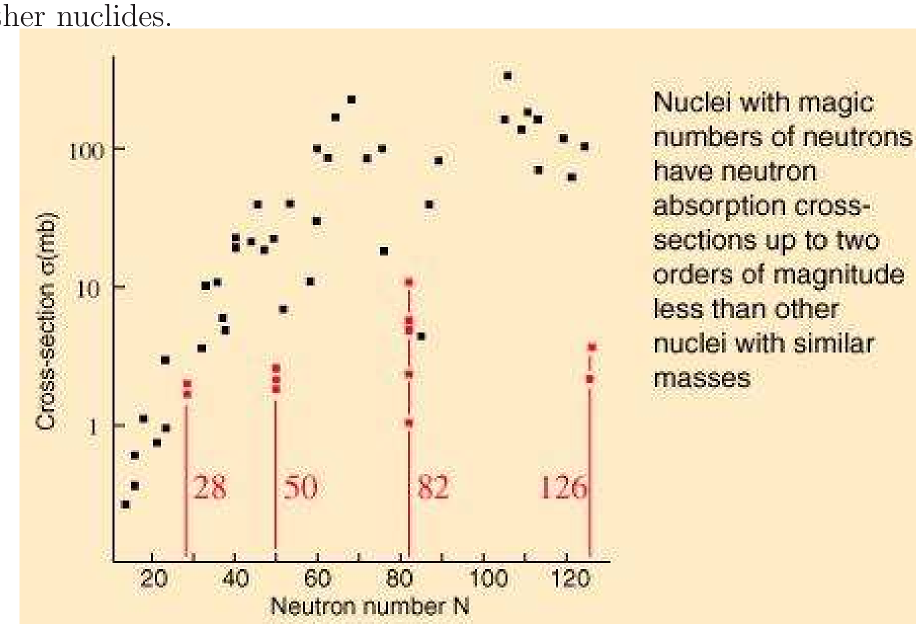
There are other special features of magic nuclei:

- The neutron (proton) separation energies (the energy required to remove the last neutron (proton)) peaks if N (Z) is equal to a magic number.

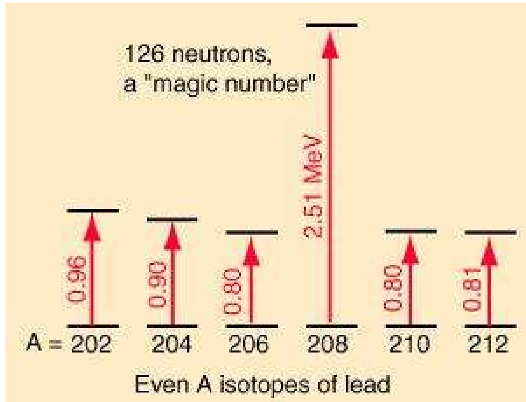


- There are more stable isotopes if Z is a magic number, and more stable isotones if N is a magic number.

- If N is magic number then the cross-section for neutron absorption is much lower than for other nuclides.



- The energies of the excited states are much higher than the ground state if either N or Z or both are magic numbers.



- Elements with Z equal to a magic number have a larger natural abundance than those of nearby elements.

5.2 Shell Model

These magic numbers can be explained in terms of the Shell Model of the nucleus, which considers each nucleon to be moving in some potential and classifies the energy levels in terms of quantum numbers $n\,l\,j$, in the same way as the wavefunctions of individual electrons are classified in Atomic Physics.

For a spherically symmetric potential the wavefunction (neglecting its spin for the moment) for any nucleon whose coordinates from the centre of the nucleus are given by polar coordinates (r, θ, ϕ) is of the form

$$\Psi_{nlm} \;=\; R_{nl}(r) Y_l^m(\theta, \phi).$$

The energy eigenvalues will depend on the principle quantum number, n, and the orbital angular momentum, l, but are degenerate in the magnetic quantum number m. These energy levels come in 'bunches' called "shells" with a large energy gap just above each shell.

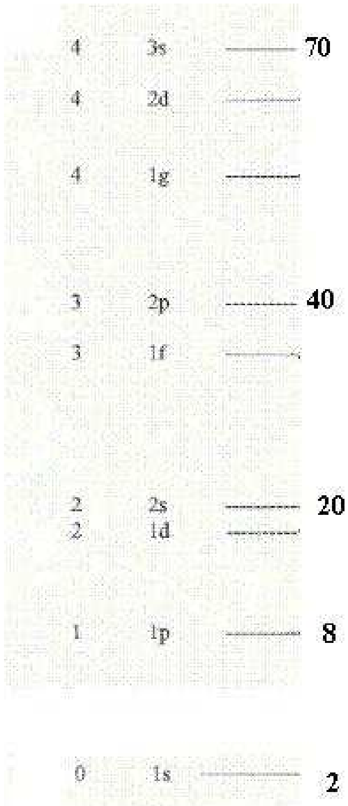
In their ground state the nucleons fill up the available energy levels from the bottom upwards with two protons (neutrons) in each available proton (neutron) energy level.

Unlike Atomic Physics we do not even understand in principle what the properties of this potential are - so we need to take a guess.

A simple harmonic potential (i.e. $V(r) \propto r^2$) would yield equally spaced energy levels and we would not see the shell structure and hence the magic numbers.

It turns out that once again the Saxon-Woods model is a reasonable guess, i.e.

$$V(r) \;=\; -\frac{V_0}{1 + \exp\left(((r-R)/\delta)\right)}$$



For such a potential it turns out that the lowest level is $1s$ (i.e. $n = 1$, $l = 0$) which can contain up to 2 protons or neutrons. Then comes $1p$ which can contain up to a further 6 protons (neutrons). This explains the first 2 magic numbers (2 and 8). Then there is the level $1d$, but this is quite close in energy to $2s$ so that they form the same shell. This allows a further 2+10 protons (neutrons) giving us the next magic number of 20.

The next two levels are $1f$ and $2p$ which are also quite close together and allow a further 6+14 protons (neutrons). This would suggest that the next magic number was 40 - but experimentally it is known to be 50.

The solution to this puzzle lies in the spin-orbit coupling. Spin-orbit coupling - the interaction between the orbital angular momentum and spin angular momentum occurs in Atomic Physics. In Atomic Physics, the origin is magnetic and the effect is a small correction. In the case of nuclear binding the effect is about 20 times larger, and it comes from a term in the nuclear potential itself which is proportional to $\mathbf{L} \cdot \mathbf{S}$, i.e.

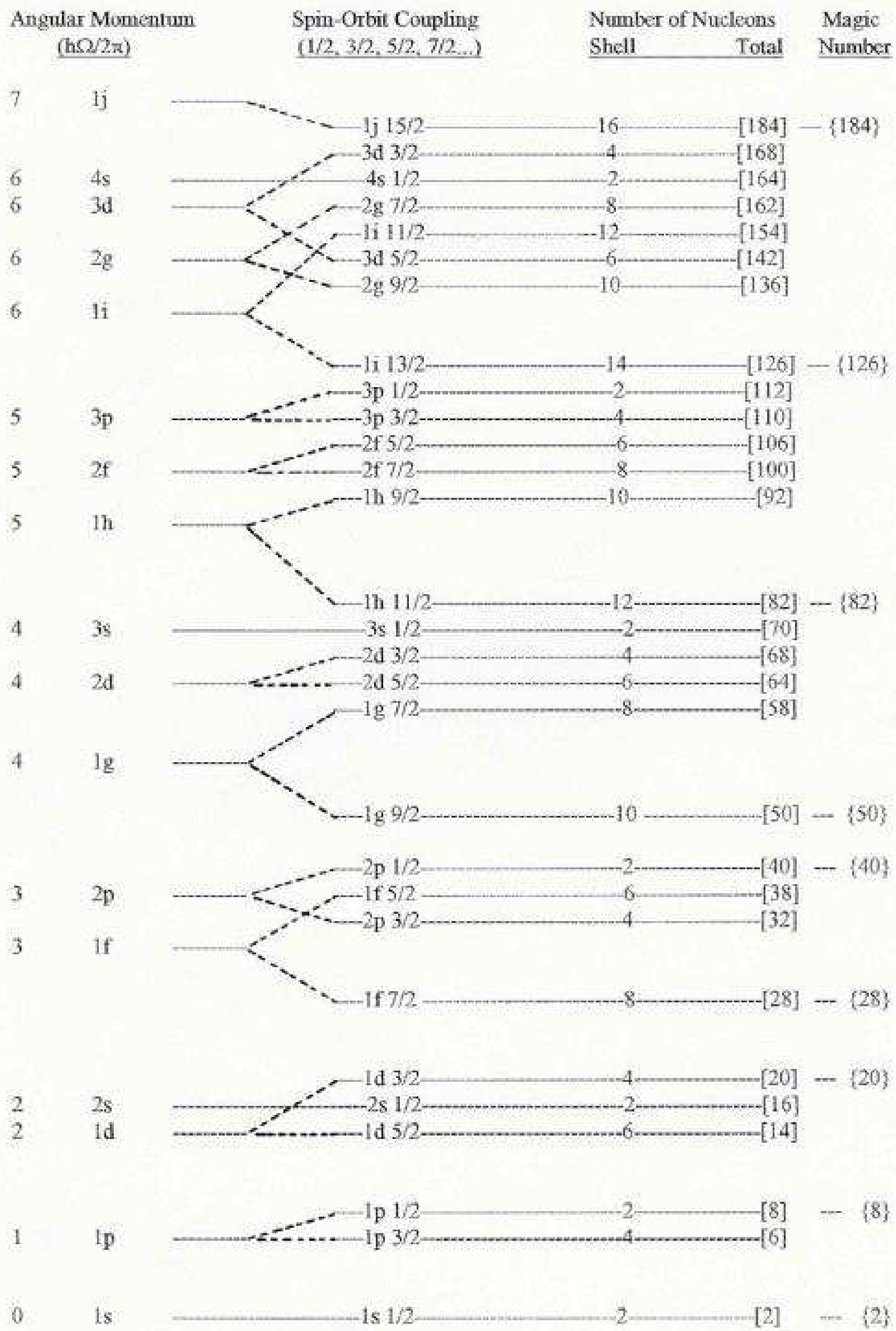
$$V(r) \ \rightarrow \ V(r) \, + \, W(r)\mathbf{L} \cdot \mathbf{S}.$$

As in the case of Atomic Physics (j-j coupling scheme) the orbital and spin angular momenta of the nucleons combine to give a total angular momentum j which can take the values $j = l + \frac{1}{2}$ or $j = l - \frac{1}{2}$. The spin-orbit coupling term leads to an energy shift proportional to

$$j(j+1) - l(l+1) - s(s+1), \quad (s = 1/2).$$

A further feature of this spin-orbit coupling in nuclei is that the energy split is in the opposite sense from its effect in Atomic Physics, namely that states with higher j have *lower energy*.

Nuclear Shell Structure



We see that this large spin-orbit effect leads to crossing over of energy levels into different shells. For example the state above the $2p$ state is $1g$ (l=4), which splits into $1g_{\frac{9}{2}}$, $(j=\frac{9}{2})$ and $1g_{\frac{7}{2}}$, $(j=\frac{7}{2})$. The energy of the $1g_{\frac{9}{2}}$ state is sufficiently low that it joins the shell below, so that this fourth shell now consists of $1f_{\frac{7}{2}}, 2p_{\frac{3}{2}},\ 1f_{\frac{5}{2}},\ 2p_{\frac{1}{2}}$ and $1g_{\frac{9}{2}}$. The maximum occupancy of this state ($(2j+1)$ protons (neutrons) for each j) is now 8+4+6+2+10=30, which added to the previous magic number, 20, gives the next observed magic number of 50.

Further up, it is the $1h$ state that undergoes a large splitting into $1h_{\frac{11}{2}}$ and $1h_{\frac{9}{2}}$, with the $1h_{\frac{11}{2}}$ state joining the lower shell.

5.3 Spin and Parity of Nuclear Ground States.

Nuclear states have an intrinsic spin and a well defined parity, $\eta = \pm 1$, defined by the behaviour of the wavefunction for all the nucleons under reversal of their coordinates with the centre of the nucleus at the origin.

$$\Psi(-\mathbf{r_1}, -\mathbf{r_2}\cdots - \mathbf{r_A}) \ = \ \eta\Psi(\mathbf{r_1}, \mathbf{r_2}\cdots\mathbf{r_A})$$

.

The spin and parity of nuclear ground states can usually be determined from the shell model. Protons and neutrons tend to pair up so that the spin of each pair is zero and each pair has even parity ($\eta = 1$). Thus we have

- Even-even nuclides (both Z and A even) have zero intrinsic spin and even parity.

- Odd A nuclei have one unpaired nucleon. The spin of the nucleus is equal to the j-value of that unpaired nucleon and the parity is $(-1)^l$, where l is the orbital angular momentum of the unpaired nucleon.

  For example $^{47}_{22}$Ti (titanium) has an even number of protons and 25 neutrons. 20 of the neutrons fill the shells up to magic number 20 and there are 5 in the $1f_{\frac{7}{2}}$ state $(l=3,\ j=\frac{7}{2})$ Four of these form pairs and the remaining one leads to a nuclear spin of $\frac{7}{2}$ and parity $(-1)^3 = -1$.

- Odd-odd nuclei. In this case there is an unpaired proton whose total angular momentum is j_1 and an unpaired neutron whose total angular momentum is j_2. The total spin of the nucleus is the (vector) sum of these angular momenta and can take values between $|j_1 - j_2|$ and $|j_1 + j_2|$ (in unit steps). The parity is given by $(-1)^{(l_1+l_2)}$, where l_1 and l_2 are the orbital angular momenta of the unpaired proton and neutron respectively.

  For example $^6_3$Li (lithium) has 3 neutrons and 3 protons. The first two of each fill the $1s$ level and the thrid is in the $1p_{\frac{3}{2}}$ level. The orbital angular mometum of each is $l=1$ so the parity is $(-1)\times(-1) = +1$ (even), but the spin can be anywhere between 0 and 3.

5.4 Magnetic Dipole Moments

Since nuclei with an odd number of protons and/or neutrons have intrinsic spin they also in general possess a magnetic dipole moment.

The unit of magnetic dipole moment for a nucleus is the "nuclear magneton" defined as

$$\mu_N \; = \; \frac{e\hbar}{2m_p},$$

which is analogous to the Bohr magneton but with the electron mass replaced by the proton mass. It is defined such that the magnetic moment due to a proton with orbital angular momentum $\mathbf{l}$ is $\mu_N \, \mathbf{l}$.

Experimentally it is found that the magnetic moment of the proton (due to its spin) is

$$\mu_p \; = \; 2.79\mu_N \; = \; 5.58\mu_N s, \quad \left(s = \frac{1}{2} \right)$$

and that of the neutron is

$$\mu_n \; = \; -1.91\mu_N \; = \; -3.82\mu_N s, \quad \left(s = \frac{1}{2} \right)$$

If we apply a magnetic field in the z-direction to a nucleus then the unpaired proton with orbital angular momentum $\mathbf{l}$, spin $\mathbf{s}$ and total angular momentum $\mathbf{j}$ will give a contribution to the $z-$ component of the magnetic moment

$$\mu^z \; = \; (5.58 s^z \; + \; l^z) \, \mu_N.$$

As in the case of the Zeeman effect, the vector model may be used to express this as

$$\mu^z \; = \; \frac{(5.58 < \mathbf{s} \cdot \mathbf{j} > + < \mathbf{l} \cdot \mathbf{j} >)}{< \mathbf{j}^2 >} j^z \, \mu_N$$

using

$$\begin{aligned}
< \mathbf{j}^2 > \; &= \; j(j+1)\hbar^2 \\
< \mathbf{s} \cdot \mathbf{j} > \; &= \; \frac{1}{2} \left(< \mathbf{j}^2 > + < \mathbf{s}^2 > - < \mathbf{l}^2 > \right) \\
&= \; \frac{\hbar^2}{2} \left(j(j+1) \; + \; s(s+1) \; - \; l(l+1) \right) \\
< \mathbf{l} \cdot \mathbf{j} > \; &= \; \frac{1}{2} \left(< \mathbf{j}^2 > + < \mathbf{l}^2 > - < \mathbf{s}^2 > \right) \\
&= \; \frac{\hbar^2}{2} \left(j(j+1) \; + \; l(l+1) \; - \; s(s+1) \right) \qquad (5.4.1)
\end{aligned}$$

We end up with expression for the contribution to the magnetic moment

$$\mu \; = \; \frac{5.58 \left(j(j+1) \; + \; s(s+1) \; - \; l(l+1) \right) + \left(j(j+1) \; + \; l(l+1) \; - \; s(s+1) \right)}{2j(j+1)} j \, \mu_N$$

and for a neutron with orbital angular momentum l' and total angular momentum j' we get (not contribution from the orbital angular momentum because the neutron is uncharged)

$$\mu \;=\; -\frac{3.82\,(j'(j'+1)\;+\;s(s+1)\;-\;l'(l'+1))}{2j'(j'+1)}j'\,\mu_N$$

Thus, for example if we consider the nuclide $^7_3$Li for which there is an unpaired proton in the $2p_{\frac{3}{2}}$ state ($l=1,\; j=\frac{3}{2}$ then the estimate of the magnetic moment is

$$\mu \;=\; \frac{5.58\left(\frac{3}{2}\times\frac{5}{2}\;+\;\frac{1}{2}\times\frac{3}{2}\;-\;1\times 2\right)+\left(\frac{3}{2}\times\frac{5}{2}\;+\;1\times 2\;-\;\frac{1}{2}\times\frac{3}{2}\right)}{2\times\frac{3}{2}\times\frac{5}{2}}\frac{3}{2}\;=\;3.79\mu_N$$

The measured value is $3.26\mu_N$ so the estimate is not too good. For heavier nuclei the estimate from the shell model gets much worse.

The precise origin of the magnetic dipole moment is not understood, but in general they cannot be predicted from the shell model. For example for the nuclide $^{17}_{9}$F (fluorine), the measured value of the magnetic moment is $4.72\mu_N$ whereas the value predicted form the above model is $-0.26\mu_N$. !! There are contributions to the magnetic moments from the nuclear potential that is not well-understood.

5.5 Excited States

As in the case of Atomic Physics, nuclei can be in excited states, which decay via the emission of a photon (γ-ray) back to their ground state (either directly ore indirectly).

Some of these excited states are states in which one of the neutrons or protons in the outer shell is promoted to a higher energy level.

However, unlike Atomic Physics, it is also possible that sometimes it is energetically cheaper to promote a nucleon from an inner closed shell, rather than a nucleon form an outer shell into a high energy state. Moreover, excited states in which more than one nucleon is promoted above its ground state is much more common in Nuclear Physics than in Atomic Physics.

Thus the nuclear spectrum of states is very rich indeed, but very complicated and cannot be easily understood in terms of the shell model.

Most of the excited states decay so rapidly that their lifetimes cannot be measured. There are some excited states, however, which are metastable because they cannot decay without violating the selection rules. These excited states are known as "isomers", and their lifetimes can be measured.

5.6 The Collective Model

The Shell Model has its shortcomings. This is particularly true for heavier nuclei. We have already seen that the Shell Model does not predict magnetic dipole moments or the spectra of excited states very well.

One further failing of the Shell Model are the predictions of electric quadrupole moments, which in the Shell Model are predicted to be very small. However, heavier nuclei with A in the range 150 - 190 and for A > 220, these electric quadrupole moments are found to be rather large.

The failure of the Shell Model to correctly predict electric quadrupole moments arises from the assumption that the nucleons move in a spherically symmetric potential.

The Collective Model generalises the result of the Shell Model by considering the effect of a non-spherically symmetric potential, which leads to substantial deformations for large nuclei and consequently large electric quadrupole moments.

One of the most striking consequences of the Collective Model is the explanation of low-lying excited states of heavy nuclei. These are of two types

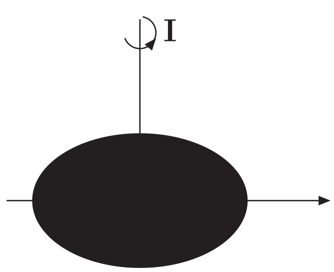
- **Rotational States:** A nucleus whose nucleon density distributions are spherically symmetric (zero quadrupole moment) cannot have rotational excitations (this is analogous to the application of the principle of equipartition of energy to monatomic molecules for which there are no degrees of freedom associated with rotation).

  On the other hand a nucleus with a non-zero quadrupole moment can have excited levels due to rotational perpendicular to the axis of symmetry.

  For an even-even nucleus whose ground state has zero spin, these states have energies

$$E_{\mathrm{rot}} = \frac{I(I+1)\,\hbar^2}{2\mathcal{I}}, \tag{5.6.2}$$

  where $\mathcal{I}$ is the moment of inertia of the nucleus about an axis through the centre perpendicular to the axis of symmetry.



  It turns out that the rotational energy levels of an even-even nucleus can only take even values of I. For example the nuclide $^{170}_{72}\mathrm{Hf}$ (hafnium) has a series of rotational states with excitation energies

$$\text{E (KeV)}: \quad 100, \quad 321, \quad 641$$

These are almost exactly in the ratio $2 \times 3 : \ 4 \times 5 : \ 6 \times 7$, meaning that these are states with rotational spin equal to 2, 4, 6 respectively. The relation is not exact because the moment of inertia changes as the spin increases.

We can extract the moment of inertia for each of these rotational states from eq.(5.6.2). We could express this in SI units, but more conveniently nuclear moments of inertia are quoted in MeV/c^2 fm^2, with the help of the relation

$$\hbar c \ = \ 197.3 \, \mathrm{MeV\, fm}.$$

Therefore the moment inertia of the $I = 2$ state, whose excitation energy is 0.1 MeV, is given (inseting $I = 2$ into eq.(5.6.2) by

$$\mathcal{I} \ = 2 \times 3 \times \frac{\hbar^2 c^2}{2c^2 E_{\mathrm{rot}}} \ = \ \frac{6}{2} \frac{197.3^2}{0.1} \ = \ 1.17 \times 10^6 \, \mathrm{MeV/c^2\, fm^2}$$

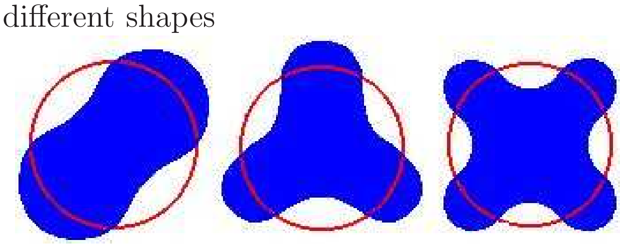
For odd-A nuclides for which the spin of the ground state I_0 is non-zero, the rotational levels have excitation levels of

$$E_{\mathrm{rot}} \ = \ \frac{1}{2\mathcal{I}} \left(I(I+1) \ - \ I_0(I_0+1) \right) \hbar^2,$$

where I can take the values $I_0 + 1, \ I_0 + 2$ etc. For example the first two rotational excitation energies of $^{143}_{60}$Nd (neodynium), whose ground state has spin $\frac{7}{2}$, have energies 128 KeV and 290 KeV. They correspond to rotational levels with nuclear spin $\frac{9}{2}$ and $\frac{11}{2}$ respectively. The ratio of these two excitation energies (2.27) is almost exactly equal to

$$\frac{\frac{11}{2} \times \frac{13}{2} - \frac{7}{2} \times \frac{9}{2}}{\frac{9}{2} \times \frac{11}{2} - \frac{7}{2} \times \frac{9}{2}} \ = \ 2.22$$

- **Shape oscillations:** These are modes of vibration in which the deformation of the nucleus oscillates - the electric quadrupole moment oscillates about its mean value. It could be that this mean value is very small, in which case the nucleus is oscillating between an oblate and a prolate spheroidal shape. It is also possible to have shape oscillations with different shapes

  The small oscillations about the equilibrium shape perform simple harmonic motion. The energy levels of such modes are equally spaced. Thus an observed sequence of equally spaced energy levels within the spectrum of a nuclide is interpreted as a manifestation of such shape oscillations.

Chapter 6

Radioactivity

Some nuclides have a far higher binding energy than some of its neighbours. When this is the case it is often energetically favourable for a nuclide with a low binding energy ("parent nucleus") to decay into one with a higher binding energy ("daughter nucleus"), giving off either an α-particle, which is the a ${}^{4}_{2}$He (helium) nucleus (α-decay) or an electron (positron) and another very low mass particle called a "antineutrino" ("neutrino"). This is called "β-decay". The difference in the binding energies is equal to the kinetic energy of the decay products

A further source of radioactivity arises when a nucleus in a metastable excited state ("isomer") decays directly or indirectly to its ground state emitting one or more high energy photons (γ-rays).

6.1 Decay Rates

The probability of a parent nucleus decaying in one second is called the "decay constant", (or "decay rate") λ. If we have $N(t)$ nuclei then the number of 'expected' decays per second is $\lambda\, N(t)$. The number of parent nuclei decreases by this amount and so we have

$$\frac{dN(t)}{dt} = -\lambda N(t). \tag{6.1.1}$$

This differential equation has a simple solution - the number of parent nuclei decays exponentially -

$$N(t) = N_0 e^{-\lambda t},$$

where N_0 is the initial number of parent nuclei at time $t = 0$.

The time taken for the number of parent nuclei to fall to $1/e$ of its initial value is called the "mean lifetime", τ of the radioactive nucleus, and we can see from eq.(6.1.1) that

$$\tau = \frac{1}{\lambda}.$$

Quite often one talks about the "half-life", $\tau_{\frac{1}{2}}$ of a radioactive nucleus, which is the time taken for the number of parent nuclei to fall to one-half of its initial value. From eq.(6.1.1) we can also see that

$$\tau_{\frac{1}{2}} = \frac{\ln 2}{\lambda} = \ln 2\,\tau.$$

6.2 Random Decay

It was stated above that the "expected" number of decays per second would be $\lambda N(t)$. This does not mean that there will always be precisely this number of decays per second.

Radioactive decay is a random process with a *probability* λ that any one nucleus will decay in one second.

The laws of random distributions tell us that if the expected number of events in a given period of time is ΔN, then the 'error' on this number is $\sqrt{\Delta N}$. More precisely there is a 68% probability that the number of events will be in the range

$$\Delta N - \sqrt{\Delta N} \rightarrow \Delta N + \sqrt{\Delta N}.$$

This means that if we want to measure the decay constant (lifetime, half-life) to within an accuracy of ϵ, we need to collect at least $1/\epsilon^2$ decays.

For example, suppose we have a sample with 10^{12} radioactive nuclei with a mean lifetime of about 10^{10} seconds and we want to measure this lifetime then in 1 second we predict that there will be (with 68% certainty) between

$$\frac{10^{12}}{10^{10}} - \sqrt{\frac{10^{12}}{10^{10}}} = 100 - 10 = 90 \text{ and } \frac{10^{12}}{10^{10}} + \sqrt{\frac{10^{12}}{10^{10}}} = 100 + 10 = 110,$$

decays per second. So if we want to determine the lifetime to better than 1% we need to observe the decays for 100 secs, for which we expect to have between 9900 and 10100 decays.

One decay per second is a unit of radioactivity known as the Bequerel (Bq) after the person who discovered radioactivity. Radioactivity is more often measured in Curies where one Curie is 3.7×10^{10} decays per second. This is the number of decays per second of one gram of $^{226}_{88}$Ra (radium).

What is the half-life of $^{226}_{88}$Ra?
Neglecting the binding energy the mass of Ra nucleus is

$$M_{Ra} = 88m_p + (226 - 88)m_n = 3.77 \times 10^{-25}\,kg$$

The number of nuclei in one gram is

$$N_0 = \frac{10^{-3}}{3.77 \times 10^{-25}} = 2.67 \times 10^{21}.$$

Therefore of the number of decays per second is 3.7×10^{10} for 2.67×10^{21} nuclei of Ra, we have for the decay constant

$$\lambda = \frac{3.7 \times 10^{10}}{2.67 \times 10^{21}} = 1.39 \times 10^{-11} \ s^{-1},$$

which gives us a half-life of

$$\tau_{\frac{1}{2}} = \frac{\ln 2}{\lambda} = \frac{0.693}{1.39 \times 10^{-11}} = 5 \times 10^{10} \ s \ \ (1620 \text{ yr})$$

6.3 Carbon Dating

Living organisms absorb the isotope of carbon ${}^{14}_{6}\mathrm{C}$, which is created in the atmosphere by cosmic ray activity. The production of ${}^{14}_{6}\mathrm{C}$ from cosmic ray bombardment exactly cancels the rate at which thatbisotope decays so that the global concentration of ${}^{14}_{6}\mathrm{C}$ remians constant.

A sample of carbon taken from a living organism will have a concentration of one part in 1.3×10^{12}, and it is being continually rejuvenated, by exchanging carbon with the environment (either by photosynthesis or by eating plants which have undergone photosynthesis or by eating other animals that have eaten such plants.)

On the other hand a sample of carbon from a dead object cannot exchange its carbon with the environment and therefore cannot rejuvenate its concentration of ${}^{14}_{6}\mathrm{C}$.

${}^{14}_{6}\mathrm{C}$ decays radioactively into ${}^{14}_{7}\mathrm{N}$ (nitrogen), via β-decay with a half-life of 5730 years.

Thus by measuring the concentration of the isotope ${}^{14}_{6}\mathrm{C}$ in a fossil sample using techniques of mass spectroscopy, the age of the fossil can be determined.

6.4 Multi-modal Decays

A radioactive nucleus can sometimes decay into more than one channel, each of which has its own decay constant.

An example of this is ${}^{212}_{83}\mathrm{Bi}$ (bismuth) which can either decay as

$${}^{212}_{83}\mathrm{Bi} \rightarrow {}^{208}_{81}\mathrm{Ti} + \alpha$$

or

$${}^{212}_{83}\mathrm{Bi} \rightarrow {}^{212}_{84}\mathrm{Po} + e^{-} + \bar{\nu}$$

with a total mean lifetime of 536 secs. Ratio of ${}^{208}_{81}\mathrm{Ti}$ (titanium) to ${}^{212}_{84}\mathrm{Po}$ (polonium) from these decays is 9:16 What are the decay constants λ_1 and λ_2 for each of these decay modes? The rate of change of the number of parent nuclei is given by

$$\frac{dN(t)}{dt} = -\lambda_1 N(t) - \lambda_2 N(t),$$

with solution

$$N(t) \;=\; N_0 e^{-(\lambda_1+\lambda_2)t}.$$

From the total lifetime we have

$$\lambda_1 + \lambda_2 \;=\; \frac{1}{536} \;=\; 1.86\times 10^{-3}\, s^{-1}$$

The ratio of the number of decay products is equal to the ratio of the decay constants, i.e.

$$\frac{\lambda_1}{\lambda_2} \;=\; \frac{9}{16}$$

This gives us

$$\lambda_1 \;=\; 6.8\times 10^{-4}\; s^{-1}.$$
$$\lambda_2 \;=\; 11.8\times 10^{-3}\; s^{-1}.$$

6.5 Decay Chains

It is possible that a parent nucleus decays, with decay constant λ_1 into a daughter nucleus, which is itself radioactive and decays (either into a stable nuclide or into another radioactive nuclide) with decay constant λ_2. An example of this is

$$^{210}_{83}\mathrm{Bi} \stackrel{\beta}{\rightarrow} {}^{210}_{84}\mathrm{Po} \stackrel{\alpha}{\rightarrow} {}^{206}_{82}\mathrm{Pb}$$

The mean lifetime for the first stage of decay is 7.2 days and the mean lifetime for the second stage is 200 days.

If at time t we have $N_1(t)$ nuclei of the parent nuclide and $N_2(t)$ nuclei of the daughter nuclide, then for $N_1(t)$ we simply have

$$\frac{dN_1(t)}{dt} \;=\; -\lambda_1 N_1(t) \qquad (6.5.2)$$

and therefore,

$$N_1(t) = N_1(0)e^{-\lambda_1 t}, \qquad (6.5.3)$$

whereas for N_2 there is a production mechanism which contributes a rate of increase of N_2 equal to the rate of *decrease* of N_1. In addition there is a contribution to the rate of decrease of N_2 from its decay process, so we have

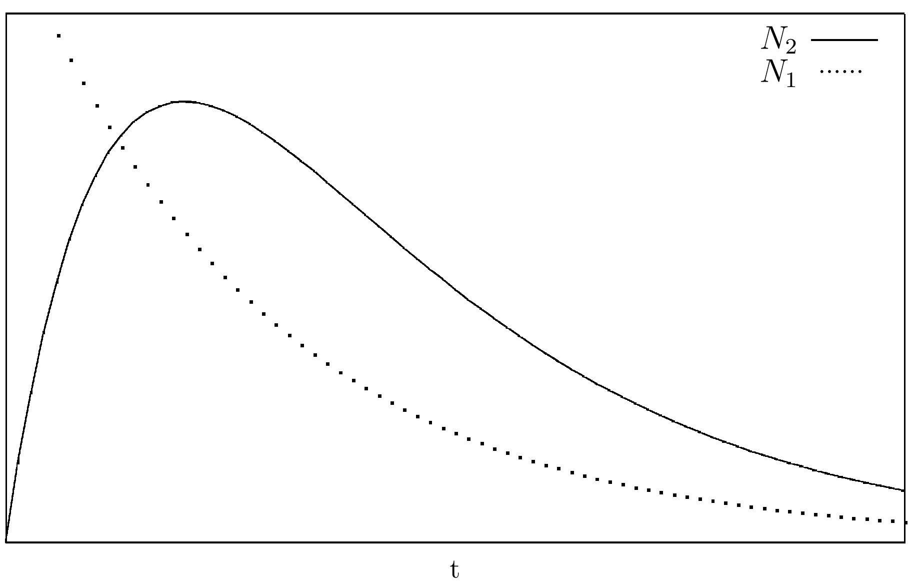
$$\frac{dN_2(t)}{dt} \;=\; \lambda_1 N_1(t) - \lambda_2 N_2(t) \qquad (6.5.4)$$

Inserting the solution of eq.(6.5.2) into eq.(6.5.4) gives

$$\frac{dN_2(t)}{dt} \;=\; \lambda_1 N_1(0)e^{-\lambda_1 t} - \lambda_2 N_2(t).$$

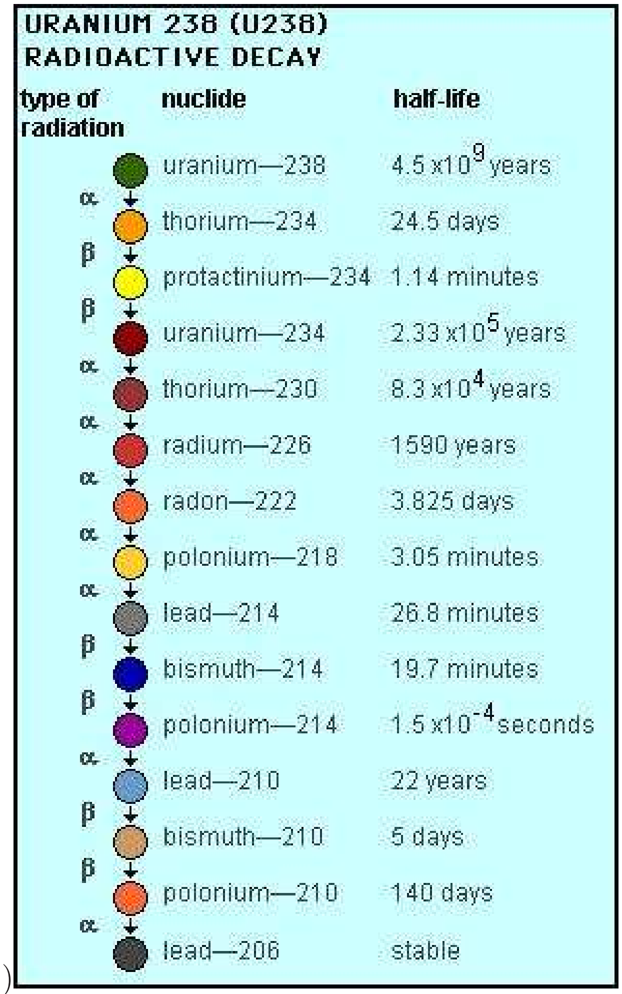
This is an inhomogeneous differential equation whose solution with $N_2(0) = 0$ is given by

$$N_2(t) \;=\; N_1(0)\frac{\lambda_1}{(\lambda_2 - \lambda_1)}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$



What is happening is that initially as the parent decays the quantity of the daughter nuclide grows faster than it decays. But after some time the available quantity of the parent nuclide is depleted so the production rate decreases and the decay rate of the daughter nuclide begins to dominate so that the quantity of the daughter nuclide also decreases.

Some heavy nuclides have a very long decay chain, decaying at each stage to another unstable nuclide before eventually reaching a stable nulcide. An example of this is ${}^{238}_{92}\mathbf{U}$, which decays in no fewer than 14 stages - eight by α-decay and six by β-decay before reaching a stable isotope of Pb. The lifetimes for the individual stages vary from around 10^{-4} s. to 10^9 years.



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In such cases, if the first parent is very long-lived, so that the number of parent nuclei does not decrease much, it is possible to reach what is known as "secular equilibrium", in which the quantities of various daughter nuclei remains unchanged. This happens when the numbers of nuclei in the chain N_A, $N_B\, N_C \cdots$ are in the ratio

$$\lambda_A N_A \;=\; \lambda_B N_B, \quad \text{etc.},$$

where λ_A, $\lambda_B \cdots$ are the decay rates for these nuclides. What is happening here is that the rate of production of daughter B, is the rate of decay of A, which is $\lambda_A\, N_A$ and this is equal to $\lambda_B\, N_B$, the rate of decay of B, so the quantity of B nuclei remains unchanged.

6.6 Induced Radioactivity

It is possible to convert a nuclide which is not radioactive into a radioactive one by bombarding it with neutrons or other particles. The stable nuclide (sometimes) absorbs the projectile in order to become an unstable, radioactive nucleus.

For example bombarding ${}^{23}_{11}\mathrm{Na}$ (sodium) with neutrons can convert the nuclide to ${}^{24}_{11}\mathrm{Na}$, which is radioactive and decays via β-decay to ${}^{24}_{12}\mathrm{Mg}$ (magnesium).

In this case if we assume that the rate at which the radioactive nuclide (with decay constant λ) is being generated is R, then the number of such nuclei is given by the differential equation

$$\frac{dN(t)}{dt} = R - \lambda N$$

If at time $t = 0$ the number of these nuclei is zero (i.e. we start the bombardment at $t = 0$) then the solution to this differential equation is

$$N(t) = \frac{R}{\lambda}\left(1 - e^{-\lambda t}\right)$$

This starts at zeros and then grows so that asymptotically

$$R = \lambda N,$$

which is the equilibrium state in which the production rate R is equal to the decay rate λN.

Chapter 7

Alpha Decay

α- decay is the radioactive emission of an α-particle which is the nucleus of ${}^4_2$He, consisting of two protons and two neutrons. This is a very stable nucleus as it is doubly magic. The daughter nucleus has two protons and four nucleons fewer than the parent nucleus.

$$ {}^{(A+4)}_{(Z+2)}\{P\} \;\rightarrow\; {}^{A}_{Z}\{D\} \;+\; \alpha. $$

7.1 Kinematics

The "Q-value" of the decay, Q_α is the difference of the mass of the parent and the combined mass of the daughter and the α-particle, multiplied by c^2.

$$ Q_\alpha \;=\; (m_P \;-\; m_D \;-\; m_\alpha)\, c^2. $$

The mass difference between the parent and daughter nucleus can usually be estimated quite well from the Liquid Drop Model. It is also equal to the difference between the sum of the binding energies of the daughter and the α-particles and that of the parent nucleus.

The α-particle emerges with a kinetic energy T_α, which is slightly below the value of Q_α. This is because if the parent nucleus is at rest before decay there must be some recoil of the daughter nucleus in order to conserve momentum. The daughter nucleus therefore has kinetic energy T_D such that

$$ Q_\alpha \;=\; T_\alpha \;+\; T_D $$

The momenta of the α-particle and daughter nucleus are respectively

$$ p_\alpha \;=\; \sqrt{2 m_\alpha T_\alpha}, $$

$$ p_D \;=\; -\sqrt{2 m_D T_D}, $$

where m_D is the mass of the daughter nucleus (we have taken the momentum of the α-particle to be positive). Conserving momentum implies $p_\alpha + p_D = 0$ which leads to

$$ T_D = \frac{m_\alpha}{m_D} T_\alpha, $$

and neglecting the binding energies, we have

$$\frac{m_\alpha}{m_D} = \frac{4}{A},$$

where A is the atomic mass number of the daughter nucleus. We therefore have for the kinetic energy of the α-particle

$$T_\alpha = \frac{A}{(A+4)} Q_\alpha.$$

Example:
The binding energy of ${}^{214}_{84}\mathrm{Po}$ is 1.66601 GeV, the binding energy of ${}^{210}_{82}\mathrm{Pb}$ (lead) is 1.64555 GeV and the binding energy of ${}^{4}_{2}\mathrm{He}$ is 28.296 MeV. The Q-value for the decay

$${}^{214}_{84}\mathrm{Po} \rightarrow {}^{210}_{82}\mathrm{Pb} + \alpha,$$

is therefore

$$Q_\alpha = 1645.55 + 28.296 - 1666.02 = 7.83\,\mathrm{MeV}.$$

The kinetic energy of the α-particle is then given by

$$T_\alpha = \frac{210}{214} \times 7.83 = 7.68\,\mathrm{MeV}.$$

Sometimes the α-particles emerge with kinetic energies which are somewhat lower than this prediction. Such α-decays are accompanied by the emission of γ-rays. What is happening is that the daughter nucleus is being produced in one of its excited states, so that there is less energy available for the α-particle (or the recoil of the daughter nucleus).

Example:
The binding energy of ${}^{228}_{90}\mathrm{Th}$ (thorium) is 1.743077 GeV, the binding energy of ${}^{224}_{88}\mathrm{Ra}$ (radium) is 1.720301 GeV and the binding energy of ${}^{4}_{2}\mathrm{He}$ is 28.296 MeV. The Q-value for the decay

$${}^{228}_{90}\mathrm{Th} \rightarrow {}^{224}_{88}\mathrm{Ra} + \alpha,$$

is therefore

$$Q_\alpha = 1720.301 + 28.296 - 1743.077.02 = 5.52\,\mathrm{MeV}.$$

The kinetic energy of the α-particle is then given by

$$T_\alpha = \frac{224}{228} \times 5.52 = 5.42\,\mathrm{MeV}.$$

$\alpha-$particles are observed with this kinetic energy, but also with kinetic energies 5.34, 5.21, 5.17 and 5.14 MeV.

From this we can conclude that there are excited states of ${}^{224}_{88}\mathrm{Ra}$ with energies of 0.08, 0.21, 0.25 and 0.28 MeV. The α-decay is therefore accompanied by γ-rays (photons) with energies equal to the differences of these energies.

It is sometimes possible to find an α-particle whose energy is *larger* than that predicted from the Q-value. This occurs when the parent nucleus is itself a product of a decay from a further ('grand'-)parent. In this case the parent α-decaying nucleus can be produced in one of its excited states. In most cases this state will decay to the ground state by emitting γ-rays before the α-decay takes places. But in some cases where the excited state is relatively long-lived and the decay constant for the α-decay is large the excited state can α-decay directly and the Q-value for such a decay is larger than for decay form the ground state by an amount equal to the excitation energy.

In the above example of α-decay from $^{214}_{84}$Po (polonium) the parent nucleus is actually unstable and is produced by β-decay of $^{214}_{83}$Bi (bismuth). $^{214}_{84}$Po has excited states with energies 0.61, 1.41, 1.54, 1.66 MeV above the gound state. Therefore as well as an α-decay with Q-value 7.83 MeV, calculated above, there are α-decays with Q-values of 8.44, 9.24, 9.37 and 9.49 MeV.

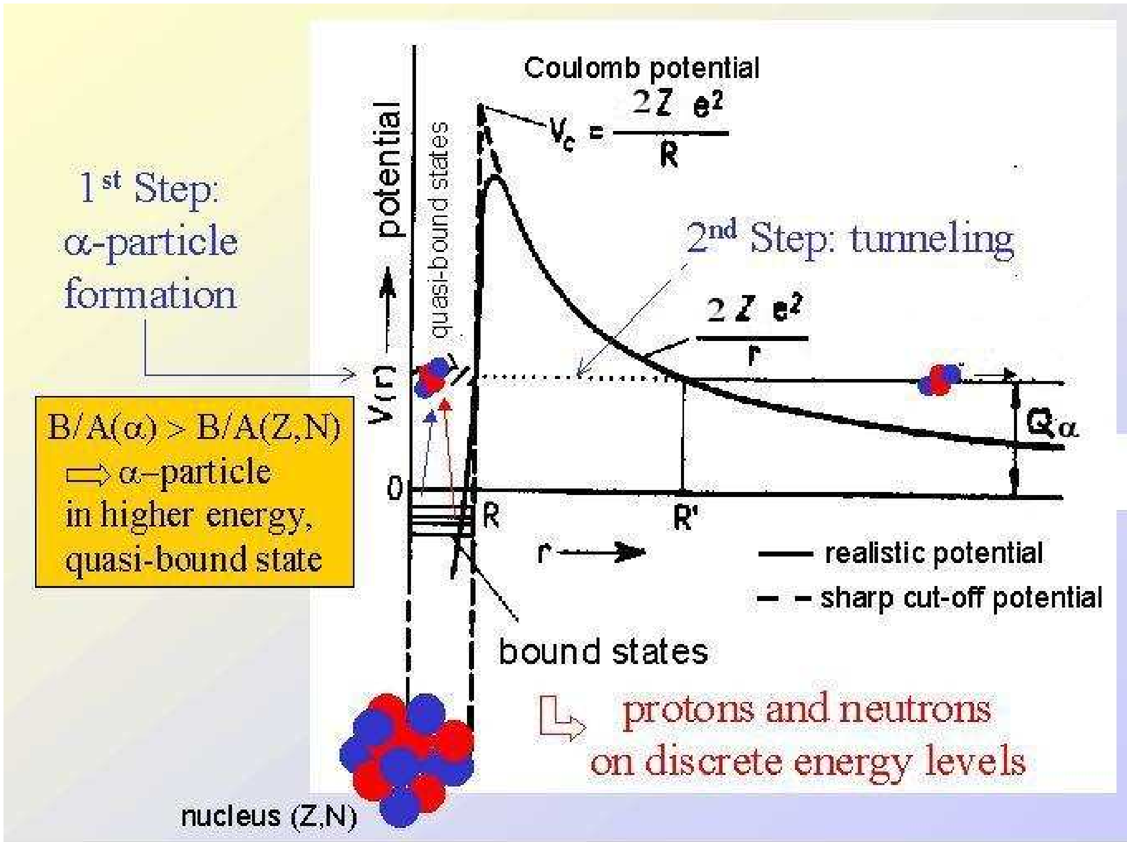
7.2 Decay Mechanism

The mean lifetime of α-decaying nuclei varies from the order of 10^{-7} secs to 10^{10} years.

We can understand this by investigating the mechanism for α-decay.

What happens is that two protons from the highest proton energy levels and two neutrons from the highest neutron energy levels combine to form an α-particle inside the nucleus - this is known as a "quasi-bound-state". It acquires an energy which is approximately equal to Q_α (we henceforth neglect the small correction due to the recoil of the nucleus).

The α-particle is bound to the potential well created by the strong, short-range, nuclear forces. There is also a Coulomb repulsion between this 'quasi-' α-particle and the rest of the nucleus.



Together these form a potential barrier, whose height, V_c, is the value of the Coulomb potential at the radius, R, of the nucleus (where the strong interactions are rapidly attenuated).

$$V_c = \frac{2Ze^2}{4\pi\epsilon_0 R},$$

where Ze is the electric charge of the daughter nucleus.

The barrier extends from $r = R$, the nuclear radius to $r = R'$, where

$$Q_\alpha = \frac{2Ze^2}{4\pi\epsilon_0 R'}.$$

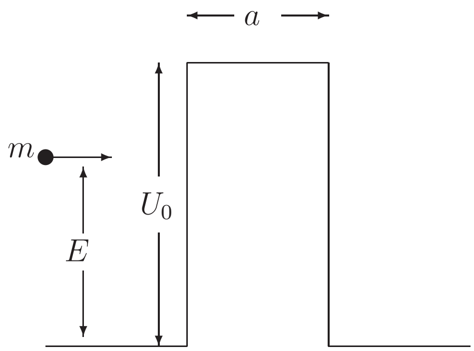
Beyond R' the α-particle has enough energy to escape.

Using classical mechanics, the α-particle does not have enough energy to cross this barrier, but it *can* penetrate through via quantum tunnelling.

For a square potential of height U_0 and width a, the tunnelling probability for a particle with mass, m and energy E, is approximately given by

$$T = \exp\left(-2\sqrt{2m(U_0 - E)}\frac{a}{\hbar}\right).$$

It is this exponential which varies very rapidly with its argument, that is responsible for the huge variation in α-decay constants.



This formula applies to a potential barrier of constant height U_0, whereas for α-decay the potential inside the barrier is

$$U(r) \ = \ \frac{2Ze^2}{4\pi\epsilon_0 r}.$$

The result of this is that the exponent in the above expression is replaced by the integral

$$-\frac{2}{\hbar}\int_R^{R'} \sqrt{2m_\alpha\left(\frac{2Ze^2}{4\pi\epsilon_0 r} - Q_\alpha\right)}\, dr$$

Finally we need to multiply the transition probability by the number of times per sec that the α-particle 'tries' to escape, which is how often it can travel from the centre to the edge of the nucleus and back. This is approximately given by

$$\frac{v}{2R},$$

where $v = \sqrt{2Q_\alpha/m_\alpha}$, is the velocity of the α-particle inside the nucleus.

When all this is done we arrive at the approximate result

$$\ln\lambda \ = \ f \ - \ g\,\frac{Z}{\sqrt{Q_\alpha}},$$

where

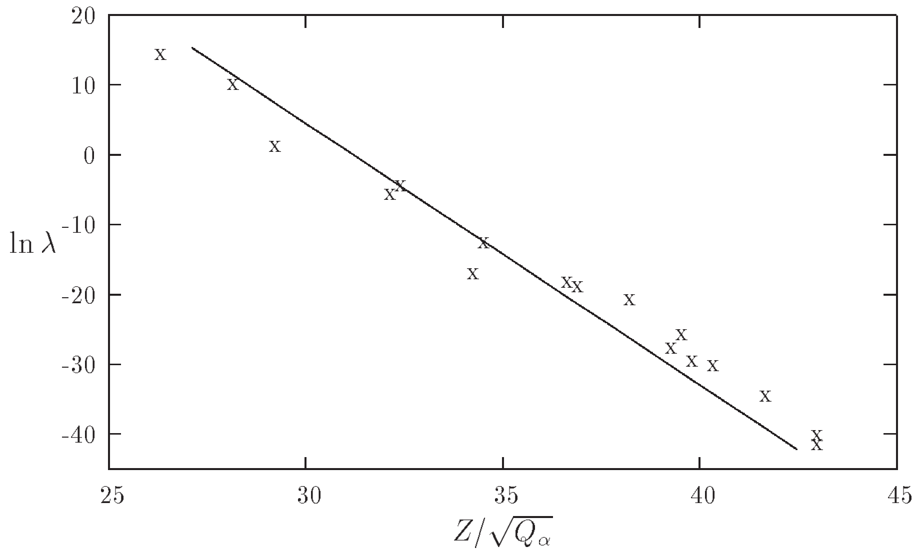
$$g \ = \ 2\sqrt{2}\pi\alpha\sqrt{m_\alpha\, c^2} \ = \ 3.97\,\mathrm{MeV}^{1/2},$$

and

$$f \ = \ \ln\left(\frac{v}{2R}\right) \ + \ 8\sqrt{RZ\alpha m_\alpha c/\hbar}.$$

f varies somewhat for different nuclei but is approximately equal to 128.

This very crude approximation agrees reasonably well with data



We see that as the quantity $Z/\sqrt{Q_\alpha}$ varies over the range 25 - 45, the logarithm of the decay constant varies over a similar range from -45 to 15, but this implies a range of lifetimes from e^{-15} to e^{45} secs (less than a microsecond to longer than the age of the Universe)

Chapter 8

Beta Decay

β-decay is the radioactive decay of a nuclide in which an electron or a positron is emitted.

$$ {}^A_Z\{P\} \ \rightarrow \ {}^A_{(Z+1)}\{D\} + \ e^- \ + \bar{\nu}, $$

or

$$ {}^A_Z\{P\} \ \rightarrow \ {}^A_{(Z-1)}\{D\} + \ e^+ \ + \nu. $$

The atomic mass number is unchanged so that these reactions occur between "isobars".

The electron (or positron) does not exist inside the nucleus but is created in the reaction

$$ n \ \rightarrow \ p + \ e^- \ + \bar{\nu}. $$

In fact the neutron has a mass that exceeds the sum of the masses of the proton plus the electron so that a free neutron can undergo this decay with a lifetime of about 11 minutes.

Inside a nucleus such a decay is not always energetically allowed because of the difference in the binding energies of the parent and daughter nuclei. When a neutron is converted into a proton the Coulomb repulsion between the nucleons increases - thereby decreasing the binding energy. Moreover there is a pairing term in the semi-empirical mass formula that favours even numbers of protons and neutrons and a symmetry term that tells us that the number of protons and neutrons should be roughly equal.

β-decay is energetically permitted provided the mass of the parent exceeds the mass of the daughter plus the mass of an electron.

$$ M(Z,A) \ > \ M((Z+1),A) + m_e, $$

for electron emission, and

$$ M(Z,A) \ > \ M((Z-1),A) + m_e, $$

for positron emission. In the latter case a proton is converted into a more massive neutron, but the binding energy of the daughter may be such that the total nuclear mass of the daughter is less than that of the parent by more than the electron mass, m_e.

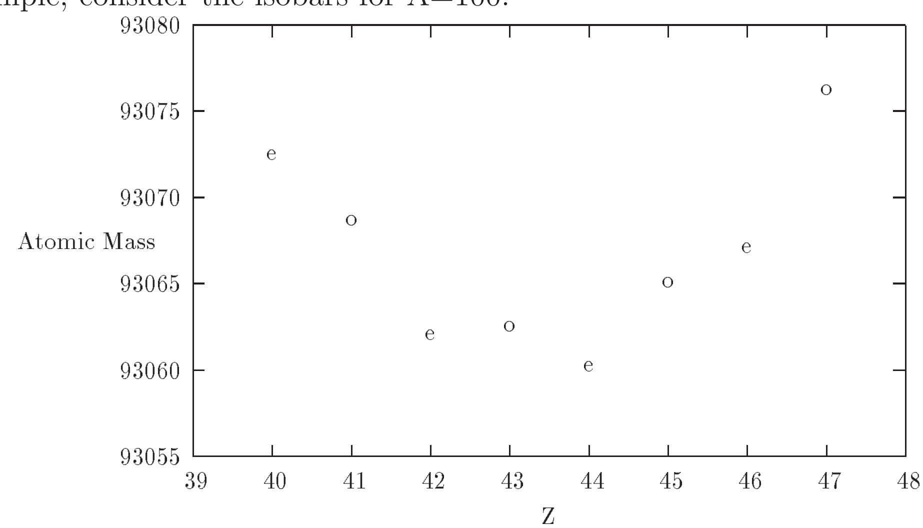
The mass of the electron can be included directly by comparing atomic masses, since a neutral atom always has Z electrons. Thus we require

$$\mathcal{M}(Z, A) \; > \; \mathcal{M}((Z+1), A)$$

for electron emission. The atomic (as opposed to nuclear) mass included the masses of the electrons. However, this will not work for positron emission, for which Z *decreases* by one unit.

For nuclei with even A, it turns out that because of the pairing term in the binding energy, nuclides with odd numbers of protons and neutrons (odd-odd nuclides) are nearly always unstable against β- decay. On the other hand, even-even nuclides can also sometimes be unstable against β-decay if the number of neutrons in a particular isobar is too large or too small for stability.

For example, consider the isobars for A=100.



We note that all the odd-odd nuclides marked "o" have a larger atomic mass than one of the adjacent even-even (marked "e") nuclides and that for the case of Z=43, *both* electron and positron emission are energetically allowed so that this nuclide (Tc – Technetium) can decay either by electron emission to Z=44 (Ru – Ruthenium) or by positron emission to Z=42 (Mo – Molibdenium). Moreover, the even-even Z=40 nuclide (Zr – Zirconium) can decay by electron emission to Z=41 (Nb – Niobium).

For nuclei with odd A there is either an even number of neutrons *or* an even number of protons. In this case the pairing term does not change from isobar to isobar and the question of stability relies on the balance between the symmetry term which prefers equal numbers of protons and neutrons and the Coulomb terms which prefers fewer protons. For such nuclides there is only one stable isobar, with some atomic number Z_A. This means that the isobars with atomic number $Z > Z_A$ have too many protons for stability can always β-decay emitting a positron, whereas isobars with $Z < Z_A$ have too many neutrons, and can undergo β-decay emitting an electron. The value of Z_A for a given A can be obtained by minimizing the *atomic* mass (including the masses of the electrons) from the semi-empirical mass formula. This gives

$$Z_A = A\frac{2a_A + (m_n - m_p - m_e)c^2/2}{4a_A + a_C A^{2/3}},$$

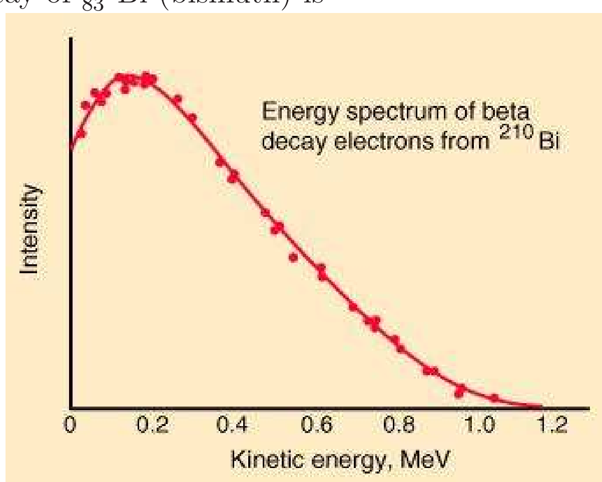
where a_A and a_C are the coefficients of the asymmetry term and Coulomb term in the semi-empirical mass formula.

8.1 Neutrinos

As in the case of α-decay the difference between the mass of the parent nucleus, m_P and the mass of the daughter, m_D plus the electron is the Q-value for the decay, Q_β,

$$Q_\beta = (m_P - m_D - m_e)c^2,$$

and in this case the recoil of the daughter can be neglected because the electron is so much lighter than the nuclei. We would expect this Q-value to be equal to the kinetic energy of the emitted electron, but what is observed is a spectrum of electron energies up to a maximum value which is equal to this Q-value. For example the intensity of electrons with different energies form the β-decay of $^{210}_{83}$Bi (bismuth) is



There is a further puzzle. Since the number of spin-$\frac{1}{2}$ nucleons is the same in the parent and daughter nuclei, the difference in the spins of the parent and daughter nuclei must be an integer. But the electron also has spin-$\frac{1}{2}$, so there appears to be a violation of conservation of angular momentum here.

The solution to both of these puzzles was provided in 1930 by Pauli who postulated the existence of a massless neutral particle with spin-$\frac{1}{2}$ which always accompanies the electron in β-decay. This was called a neutrino. Neutrinos interact very weakly with matter and so they were not actually detected until 1953 (by Reines and Cowan). The fact that the neutrino has spin-$\frac{1}{2}$ means that the total angular momentum can be conserved (if necessary the electron-antineutrino system has orbital angular momentum) and the Q-value is the sum of the energies of the electron and antineutrino. The kinetic energy of the electron can vary from zero (strictly arbitrarily small) where all the Q-value is taken by the antineutrino (the momentum being conserved by the small recoil of the daughter nucleus) to the Q-value in which case the energy carried off by the antineutrino is negligible.

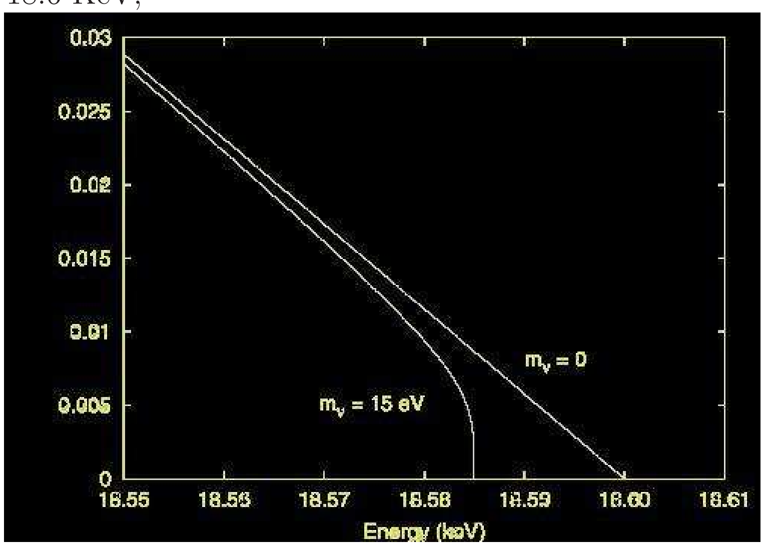
Electrons and neutrinos are examples of “leptons” which are particles that do not interact under the strong nuclear forces - they are not found inside nuclei.

By convention, electrons and neutrinos are assigned a “lepton number” of 1, which means that positrons and antineutrinos have a lepton number of -1. Lepton number is conserved so that it is actually an antineutrino that is emitted together with electron emission β-decay and a neutrino together with positron emission.

The fact that the neutrino has (almost) zero mass is deduced by examining the end-point of the electron energy spectrum. For example for the decay

$$ {}^{3}_{1}\mathrm{H} \;\rightarrow\; {}^{3}_{2}\mathrm{He} + e^{-} + \bar{\nu}, $$

with a Q-value of 18.6 KeV,



For a massless neutrino its total (relativistic) energy can be arbitrarily small and the electron can carry energy up to the Q-value. If the neutrino has a mass, m_ν then the minimum energy that it can have is $m_\nu c^2$, and the electron energy spectrum drops off sharply at the end-point.

It is now known that neutrinos *do* have a tiny mass. The first hint of this was during the observation of the Supernova in 1987, when a burst of neutrinos were observed a few seconds after the burst of γ-rays, implying that the neutrinos had not travelled form the Supernova with exactly the speed of light. This was confirmed by neutrino observation experiments at the Kamiokande neutrino detector in Japan in 1999. However the mass of the neutrino is almost certainly smaller than 0.1 eV/c^2 (compared with the electron mass of 0.511 MeV/c^2). For our purposes we may neglect the neutrino mass.

8.2 Electron Capture

Nuclei which can β-decay emitting a positron and an neutrino, can also decay by another mechanism.

$$e^- + {}^A_Z\{P\} \rightarrow {}^A_{(Z-1)}\{D\} + \nu.$$

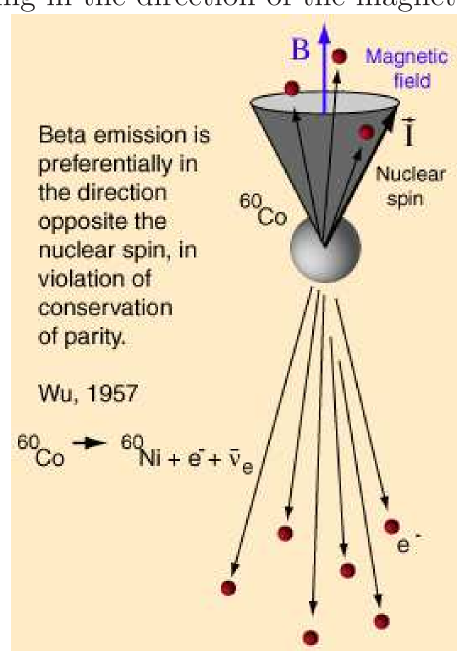
What happens here is that an atom can absorb an electron from one of the inner shells (usually the innermost shell, which is called the "K-shell") and be converted into an atom with one lower atomic number. The energy is entirely carried away by the neutrino and is nearly always undetected because neutrinos interact so weakly with matter.

8.3 Parity Violation

β-decay exhibits a further peculiarity. This was discovered in 1957 by C.S. Wu who observed the decay of radioactive cobalt into nickel

$${}^{60}_{27}\mathrm{Co} \rightarrow {}^{60}_{28}\mathrm{Ni} + e^- + \bar{\nu}.$$

The cobalt sample was kept a low temperature and placed in a magnetic field so that the spin of the cobalt was pointing in the direction of the magnetic field.



She discovered that most of the electrons emerged in the opposite direction from the applied magnetic field. If we write $\mathbf{s}$ for the spin of the parent nucleus and $\mathbf{p_e}$ for the momentum of an emitted electron, this means that the average value of the scalar product $\mathbf{s} \cdot \mathbf{p_e}$ was negative. In order to balance the momentum the antineutrinos are usually emitted in the direction of the magnetic field, so that the average value of $\mathbf{s} \cdot \mathbf{p}_{\bar{\nu}}$ was positive.

Under the parity operation

$$\mathbf{r} \rightarrow -\mathbf{r}$$

and

$$\mathbf{p} \rightarrow -\mathbf{p}$$

but angular momentum which is defined as a vector product

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

is unchanged under parity

$$\mathbf{L} \rightarrow \mathbf{L}.$$

Spin is an internal angular momentum and so it also is unchanged under parity.

But this means that the scalar product $\mathbf{s} \cdot \mathbf{p_e}$ does change under parity

$$\mathbf{s} \cdot \mathbf{p_e} \rightarrow -\mathbf{s} \cdot \mathbf{p_e}$$

so that the fact that this quantity has a non-zero average value (or expectation value in quantum mechanics terms) means that the mechanism of β-decay violates parity conservation

If we viewed the above diagram in the corner of a mirrored room so that all the directions were reversed the spin would point in the same direction, but the electron direction would be reversed so that in that world the electrons would prefer to emerge in the direction of the magnetic field.

The spin of the daughter nucleus $^{60}_{28}$Ni is 4 (it is produced in an excited state) whereas that of the parent $^{60}_{27}$Co was 5, so that in order to compensate for unit of angular momentum lost (in the direction of the magnetic field) the angular momentum the antineutrinos and electrons have their spins in the direction of the magnetic field. This means that the antineutrinos have a spin component $+\frac{1}{2}$ in their direction of motion (in units of $\hbar$) whereas the electrons have a spin component $-\frac{1}{2}$ in their direction of motion. The sign of the component of the spin of a particle in its direction of motion is called the "helicity" of the particle. Neutrinos always have negative helicity (antineutrinos always have positive helicity). An electron can have component of spin either $+\frac{1}{2}$ or $-\frac{1}{2}$ in its direction of motion (either positive or negative helicity). However, the electrons emitted in β-decay usually have negative helicity (positrons emitted in β-decay usually have positive helicity). This means that the mechanism responsible for β-decay (called the "weak interaction") distinguish between positive and negative helicity and therefore violate parity.

Chapter 9

Gamma Decay

As we have seen γ-decay is often observed in conjunction with α- or β-decay when the daughter nucleus is formed in an excited state and then makes one or more transitions to its ground state, emitting a photon whose energy is equal to the energy difference between the initial and final nuclear state. These energy differences are usually of order 100 KeV so the photon is well in the γ-ray region of the electromagnetic spectrum.

The lifetime of excited nuclear states is usually of the order of $10^{-13} - 10^{-12}$ s., so the lifetime is far too short to be measured.

The decay rate (inverse of the mean lifetime) depends on the energy of the photon emitted and the 'type' of radiation.

As in the case of Atomic Physics the transition amplitude is proportional to the matrix element of the electric field between the initial and final wavefunctions of the nucleon that makes the transition. This electric field has a space dependence that may be written

$$E = E_0 e^{i\mathbf{k}\cdot\mathbf{r}},$$

where $\mathbf{k}$ is the wavevector of the emitted photon. For photons of energy 100 KeV and a nucleus of radius a few fm, $\mathbf{k}\cdot\mathbf{r}$ is much less than 1 ($\mathbf{kr} = \frac{\mathbf{p}_k\mathbf{r}}{\hbar} = \frac{Er\cos\theta}{c\hbar} \simeq \frac{10^5\text{ev}\times 1\text{fm}}{1.973\times 10^8\text{ev}\times 1\text{fm}} \simeq 10^{-3}$) and it is sufficient to expand this exponential to first order.

The transition amplitude is therefore proportional to

$$A \propto \int \Psi_f^*(\mathbf{r})\mathbf{k}\cdot\mathbf{r}\Psi_i(\mathbf{r})d^3\mathbf{r},$$

where Ψ_i and Ψ_f are the initial and final wavefunctions of the proton that makes the transition. This is called "electric dipole" transition (there is no "electric monopole" transition from the first term in the expansion of the exponential because $\Psi_f(\mathbf{r})$ and $\Psi_i(\mathbf{r})$ are orthogonal wavefunctions).

The rate for such transition is well approximated by the formula

$$\lambda = 10^5 E_\gamma^3 A^{2/3},$$

where E_γ is the energy of the photon in KeV. The factor of $A^{2/3}$ is understood from the fact that the transition amplitude is proportional to the nuclear radius, which is in turn proportional to $A^{1/3}$ (the transition rate is proportional to the square of the amplitude). For photons with energy of order 100 KeV and A of order 100 this gives $2.5 \times 10^{12}\, s^{-1}$.

However, for the above electric dipole matrix element to be non-zero we require certain conditions on the spin and parity of the initial and final states. As in Atomic Physics, the photon carries away one unit of angular momentum, so that the initial and final nuclear spins have to obey the selection rule

$$\Delta I \;=\; 0, \pm 1 \;\; (I=0 \to I=0 \text{ forbidden})$$

Furthermore since $\mathbf{r}$ is odd under parity reversal, we require the initial and final sates to be of opposite parity, which means that the orbital angular momentum changes by one unit.

If the parity of the initial and final states are the same then the transition is still allowed, but this means that the photon carries away the angular momentum by flipping the spin of the nucleon that makes the transition. For this to happen the magnetic moment of the nucleon interacts with the magnetic field component of the electromagnetic wave associated with the emitted photon. This is called a "magnetic dipole transition" amplitude, and for such a process the transition amplitude is suppressed relative to the amplitude for a typical electric dipole transition by about a factor of

$$\frac{\hbar c}{m_p R}$$

which is about 0.1 for a nucleus of radius a few fm. (and therefore .01 suppression of the decay rate).

Transitions between nuclear states in which the photon is required to carry off more than one unit of angular momentum are permitted. This is because the photon can acquire orbital angular momentum relative to the recoiling nucleus. Thus the total angular momentum change, L in a nuclear transition can take the values

$$|I_i - I_f| \;\leq\; L \;\leq\; |I_i + I_f|,$$

where I_i and I_f are the initial and final nuclear spins. However there is a price to pay in terms of transition rates. For each increase in L there is a suppression in the transition amplitude of kR, because these higher multipole transitions arise from higher orders in the expansion of $\exp(i\mathbf{k}\cdot\mathbf{r})$. For a nucleus of radius a few fm and a photon energy of 100 KeV is a factor of 10^{-3} (so a factor of 10^{-6} in the rate). There is a further suppression factor for higher values of L. A transition will proceed by the lowest allowed value of L.

This is also subject to selection rules for th parity difference between initial and final states, namely

$$\Delta P \;=\; (-1)^L,$$

for electric transitions (written E{L}) with angular momentum L, and and

$$\Delta P \;=\; (-1)^{L-1},$$

for the (even further suppressed) magnetic transitions.

Thus from the initial and final nuclear spins and parities we can determine the "multipolarity" of the transition and whether it is electric or magnetic.

Here are some examples

$$\begin{aligned}2^+ &\rightarrow 1^-, \quad \text{E1}, \\ 2^+ &\rightarrow 1^+, \quad \text{M1}, \\ 3^+ &\rightarrow 1^-, \quad \text{M2}, \\ 3^+ &\rightarrow 1^+, \quad \text{E2}.\end{aligned}$$

Most electromagnetic transitions from an excited state to the ground state have a lifetime which is too short to be measured (less than 1 μs). However, in the Shell Model the energy levels sometimes arrange themselves such that there is a very high spin excited state next to a low spin ground state or vice versa. Such a transition is only permitted by a high multipolarity transition and therefore proceeds very slowly. The excited states then live long enough for their lifetime to be measured and can even be as long a several years. An example is the nuclide ${}^{137}_{56}$Ba (barium) which has an excited state with spin and parity $\frac{11}{2}^-$ next to a ground state of $\frac{3}{2}^+$. The transition is M4 and the excited state has a mean lifetime of around 200 s. These metastable excited states are called "isomers" and there are regions of the Periodic Table known as "islands of isomers" where such metastable excited states are quite common.

9.1 The Mössbauer Effect

In Atomic Physics, it is possible to excite atoms into their excited states by bombarding them with photons with the resonant frequencies, i.e. with energies equal to the energies between the ground state and the excited states.

In nuclei this is not usually possible. The reason for this is to with the small nuclear recoil. The energy of the emitted photon, E_γ is not exactly equal to the excitation energy E_0. The photon carries momentum E_γ/c and so the recoiling nucleus must have equal and opposite momentum . Consequently it acquires a recoil kinetic energy of

$$T = \frac{E_\gamma^2}{2M_N c^2},$$

where M_N is the nuclear mass. The de-excitation energy E_0 is the sum of the photon energy plus this kinetic energy

$$E_0 = E_\gamma + \frac{E_\gamma^2}{2M_N c^2},$$

which has approximate solution

$$E_\gamma \;=\; E_0 \left(1 - \frac{E_0}{2M_N c^2}\right)$$

For a photon of energy 100 KeV and a nucleus with A=100, this recoil energy is about 0.05 eV. Furthermore if we now use the emitted photon to bombard a similar nuclide with the hope of exciting it, we find that the target nucleus also recoils so that the energy that it can absorb in its own rest frame, E_0' is given by

$$E_0' \;=\; E_\gamma \left(1 - \frac{E_0}{2M_N c^2}\right) \approx E_0 \left(1 - \frac{E_0}{M_N c^2}\right),$$

so that E_0' falls short of E_0 by about 0.1 eV (in the above example).

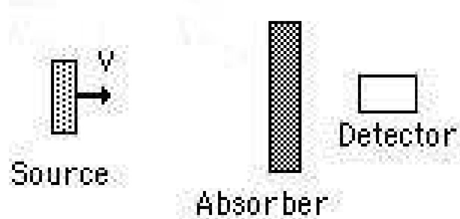
This may not seem much for a photon of energy 100 KeV, but the problem is that even for fast decaying excited states with lifetimes, τ, of about 10^{-12} s., the line-width is given by

$$\Gamma \;=\; \frac{\hbar}{\tau} \approx 10^{-3} \text{ eV},$$

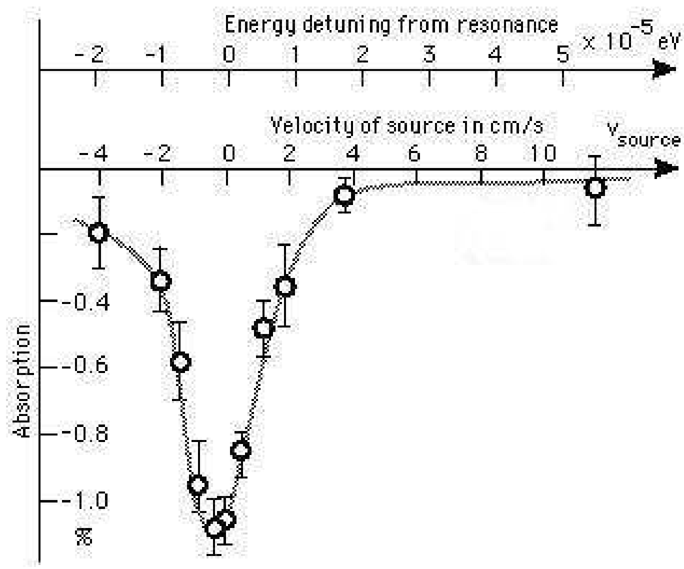
so the difference between the excitation energy E_0 and the energy E_0 that the recoiling nucleus can absorb is much larger larger than the width of the photon, thereby making the absorption impossible.

The way out of this was discovered by Mössbauer. If the source and target nuclei are both fixed in a crystal lattice then the recoil momentum can be taken up by the entire crystal (who mass is many orders of magnitude larger than that of the nucleus) and the recoil energy is negligible.

This is called the Mössbauer effect and it provides an extremely accurate method for measuring the widths of nuclear transitions.



The source and target are both fixed to a crystal and if the source is stationary the intensity of $\gamma-$rays reaching the detector is small because most of them are absorbed by the target.



If the source is moving by as little as a few cm per second there is an increase in the intensity of γ-rays reaching the detector, because the Doppler effect of the γ-rays from the source causes the incident photons to be just off-resonance. Line widths can be measured this way to an accuracy of 10^{-5} eV. If the source is moved with velocity v then using the Doppler shift the difference $\Delta\lambda$ between the wavelength of the emitted photon (in the rest frame of th emitter) and the wavelength of the absorbed photon is

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c},$$

(we are able to use the non-relativitic Doppler effect for such small velocities). In terms of photon energies we may write this as

$$\frac{\Delta E}{E} = \frac{v}{c}.$$

Now if for this velocity the absorption has fallen to approximately one half of the peak absorption (for $v = 0$) then this value of ΔE corresponds to the half width, $\frac{1}{2}\Gamma$ of the spectral line. Thus we end up with an expression for the line-width for a photon of energy E

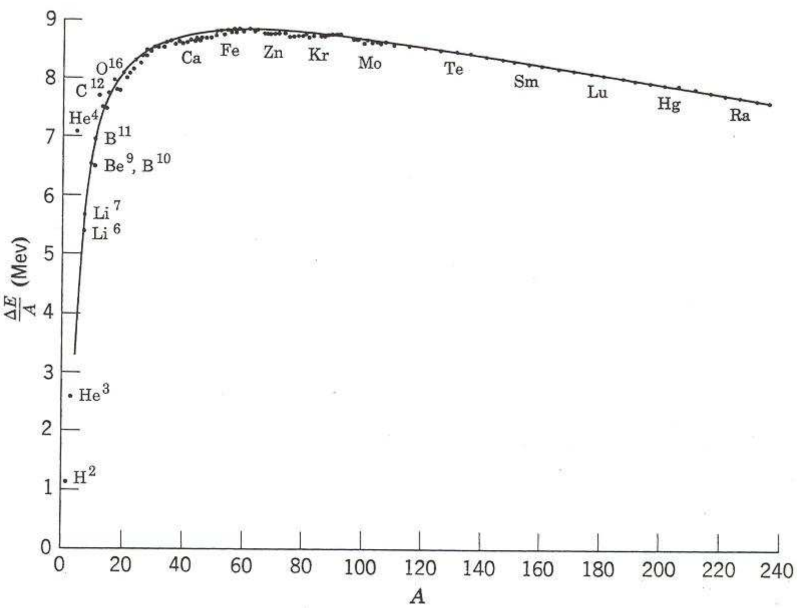
$$\Gamma = 2E\frac{v_{1/2}}{c},$$

where $v_{1/2}$ is the velocity for which the absorption falls to one-half of its peak value.

Chapter 10

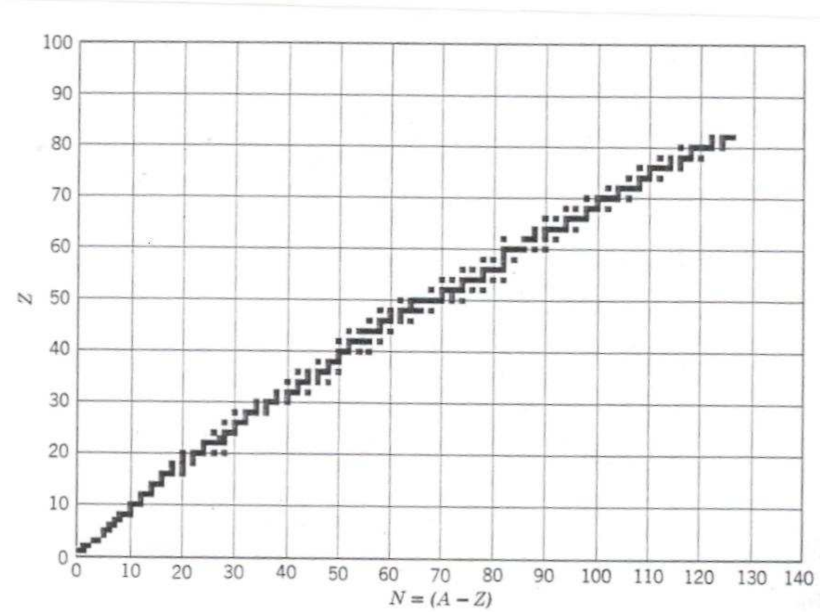
Nuclear Fission

If we look again at the binding energies (per nucleon) for different nuclei, we note that the heavier nuclei have a smaller binding energy than those in the middle of the Periodic Table.



This means that it is energetically favourable for a heavy nucleus (with A greater than about 100) to split into two fragments of smaller nuclei, thereby releasing energy which goes into the kinetic energy of the fragments. This process is called "nuclear fission".

If we also recall the stability curve, which plots the number of protons against the number of neutrons for the stable nuclides



we observe that the heavier nuclei prefer more neutrons (compared with the number of protons) than the lighter ones. This tells us that along with the fission process there will be some 'spare' neutrons emitted. These neutrons will also take up some of the energy released. There are usually 3 or 4 neutrons emitted per fission reaction.

An example of nuclear fission is

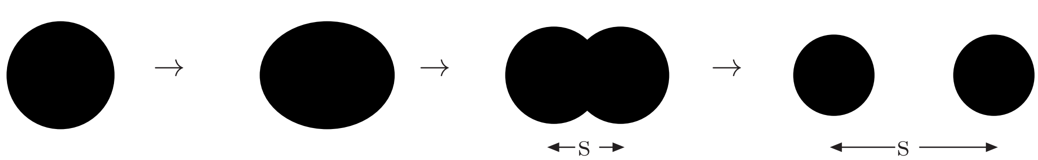
$$^{238}_{92}\mathrm{U} \;\rightarrow\; ^{145}_{57}\mathrm{La} + \,^{90}_{35}\mathrm{Br} + 3n$$

Note that the fission fragments do *not* have atomic mass numbers close to one half of the atomic mass number of the parent, but rather their atomic mass numbers are separated by about 50. This is normally the case - (the reason for this is not well-understood).

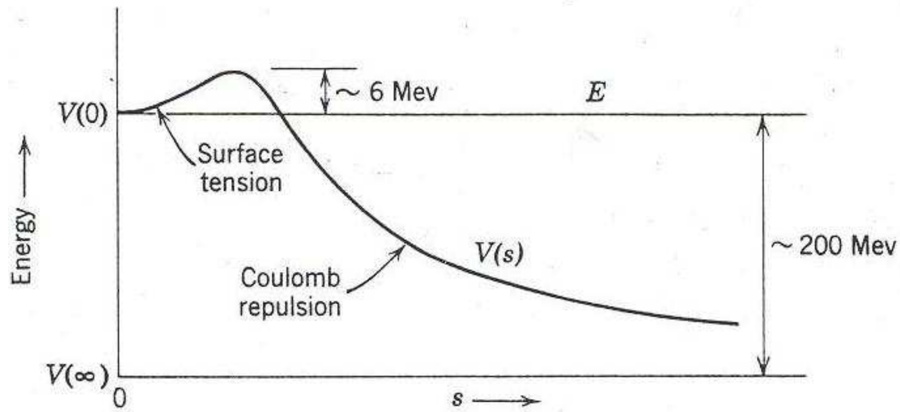
The binding energies of $^{238}_{92}$U, $^{145}_{57}$La and $^{90}_{35}$Br are 1803 Mev, 1198 MeV and 763 MeV respectively. This means that this reaction releases 1198+763-1803 = 158 MeV of energy.

Such spontaneous fission processes are known in Nature, but they are very rare. The mean lifetime for the above process is about 10^{17} years, compared to the mean lifetime for α-decay from the same nuclide which is about 10^{11} years.

The reason for this is that in order to split into two parts the nucleus must first undergo a deformation of its shape (a 'stretching') into an ellipsoidal shape and then to develop a 'neck' in the middle before it finally breaks into two nuclei.



In its deformed state there are two forces acting on the nucleus. One is an increased surface energy (surface tension of a liquid drop - which explains why liquid drops are spherical), and the other is the Coulomb repulsion between the fission fragments. Together these produce a potential barrier.

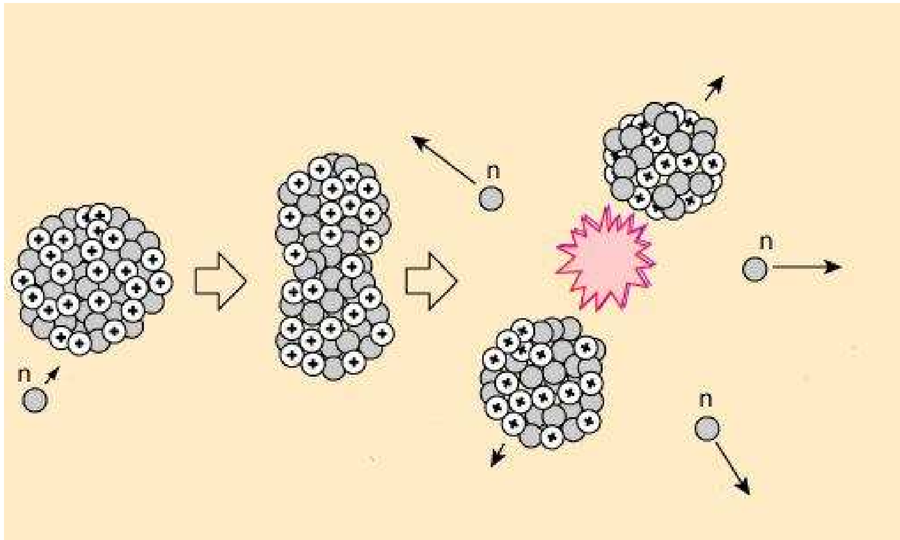


As in the case of α-decay, for spontaneous fission to take place the fissions fragments must undergo quantum tunnelling through this barrier. The height of the barrier is about 6 MeV. This is the same as the case for α-decay, but if we recall that the tunneling probabililty

$$T \sim \exp\left\{-\frac{2}{\hbar}\int_R^{R'}\sqrt{2M\left(\frac{ZZ'}{4\pi\epsilon_0 r}-Q\right)}\,dr\right\},$$

where M is the mass of the emitted particle, we see that the tunnelling probability for an α-particles is much larger than the probability for tunnelling of a much heavier fission product.

Much more likely is induced fission. In this case the parent nucleus is bombarded with a neutron. If the parent absorbs the neutron, the neutron binds to the parent, releasing energy (the binding energy of the neutron) in the form of vibrational energy, which could be more than the $\approx$ 6 MeV required to overcome the potential barrier.



Where the binding energy of the extra neutron is insufficient to overcome the potential barrier, the incident neutron needs to have a minimum kinetic energy in order to be able to induce fission. In the case of $^{238}_{92}$U (uranium) the binding energy of the extra neutron falls about 1 MeV short of the required energy. This means simply that fission is only induced by a neutron of kinetic energy greater than 1 MeV. On the other hand, the isotope $^{235}_{92}$U has one unpaired neutron. When the nucleus absorbs an extra neutron it pairs up with this unpaired neutron and there is extra binding energy from the pairing term. This is sufficient to release sufficient energy to give the nucleus sufficient vibrational energy to overcome the potential barrier and fission occurs for this isotope for *any* incident neutron.

Despite the fact that three or four neutrons are emitted in the fission reaction, the fission fragments still contain more neutrons than their stable isobars. This means that the fission fragments are usually unstable against β-decay.

For example in the case of fission of $^{238}_{92}$U, the stable isobar with A=145 is $^{145}_{60}$Nd (neodynium), which means that the fragment $^{145}_{57}$La (lanthanum) decays in three steps emitting and electron and an antineutrino at each step, until it reaches a stable nuclide. The stable isobar of with A=90 is $^{90}_{40}$Zr (zirconium) so the fission fragment $^{90}_{35}$Br (bromine) decays in a five stage β-decay chain.

These β-decay chains release further energy almost all of which is carried away by the energies of the electrons and antineutrinos.

Direct emission of a neutron from a nuclide with too many neutrons for stability is unlikely - the point here is that there is no Coulomb repulsion and so the surface energy tends to keep the neutron bound to the parent. However, it does sometimes occur. For example, the fission fragment $^{90}_{35}$Br produces $^{90}_{36}$Kr (krypton) in its first stage of β-decay, which can be produced in an excited state with sufficient energy to overcome the surface energy. This excited state can emit a neutron directly to become $^{89}_{36}$Kr. This isobar is still unstable against β-decay the stable isobar being $^{89}_{39}$Y, so the $^{89}_{36}$Kr decays in three stages.

The neutrons which are emitted in a fission reaction can be absorbed by another parent nucleus which then itself undergoes induced fission. In the case of ${}^{238}_{92}\mathrm{U}$, however, the three neutrons that emerge come out with an energy of less than 1 MeV (most of the 158 MeV released in this reaction goes into the kinetic energy of the fission fragments), so that these neutrons do not have enough energy to induce further fission of this nuclide. However, if there is a substantial concentration of the rarer isotope ${}^{235}_{92}\mathrm{U}$, then these spare neutrons can be captured by the nuclei of ${}^{235}_{92}\mathrm{U}$ and this can indeed induce fission, because in this case there is no energy threshold below which fission is not induced.

This is the principle of the "chain reaction".

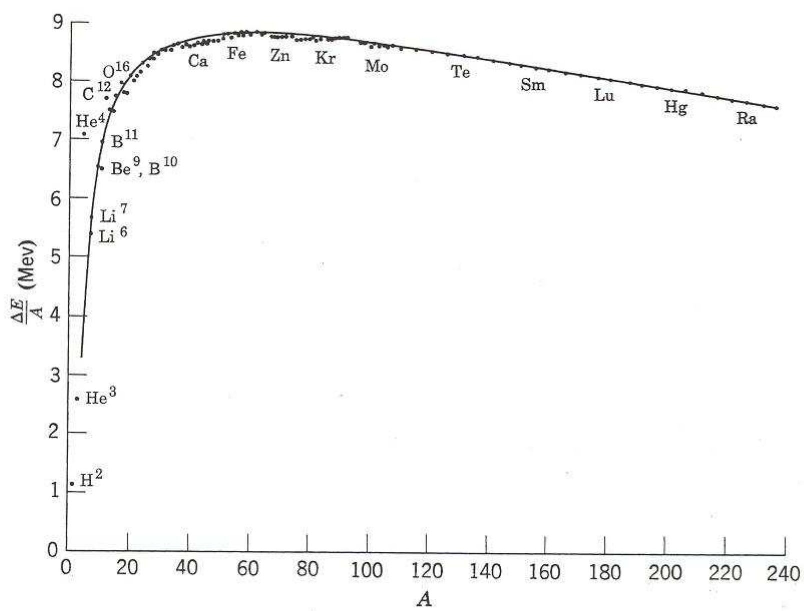
Let k be the number of neutrons produced in a sample of fissile material at stage n of this chain divided by the number of neutrons produced at stage $n-1$. This number will depend on how many of the neutrons produced at stage $n-1$ are absorbed by a nucleus that can undergo induced fission.

- If $k < 1$ the the chain reaction will simply fizzle out and the process will halt very quickly. This is what happens in natural uranium ore, in which the concentration of ${}^{235}_{92}\mathrm{U}$ is so small that the probability of one of the neutrons produced is absorbed by this isotope is very small.
- If $k > 1$, then the chain reaction will grow until all the fissile material is used up (atomic bomb). This is achieved by enriching the natural ore so that there is a sufficiently large concentration of ${}^{235}_{92}\mathrm{U}$. For a spherical sample the value of k grows as the neutron absorption probability which grows with the radius of the sphere. The mass of the uranium must therefore exceed some "critical mass" in order for the chain reaction to occur.
- If $k = 1$ then we have a controlled reaction. This is needed in a nuclear reactor. The absorption is controlled by interspersing the uranium with cadmium or boron rods that absorb most of the neutrons (cadmium and boron have a high neutron capture cross-section). The reaction is controlled automatically by moving the rods in and out so that the value of k is kept equal to one.

Chapter 11

Nuclear Fusion

If we look again at the binding energies (per nucleon) for different nuclei, we note also that the lightest nuclei have a much smaller binding energy per nucleon than those in the middle of the Periodic Table.



Much more energy per nucleon can be released by fusion of two of these light nuclei to form a heavier nucleus, than in the case of fission.

For example, if we consider the fusion of a deuteron and a hydrogen nucleus into helium

$$ {}^2_1\mathrm{H} + {}^1_1\mathrm{H} \ \rightarrow \ {}^3_2\mathrm{He} + \gamma $$

The γ is emitted because the helium is formed in an excited state. The mass of a deuteron is 3.34358×10^{-27} kg, that of a proton is 1.67262×10^{-27} kg, and for ${}^3_4$He the mass is 5.00832×10^{-27} kg. Using $E = \Delta mc^2$ where Δm is the mass difference between the initial and final states, and converting into MeV, we find that this reaction releases 4.4 MeV

$$ E = (3.34358 + 1.67262 - 5.00832) \times 10^{-27}\mathrm{kg} \times (3 \cdot 10^8 \mathrm{m/s})^2 / 1.6021^{-19}\mathrm{J/eV} \simeq 4.4\mathrm{MeV} $$

which is carried off in the energy of the γ and the kinetic energy of the helium nucleus. The energy released per fusion reaction is usually much less than that released in a typical fission reaction. However, the energy released per nucleon and therefore the energy released per unit mass is very much greater.

Many fusion reactions are a little more complicated than this, for the opposite reason that fission products are unstable against β-decay. The fusion products usually have *too few* neutrons to form a stable nucleus and one of the protons converts into a neutron, emitting a positron and a neutrino. For example, there is no bound state of two protons, i.e. ${}^2_2\mathrm{He}$ does not exist. Therefore for proton-proton fusion

$$ {}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{H} + e^+ + \nu $$

Such fusion reactions do not occur spontaneously. The reason for this is that the nuclei are positively charged so that they repel each other and they have to overcome the Coulomb barrier in order to be able to get close enough to be able to fuse. An approximate estimate of the energy required for two protons to fuse is

$$ E = \frac{e^2}{4\pi\epsilon_0 R} $$

where R is a typical nuclear radius. Inserting 4 fm for this radius we get an energy of around 350 KeV.

In order for protons to have an average energy associated with the degree of freedom corresponding to motion in the direction of the target proton, we would need a temperature T such that

$$ E = \frac{1}{2}kT $$

Putting in numbers we get a required temperature of 4×10^9 K.

In practice, fusion can take place at temperatures which are considerably lower than this. Fusion takes place in the core of the sun whose temperature is a mere 1.3×10^7 K. The reason for this is twofold

1. The above calculation of temperature determines the *average* energy per proton. But we know that these energies are distributed according to the Maxwell-Boltzmann distribution, which has a tail (albeit exponentially suppressed) and this tail means that there are some particles whose energy is much larger than the average.

2. It is not necessary for the incident protons to have sufficient energy to overcome the Coulomb barrier entirely. The protons can also get through the barrier by quantum tunnelling, provided the barrier height is not too high above the kinetic energy of the incoming particle.

The fusion in the sun and other stars, which is their source of energy, works in cycles. A cycle is a series of stages of fusion in which the initial particles are protons, but in subsequent stages the product of a previous stage fusion can fuse with another proton to form yet another fusion product. At the end of the cycle all the intermediate fusion products have disappeared leaving a stable fusion product, usually ${}^4_2$He.

The most common such cycle is the so-called proton cycle:

$$\begin{aligned} {}^1_1\mathrm{H} + {}^1_1\mathrm{H} &\rightarrow {}^2_1\mathrm{H} + e^+ + \nu \quad (\times 2) \\ {}^1_1\mathrm{H} + {}^2_1\mathrm{H} &\rightarrow {}^3_2\mathrm{He} + \gamma \quad (\times 2) \\ {}^3_2\mathrm{He} + {}^3_2\mathrm{He} &\rightarrow {}^4_2\mathrm{He} + {}^1_1\mathrm{H} + {}^1_1\mathrm{H} \end{aligned}$$

The third step in this cycle has as its initial state the result of two instances of the first two steps. At the end of the cycle there is no ${}^3_2$He and if we balance the number of initial and final protons we see that a net four protons have been turned into one ${}^4_2$He nucleus and two positrons, two neutrinos and two photons:

$$4{}^1_1\mathrm{H} \rightarrow^4_2 \mathrm{H} + 2e^+ + 2\nu + 2\gamma$$

The total energy released by this process is 24.7 MeV (one can calculate it using $m_{{}^4_2\mathrm{H}} = 6.64466 \times 10^{-27}\mathrm{kg}, \mathrm{m_e} = 9.109 \times 10^{-31}\mathrm{kg}$).

Other cycles also occur within the sun. An example is the carbon cycle

$$\begin{aligned} {}^{12}_6\mathrm{C} + {}^1_1\mathrm{H} &\rightarrow {}^{13}_7\mathrm{N} + \gamma \\ {}^{13}_7\mathrm{N} &\rightarrow {}^{13}_6\mathrm{C} + e^+ + \nu \\ {}^{13}_6\mathrm{C} + {}^1_1\mathrm{H} &\rightarrow {}^{14}_7\mathrm{N} + \gamma \\ {}^{14}_7\mathrm{N} + {}^1_1\mathrm{H} &\rightarrow {}^{15}_8\mathrm{O} + \gamma \\ {}^{15}_8\mathrm{O} &\rightarrow {}^{15}_7\mathrm{N} + e^+ + \nu \\ {}^{15}_7\mathrm{N} + {}^1_1\mathrm{H} &\rightarrow {}^{12}_6\mathrm{C} + {}^4_2\mathrm{He} \end{aligned}$$

Two steps are β-decays of fusion products which have too few neutrons for stability and therefore one of the protons is converted into a neutron in the β-decay process. The ${}^{12}_6$C is regenerated and the net effect is again four protons have been turned into one ${}^4_2$He nucleus and two positrons, two neutrinos and three photons, with total energy released by this process is 24.7 MeV. The carbon is initially produced by the fusion of three ${}^4_2$He nuclei. The fusion processes in the carbon cycle require more energy in order to overcome the Coulomb barrier, and is therefore more likely at higher temperatures. On the other hand, once these temperatures have been reached, this cycle is more likely than the proton cycle, because in the proton cycle it is necessary for two ^{3}He nuclei to fuse together - which is unlikely because the ^{3}He nuclei are produces from previous fusion processes and their density is low. The sun is a relatively cool star for which the proton cycle dominates.

The γ-rays are initially produced at energies of around 1 MeV. They scatter against other charged particles in the sun, losing energy at each scattering (Compton effect) and eventually "thermalize", i.e. they settle at an energy (wavelength) distribution which is the black-body distribution at the temperature of the surface of the sun - a distribution with a peak in the visible light range.

For over fifty years a great deal of effort has been put into trying to produce the equivalent environment of the core temperature of the sun, in order to be able to use fusion as an energy source. The necessary temperatures have been achieved but usually for too short a time for the fusion to take place.

In 1989 Pons and Fleischman (Southampton) announced that they had observed cold fusion in a chemistry laboratory. It is now widely accepted that their interpretation of their data was erroneous.

More recently it has been suggested that when small bubbles are adiabatically suppressed they can reach very high temperatures forming a plasma. Certainly a light flash can be observed from this plasma. In 2002 it was suggested by Taleyarkhan and collaborators that the temperature could reach that required for fusion and that a γ-ray emission was observed. The experiment has recently been repeated by Suslick et. al. who have cast doubt as to whether this γ-ray is indeed associated with fusion. More information about this can be found at the Website
http://www.nature.com/news/2005/050228/full/050228-7.html

Chapter 12

Charge Independence and Isospin

If we look at mirror nuclei (two nuclides related by interchanging the number of protons and the number of neutrons) we find that their binding energies are almost the same.

In fact, the only term in the Semi-Empirical Mass formula that is *not* invariant under Z $\leftrightarrow$ (A-Z) is the Coulomb term (as expected).

$$B(A,Z) \;=\; a_V\, A \,-\, a_S\, A^{2/3} \,-\, a_C \frac{Z^2}{A^{1/3}} \,-\, a_A \frac{(Z-N)^2}{A} \,+\, \frac{\left((-1)^Z + (-1)^N\right)}{2} \frac{a_P}{A^{1/2}}$$

Inside a nucleus these electromagnetic forces are much smaller than the strong inter-nucleon forces (strong interactions) and so the masses are very nearly equal despite the extra Coulomb energy for nuclei with more protons.

Not only are the binding energies similar - and therefore the ground state energies are similar but the excited states are also similar.

As an example let us look at the mirror nuclei (Fig. 12.2) ${}^7_3$Li and ${}^7_4$Be, where we see that for all the states the energies are very close, with the ${}^7_4$Be states being slightly higher because it has one more proton than ${}^7_3$Li.

All this suggests that whereas the electromagnetic interactions clearly distinguish between protons and neutrons the strong interactions, responsible for nuclear binding, are 'charge independent'.

Let us now look at a pair of mirror nuclei whose proton number and neutron number differ by two, and also the nuclide between them. The example we take is ${}^6_2$He and ${}^6_4$Be, which are mirror nuclei. Each of these has a closed shell of two protons and a closed shell of two neutrons. The unclosed shell consists of two neutrons for ${}^6_2$He and two protons ${}^6_4$Be. the nuclide 'between' is ${}^6_3$Li which has one proton and one neutron in the outer shell.

From the principle of charge independence of the strong interactions we might have expected all three nuclides to display the same energy-level structure. We see that although there are states in ${}^6_3$Li which are close to the states of the mirror nuclei ${}^6_2$He and ${}^4_2$Be, there are also states in ${}^6_3$Li which have no equivalent in the two mirror nuclei.

We can understand this from the Pauli exclusion principle. In the case of ${}^6_2$He and ${}^4_2$Be

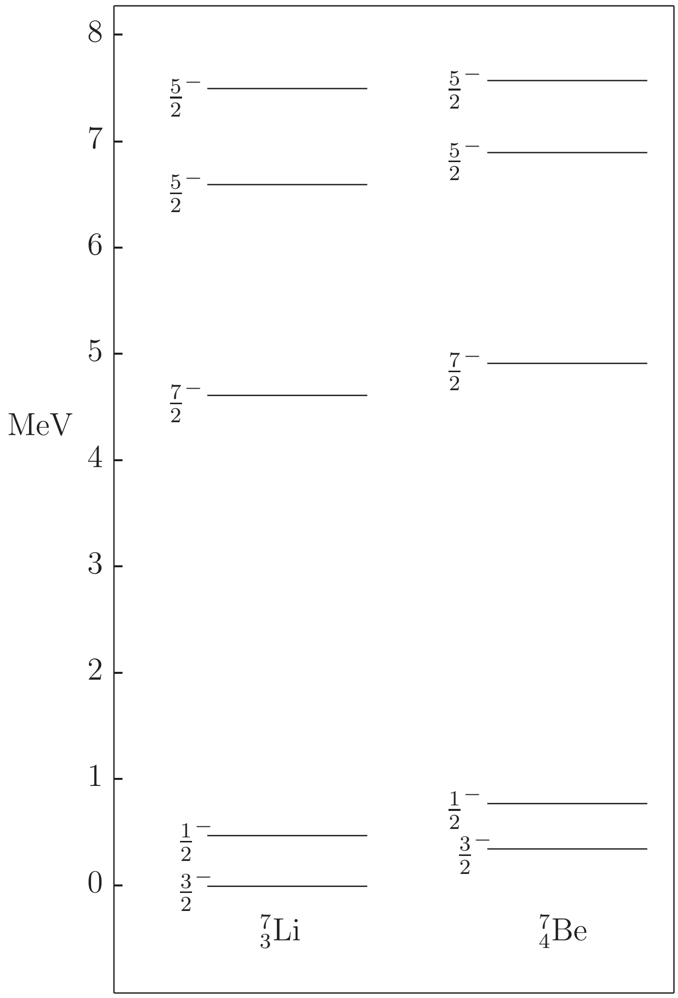


Figure 12.1: Energy states for the mirror nuclei (Fig. 12.2) $^7_3$Li and $^7_4$Be.

which have either two protons or two neutrons in the outer shell, these cannot be in the same state (with the same spin), whereas in the case of $^6_3$Li for which the nucleons in the outer shell are not identical, this principle does not apply and there are extra states, in which the neutron and proton are in the same state.

12.1 Isospin

We can express this is a more formal (mathematical), but useful way by introducing the concept of "Isospin".

If we have two electrons with z- component of their spin set to $s_z = +\frac{1}{2}$ and $s_z = -\frac{1}{2}$ (in units of $\hbar$) then we can distinguish them by applying a (non-uniform) magnetic field in the z-direction - the electrons will move in opposite directions. But in the absence of this

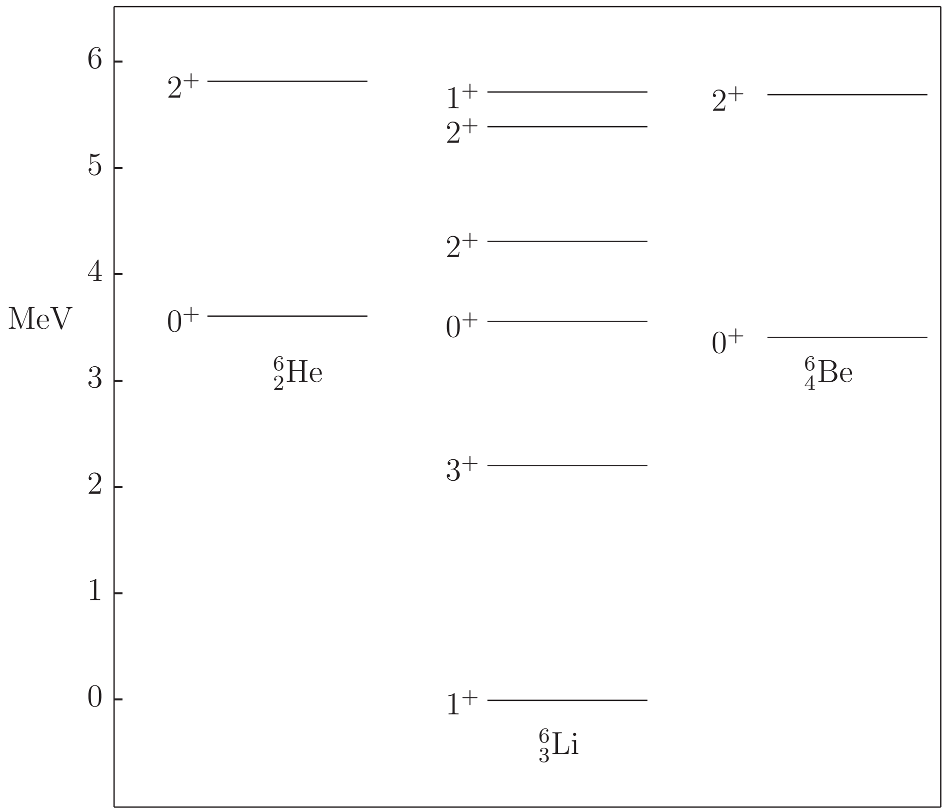


Figure 12.2: Energy states for $^6_2$He, $^7_3$Li and $^6_4$Be.

external field these two cannot be distinguished and we are used to thinking of these as two states of the same particle.

Similarly, if we could 'switch off' electromagnetic interactions we would not be able to distinguish between a proton and a neutron. As far as the strong interactions are concerned these are just two states of the same particle (a nucleon).

We therefore think of an imagined space (called an 'internal space') in which the nucleon has a property called "isospin", which is mathematically analogous to spin. The proton and neutron are now considered to be a nucleon with different values of the third component of this isospin.

Since this third component can take two possible values, we assign $I_3 = +\frac{1}{2}$ for the proton and $I_3 = -\frac{1}{2}$ for the neutron. The nucleon therefore has isospin $I = \frac{1}{2}$, in the same way that the electron has spin $s = \frac{1}{2}$, with two possible values of the third component.

As far as the strong interactions are concerned this just represents two possible quantum states of the same particle. If there were no electromagnetic interactions these particles would be totally indistinguishable in all their properties - mass, spin etc.

In the same way that angular momentum is conserved, isospin is conserved in any transition mediated by the strong interactions. This is an example of an approximate symmetry - inside the nucleus the strong forces between nuclei do not distinguish between particles with different third component of isospin and would lead to identical energy levels, but there are electromagnetic interactions which break this symmetry and lead to small differences in the energy levels of mirror nuclei.

The electromagnetic interactions couple to the electric charge, Q, of the particles and in the case of nucleons this electric charge is related to the third component of isospin by

$$Q = I_3 + \frac{1}{2}$$

Other particles can also be classified as isospin multiplets. For example there are three pions, $\pi^+_, \pi^0$, π^-, which have almost the same mass and zero spin etc. There are three of them with different charges but which behave in the same way under the influence of the strong interactions. Therefore they form an isospin multiplet with $I = 1$ and three possible third components, namely $+1$, 0 -1. In the case of pions the electric charges are equal to I_3.

Particles which are members of an isospin multiplet have the same properties, with the exception of their electric charge, i.e. they have the same spin and almost the same mass (the small mass differences being due to the electromagnetic interactions which are not isospin invariant. We will see later that particles can have other properties (call "strangeness", "charm" etc.) and members of an isospin multiplet will have the same values of these proerties as well.

In the same way that two electrons can have a total spin $S = 0$ or $S = 1$, two nucleons can have a total isospin $I = 0$ or $I = 1$, and (systems of n nucleons can have isospins up to $n/2$). For two electrons we may write the total wavefunction as

$$\Psi_{12} = \Psi(\mathbf{r}_1, \mathbf{r}_2)\chi(s_1, s_2),$$

where $\chi(s1, s2)$ is the spin part of the wavefunction. For $S = 1$ we have

$$\begin{aligned}
\chi(s_1, s_2) &= (\uparrow\uparrow), \quad S_z = +1 \\
\chi(s_1, s_2) &= \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \quad S_z = 0 \\
\chi(s_1, s_2) &= (\downarrow\downarrow), \quad S_z = -1
\end{aligned}$$

which is symmetric under interchange of the two spins, which means that by fermi statistics the spatial part of the wavefunction must be antisymmetric under the interchange of the positions of the electrons,

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = -\Psi(\mathbf{r}_2, \mathbf{r}_1),$$

or for the case of $S = 0$,

$$\chi(s_1, s_2) = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow),$$

which is antisymmetric under interchange of spins so it must be accompanied by a symmetric spatial part of the wavefunction

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = +\Psi(\mathbf{r}_2, \mathbf{r}_1).$$

In the case of two nucleons we also have a total isospin part of the wavefunction, so the complete wavefunction is

$$\Psi_{12} = \Psi(\mathbf{r}_1, \mathbf{r}_2)\chi_S(s_1, s_2)\chi_I(I_1, I_2),$$

where $\chi_I(I_1, I_2)$ is the isospin part of the wavefunction. For total isospin $I = 1$ we have

$$\begin{aligned}
\chi_I(I_1, I_2) &= (p\,p)\,, \quad I_3 = +1 \\
\chi_I(I_1, I_2) &= \frac{1}{\sqrt{2}}\,(p\,n\, +\, n\,p)\,, \quad I_3 = 0 \\
\chi_I(I_1, I_2) &= (n, n)\,, \quad I_3 = -1,
\end{aligned}$$

which is symmetric under the interchange of the isospins of the two nucleons, so that (as in the case of two electrons) it must be accompanied by a combined spatial and spin wavefunction that must be antisymmetric under simultaneous interchange of the two positions *and* the two spins. But we also have the $I = 0$ state

$$\chi_I(I_1, I_2) = \frac{1}{\sqrt{2}}\,(p\,n\, -\, n\,p)\,,$$

which is antisymmetric under the interchange of the two isospins and therefore when the nucleons are combined in this isospin state they must be accompanied by a combined spatial and spin wavefunction which is *symmetric* under simultaneous interchange of the two positions *and* the two spins.

Returning to the three nuclei ${}^6_2$He and ${}^6_4$Be and ${}^6_3$Li, the closed shells of neutrons and protons have a total isospin zero so we do not need to consider these in determining the isospin of the nuclei. We note that ${}^6_2$He has two neutrons in the outer shell so its isospin must be $I = 1$, with $I_3 = -1$ whereas the ${}^6_4$Be has two protons in the outer shell so its isospin must be $I = 1$, with $I_3 = +1$ implying that for these two nucleons the remaining part of the wavefunction (spatial and spin parts) must be antisymmetric under simultaneous interchange of the two positions and the two spins. On the other hand, the nucleus ${}^6_3$Li has one proton and one neutron in the outer shell and can therefore be either in an $I = 1$ state like the other two nuclei or in an $I = 0$ state which is not possible for the other two. The strong interactions will give rise to different energy levels depending on the total isospin of the nulceons in the outer shell (in the same way that atomic energy levels depend on the total angular momentum J). Thus we see that two of the states shown for ${}^6_3$Li can be identified as $I = 1$ states and they approximately match states for the other two nuclei, but the others are $I = 0$ states and have no counterpart in ${}^6_2$He or ${}^6_4$Be, and which have wavefunctions that are symmetric under the simultaneous interchange of the positions and the spins of the two nucleons in the outer shell.

The fact that the ground states of ${}^{6}_{2}$He and ${}^{6}_{4}$Be have spin zero and the ground state of ${}^{6}_{3}$Li has spin one, can be deduced from the isospin of these ground states. For ground state wavefunctions the orbital angular momentum, l, is zero and since the symmetry of the spatial part of the wavefunction is given by $(-1)^l$, this means that the spatial part of the wavefunction is symmetric under the interchange of the positions of the two nucleons in the outer shell. Since we know that the overall wavefunction for the two nucleons in the outer shell must be antisymmetric under interchange, because the nucleons are fermions, it follows that the isospin part and the spin part of the wavefunction must have opposite symmetry. Thus for the ground states of ${}^{6}_{2}$He and ${}^{6}_{4}$Be, which are in I=1 (symmetric) isospin states, the spin part of the wavefunction must be antisymmetric and therefore the spins of the two outer shell nucleons must combine to give spin $S = 0$, whereas for the ground state of ${}^{6}_{3}$Li which is **from the experiment known to be** in an I=0 (antisymmetric) isospin state, the spin part of the wavefunction must be symmetric and therefore the spins of the two outer shell nucleons must combine to give spin $S = 1$.

Chapter 13

Accelerators

Particles physics, also known as 'high energy physics' is the study of the fundamental forces of nature and the particles that can be found at very high energies.

The most massive particles that has been discovered so far are W-boson with a mass of 80.4 GeV/c^2, Z-boson with a mass of 91.2 GeV/c^2, and *top*-quark with a mass of 172.0 GeV/c^2. All these particles are 100 times heavier than the proton. So we need a really high energy to produce these particles.

Another way of seeing that we need high energies is to note that we wish to probe very short distances. At the very least we want to probe distances which are small compared with a typical nuclear radius, i.e.

$$x \ll 1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$$

In order to do this the uncertainty in the position, Δx must be much smaller than 1 fm, and by Heisenberg's uncertainty principle

$$\Delta x \Delta p \geq \hbar/2$$

the uncertainty in momentum Δp must obey the inequality

$$\Delta p \gg \frac{\hbar}{1\,\mathrm{fm}} = 197\,\mathrm{MeV/c}.$$

This in turn means that the momenta of the particle used as a probe must have a momentum much larger than this, and hence an energy large compare with ≈ 200 MeV.

In fact, the weak interactions have a range which is more than two orders of magnitude shorter than this and so particles used to investigate the mechanism of weak interactions have to have energies of at least 100 GeV.

In order to achieve these very high energies particles are accelerated in "accelerators". Incident particles are accelerated to these high energies and scattered against another particle. There is enough energy to smash the initial particles up and produce many other particles in the final state, some of them with considerably higher masses than the incident particles. Such scattering is called "inelastic scattering" (conversely a scattering event in which the final state particles are the same as the initial particles is called "elastic scattering". Rutherford scattering or Mott scattering are examples of elastic scattering.) The word 'elastic' here means that none of the incoming energy is used up in the production of other particles.

In elastic scattering we talk about a differential cross-section (with respect to solid angle), which is the number of particles per incident flux in a given element of solid angle. For inelastic events we can talk about the total cross-section for a particular process. For example, at the LEP accelerator (electron-positron scattering) at CERN one possible process was

$$e^+ + e^- \rightarrow W^+ + W^-,$$

in which the electron and positron annihilate each other and produce two W-bosons instead. The W-boson has a mass of 80.4 GeV/c^2, so that total centre-of-mass energies of over 160 GeV are required for this process to take place. The cross-section $\sigma_{(e^+e^- \rightarrow W^+W^-)}$ is the total number of events in which two W-bosons are produced per unit incident flux (i.e. the number of W-boson pairs produced divided by the number of particle scatterings per unit area)

It is now believed that there exist particles with masses which are an order of magnitude larger than this and modern accelerators can achieve energies of up to 1 TeV (10^{12} eV). This new energy frontier and respectively new small distances can be probed by presently the most powerful accelerator in the world – the Large Hadron Collider (LHC) – at CERN which has resumed running in November 2009 with energy 7.5 TeV.

13.1 Fixed Target Experiments vs. Colliding Beams

The total energy of a projectile particle plus the target particle depends on the reference frame. The frame that is relevant for the production of high mass particles is the centre-of-mass frame for which the projectile and target have equal and opposite momentum p. For simplicity let us suppose that the projectile and target particle are the same, or possibly particle antiparticle (e.g. proton-proton, proton-antiproton, or electron-positron) so that their masses, m are the same. This means that in this frame both the particle have the same energy, E_{CM} (since we are usually dealing with relativistic particles, this means kinetic *plus* rest energy.)

Let us construct the quantity

$$s = \left(\sum_{i=1,2} E_i\right)^2 - \left(\sum_{i=1,2} \mathbf{p}_i\right)^2 c^2$$

In the centre-of-mass frame, where the momenta are equal and opposite the second term vanishes and we have

$$s = 4E_{CM}^2,$$

i.e. s is the square of the total incoming energy in the centre of mass frame - this is a quantity that is often used in particle physics and the notation s is always used. For one particle we know that $E^2 - p^2c^2$ is equal to m^2c^4 and is therefore the same in any frame of reference even though the quantities E and $\mathbf{p}$ will be different in the two frames. Likewise the above quantity, s, is the same in any frame of reference (we say that 'it invariant under Lorentz transformations.')

In the frame in which the target particle is at rest, its energy is mc^2 and its momentum is zero, whereas the projectile has energy E_{LAB} and momentum $\mathbf{p}_{LAB}$ so that we have

$$s = \left(E_{LAB} + mc^2\right)^2 - \mathbf{p}_{LAB}^2 c^2 = E_{LAB}^2 + m^2c^4 + 2mc^2 E_{LAB} - \mathbf{p}_{LAB}^2 c^2 = 2m^2c^4 + 2mc^2 E_{LAB},$$

where in the last step we have used the relativity relation

$$E_{LAB}^2 - \mathbf{p}_{LAB}^2 c^2 = m^2c^4.$$

Equating the two expressions for s (and taking a square root we obtain the relation

$$\sqrt{s} = 2E_{CM} = \sqrt{2m^2c^4 + 2mc^2 E_{LAB}}.$$

For non-relativistic incident particles with kinetic energy $T \ll mc^2$ for which $E_{LAB} = mc^2 + T$, this gives

$$\sqrt{s} = 2E_{CM} = 2mc^2 + T,$$

as expected, but for relativistic particles the centre-of-mass energy is considerably reduced. For example, taking the proton mass be be approximately 1 GeV/c^2, the if we have an accelerator that can accelerate protons up to an energy of 100 GeV, the total centre-of-mass energy achieved is only about 15 GeV - far less than the energy required to produce a particle of mass 100 GeV/c^2.

The solution to this problem is to use colliding beams of particles. In these experiments both the initial particles involved in the scattering emerge from the accelerator and are then stored in storage rings, in which the particles move in opposite directions around the ring, with their high energies maintained by means of a magnetic field. At various point around the rings the beams intersect and scattering takes place. In this way the laboratory frame *is* the centre-of-mass frame and the full energy delivered by the accelerator can be used to produce high mass particles.

13.2 Luminosity

The luminosity $\mathcal{L}$ is the number of particle collisions per unit area (usually quoted in cm^2) per second. The number of events of a particular type which occur per second is the cross-section multiplied by the luminosity. In the example of two W-boson production at LEP the cross-section, $\sigma_{(e^+e^- \to W^+W^-)}$ is 15 pb (p=pico means 10^{-12}) and the luminosity of LEP was 10^{32} per cm^2 per second. The number of these pairs of W-bosons produced per second is given by

$$\frac{dN_{W^+W^-}}{dt} = (15 \times 10^{-12} \times 10^{-28}) \times (10^{32} \times 10^4) = 1.5 \times 10^{-3},$$

where the first term in parenthesis is the cross-section converted to m^2 and the second is the luminosity converted to $\mathrm{m}^{-2}\,\mathrm{sec}^{-1}$. So, the general formula for reaction rate, $R = dN/dt$ is

$$R = \sigma \times \mathcal{L}$$

while for integrated luminosity over the time $L = \int \mathcal{L} dt$ the number of events, N, we will observe is given by

$$N = \sigma \times L$$

.

$\mathcal{L}$ is proportional to the number 'bunches' of particles in each beam, n (typically 5-100), the revolution frequency, f(kHz-MHz), N_1, N_2 – the number of particles in each bunch ($\simeq 10^{10} - 10^{11}$) and inversely proportional to the bean cross section, A ($\mu\mathrm{m}^2$):

$$\mathcal{L} = \frac{nfN_1N_2}{A}$$

As in the case of radioactivity the cross-section is a probability for a particular event and the actual number of events observed is a random distribution with that probability. If a cross-section predicts N events over a given time-period, the error on that number is $\sqrt{N}$ (this means that there is a 68% probability that the number of events observed will lie in the region $N - \sqrt{N}$ to $N = \sqrt{N}$). To be able to measure the above cross-section at LEP to an accuracy of 1% it was necessary to collect 10000 such W-pairs, which, at a rate of 1.5×10^{-3} per sec., took about three months.

We pay a price for colliding beam experiments in terms of luminosity. For a fixed target experiment we can make an estimate of the luminosity in the case of proton-proton scattering from the fact that the incident particles are travelling almost with the speed of light. The luminosity is given by the number of protons in a column of the target of unit area and length c. For a solid whose density is $10^4\,\mathrm{kg\,m}^{-3}$, and assuming that about one half of the target material consists of protons of mass 1.67×10^{-27} kg, this comes out to about 10^{35} per cm^2 per sec. In colliding beams it is necessary to focus the incident beams as tightly as possible using magnetic fields, in order to maximize the luminosity. So far, luminosities of 10^{32} per cm^2 per sec. have been achieved, which means the reaction rate is down by three orders of magnitude compared with a fixed target experiment. However, the LHC is designed to reach a luminosity of 10^{34} per cm^2 per sec. - i.e luminosities within an order of magnitude of that obtained in fixed target experiments.

13.3 Types of accelerators

As we have discussed, the general aim of accelerators is to collide two particles at high(est) energy and create new particles from combined energy and quantum numbers or to probe inside one of the particles to see what it is made of.

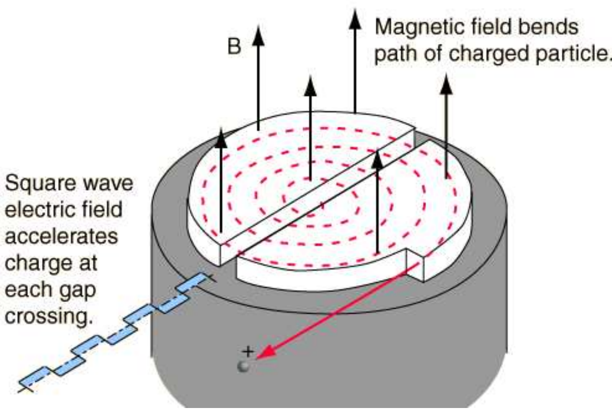
Only stable charged particles can be accelerated: such as electrons, positrons, protons, anti-proton and some ions. Potentially, the long-lived particles such as muon ($\tau \simeq 2 \times 10^{-6}$s) were discussed to be used in the future muon collider.

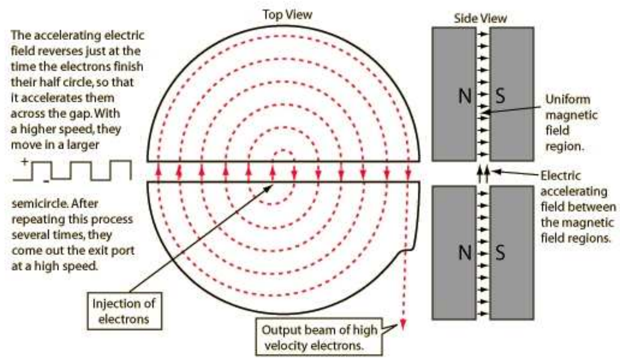
Single DC stage accelerators such as the Van de Graaff Generator can accelerate electrons and protons upto about 20 MeV.

There are two general types of modern accelerators – Circular (Cyclic) and Linear.

13.3.1 Cyclotrons

The prototype design for all circular accelerators is the cyclotron.





This is a device in which the (charged) particles to be accelerated move in two hollow metallic semi-disks (D's) with a large magnetic field B applied normal to the plane of the D's. The particles move in a spiral from the center and an alternating electric field is applied between the D's whose frequency is equal to the frequency of rotation of the charged particles, such that when the particles crosses from one of the D's to the other the electric field always acts in the direction which accelerates the particles.

A charged particle with charge e moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ experiences a force $\mathbf{F}$, where

$$\mathbf{F} \;=\; q\mathbf{v} \times \mathbf{B}.$$

When the magnetic field is perpendicular to the plane of motion of the charged particle, this force is always towards the centre and gives rise to centripetal acceleration, so that at the moment when the particles are moving in a circle of radius r

$$F \;=\; B\,e\,v \;=\; m\frac{v^2}{r}.$$

We see immediately that the angular velocity $\omega = v/r$ is constant, so that the frequency of the alternating electric field remains constant. The maximum energy that the particles can acquire depends on the radius, R, for which the velocity has its maximum value v_{max},

$$v_{max} \;=\; \frac{BeR}{m},$$

leading to a maximum kinetic energy

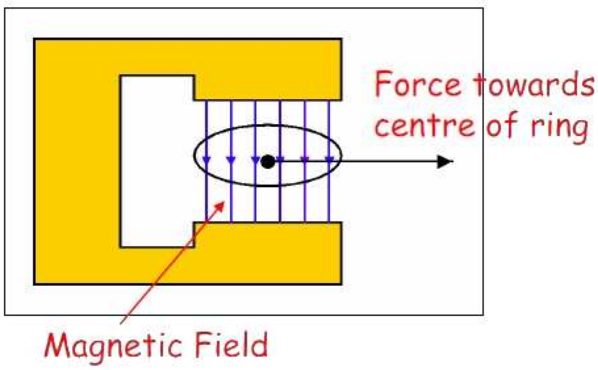
$$T_{max} \;=\; \frac{1}{2}mv_{max}^2 \;=\; \frac{B^2e^2R^2}{2m}.$$

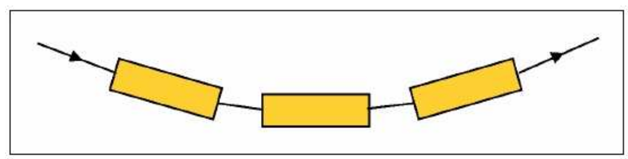
This works fine if the energy of the particle remains non-relativistic. However, in high energy accelerators the particles are accelerated to energies which are extremely relativistic - the particles are travelling very nearly with the velocity of light (at the LHC v/c will be $1 - 10^{-15}$!). Taking relativistic effects into account The angular velocity is now

$$\omega \;=\; \sqrt{1 - v^2/c^2}\frac{Be}{m}.$$

This means that as the particles accelerate, either the frequency of the applied electric field must vary - such machines are called "synchrocyclotrons" - or the applied magnetic field must be varied (or both) - such machines are called "synchrotrons".

At Synchrotron dipole magnets keep particles in circular orbit using $p = 0.3 \times B \times R$ (p in GeV/c, B in Tesla, R in meters), while quadrupole magnets used to focus the beam.





Since the bending field B is limited then the maximum energy is limited by the size of the ring. The CERN SPS (Super Proton Synchrotron) has a radius R = 1.1Km and a momentum of 450 GeV/c. Particles are accelerated by resonators (RF Cavities). The bending field B is increased with time as the energy (momentum p) increases so as to keep R constant [$p = 0.3BR$]. Electron synchrotrons are similar to proton synchrotrons except that the energy losses are greater.

One of the main limiting factors of synchrotron accelerators is the Synchrotron Radiation. A charged particle moving in a circular orbit is accelerating (even if the speed is constant) and therefore radiates. The energy radiated per turn per particle is:

$$\Delta E = \frac{4\pi e^2 \beta^2 \gamma^4}{3R}$$

where e is the charge, $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2} = E/m$, from which follows that

$$\Delta E \propto 1/m^4$$

For relativistic electrons and protons of the same momentum the ratio of energy losses are very large for electrons versus protons:

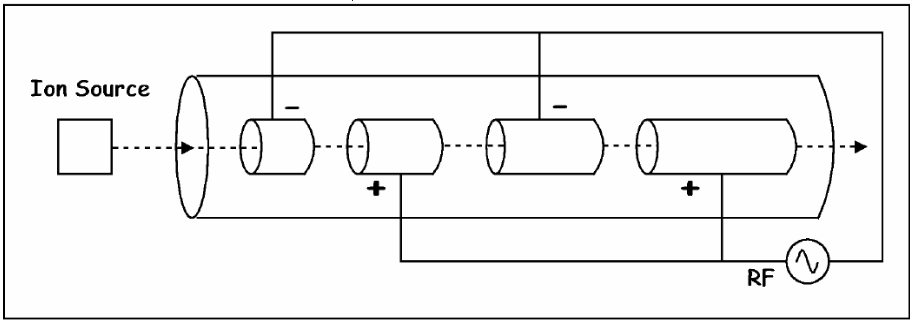
$$\frac{\Delta E_e}{\Delta E_p} = \left(\frac{m_p}{m_e}\right)^4 \simeq 10^{13}$$

.

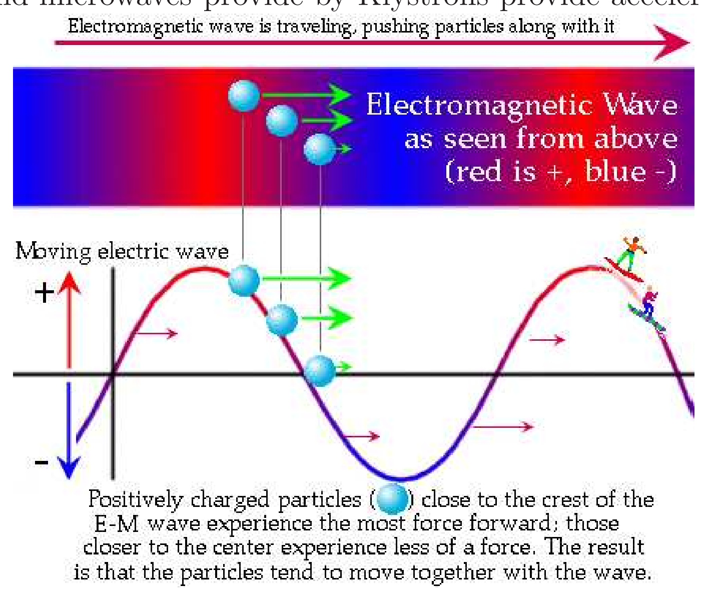
13.3.2 Linear Accelerators

The energy loss due to synchrotron radiation, can be avoided in a linear accelerator. In such a machine the particles are accelerated by means of an applied electric field along a long a tube.

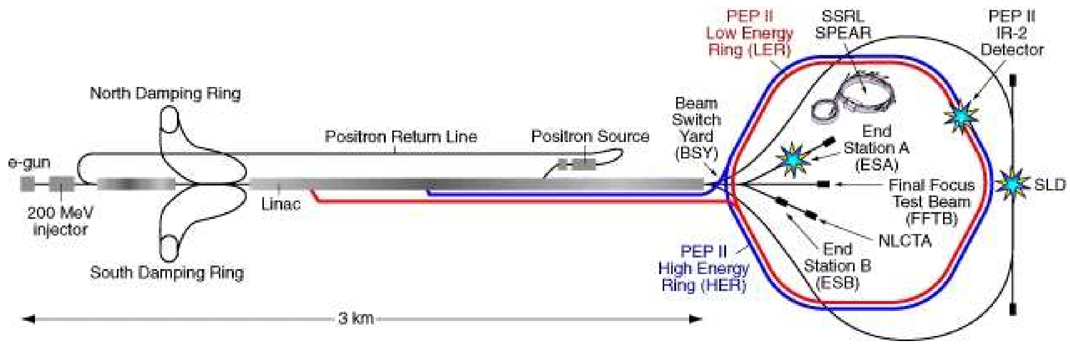
Proton Linear Accelerators (Linacs) use a succession of drift tubes of increasing length (to compensate for increasing velocity).



Particles always travel in vacuum. There is no field inside the drift tubes. External field between ends of tubes changes sign so proton always sees $-ve$ in front and $+ve$ behind. Proton linacs of 10-70m give energies of 30 to 200 MeV. Usually used as injectors for higher energy machines. Above a few MeV, electrons travel at speed of light. The 'tubes' become uniform in length and microwaves provide by Klystrons provide accelerating potential.



The largest linear collider in existence is SLAC (Stanford Linear Collider Center) in California. This is 3 km. long and accelerates both electrons and positrons up to energies of 50 GeV. It is able to accelerate both electrons and positrons simultaneously by sending an electromagnetic wave in the microwave band along the beam pipe and injecting bunches of electrons and positrons which are precisely one half wavelength apart, so that the electric field acting on the positrons is in the forward direction and so accelerates the positrons in the forward direction, whereas the electric field acting on the electrons is in the backwards direction, but because the electrons have negative charge they are also accelerated in the forward direction.



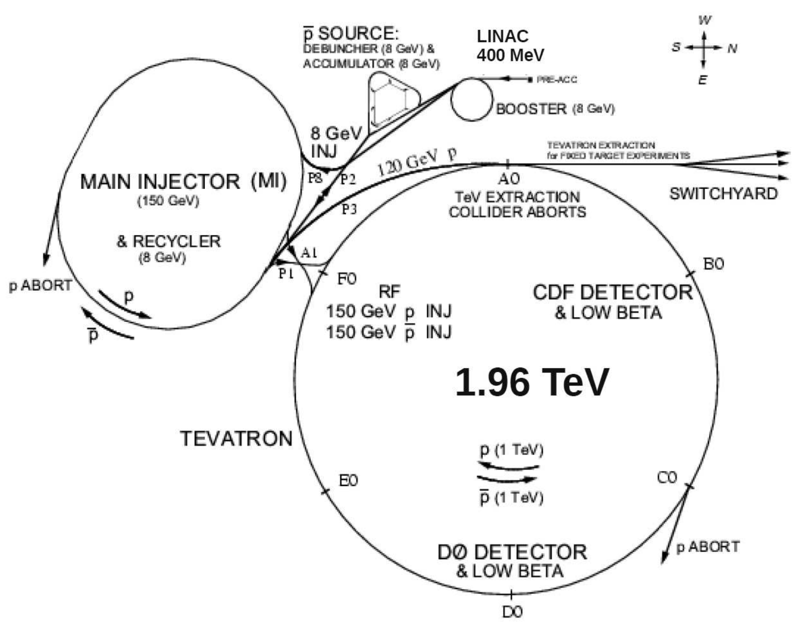
At the end of the tube the electrons and positrons are stored in a storage ring (they go around the storage ring in opposite directions under the influence of the same magnetic field) and there are intersection points where electron-positron scatterings occur.

There are plans (awaiting international approval) to build a much larger linear collider (known as ILC - the International Linear Collider) which will have a total centre-of-mass energy of 500 GeV (or perhaps even 1 TeV).

13.4 Main Recent and Present Particle Accelerators

Here are some of today's main accelerator laboratories.

- **FermiLab:**

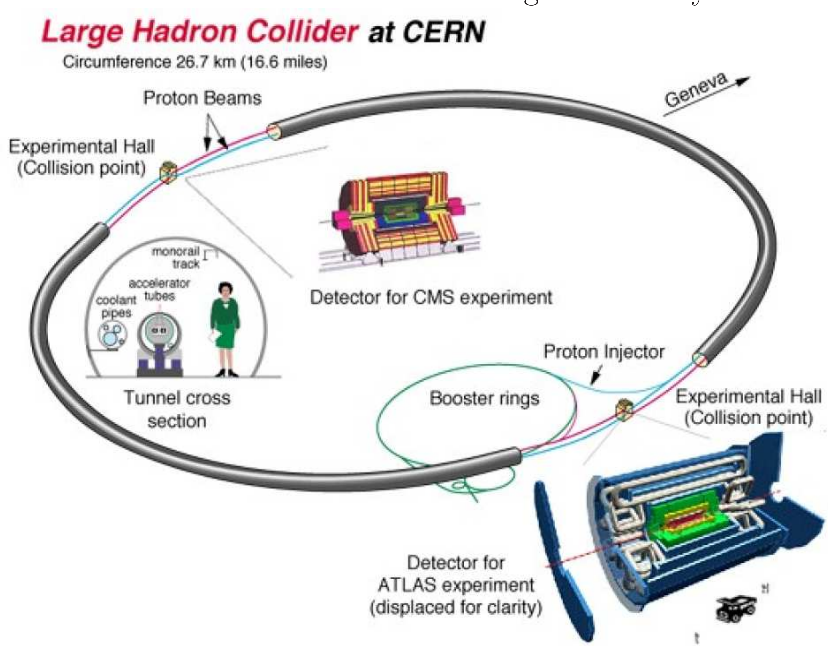


Situated just outside Chicago this is now running the Tevatron in which protons and antiprotons are each accelerated to an energy of 1.96 TeV and then move around a ring of circumference 6 km. This is a synchrotron in which very high magnetic fields are achieved using superconducting (electro-)magnets, which are capable of maintaining very large currents thereby producing large magnetic fields. The luminosity is

$10^{32}\,\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$

- **CERN:**
  Situated just outside Geneva, until 2001 the main experiment was LEP in which electrons and positrons were each accelerated to an energy of about 100 GeV, and had a luminosity of $10^{32}\,\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$. This was the largest electron synchrotron in the world with a circumference of 27 km.

  The next project at CERN is the LHC has started in September 2008. After an accident in October 2008, LHC has resumed its operation in November 2009 and now it is colliding protons against protons with energies 3.5TeV$\times$3.5TeV resulting to a total cms energy of 7 TeV. Using a specially designed magnetic field configuration, two beams of protons moving in opposite direction around the same ring is possible. In the future, protons will each be accelerated to 7 TeV and the design luminosity is $10^{34}\,\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$.



- **DESY:**
  Situated just outside Hamburg, this laboratory is running the HERA accelerator which accelerated protons to an energy of 820 GeV and electrons (or positrons) to 27 GeV. It is the only accelerator in which the initial particles are not the same - or particle-antiparticle pairs.

Table below presents summary on present and recent colliders as well as comparison of electron and proton accelerators.

Name	Particles	Energies	Where	Status
SLC	e^+e^-	50+50GeV	Stanford USA	Ended
LEP	e^+e^-	100+100GeV	CERN Geneva	Ended
Tevatron	$p\bar{p}$	980+980GeV	Fermilab USA	Ended
HERA	$e^{\pm}p$	30+820GeV	DESY Hamburg	Ended
PEP II	e^+e^-	9+3.1GeV	Stanford USA	Current
KEKB	e^+e^-	8+3.5GeV	Tsukuba Japan	Current
LHC	pp	4.0+4.0TeV	CERN Geneva	Current

Electron Machines	Proton Machines
Clean - no other particles involved than e^+e^-.	Messy – qq or $q\bar{q}$ interact and rest of p or $\bar{p}$ is junk.
Lower energy for same radius (synchrotron radiation). LEP e^+e^- ~ 200 GeV.	Higher energy for same radius. LHC (pp) in LEP tunnel ~14 TeV.
Energy of e^+e^- known.	Energy of qq or $q\bar{q}$ not known.
Fixed energy (for a given set of operating conditions).	Range of qq or $q\bar{q}$ energies for fixed pp or $p\bar{p}$ energy.
Best for detailed study.	Best for discovering new things.

Chapter 14

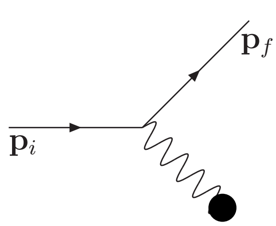
Fundamental Interactions (Forces) of Nature

Interaction	**Gauge Boson** (Force carrier)	**Gauge Boson Mass**	**Interaction Range**
Strong	Gluon	0	short-range (a few fm)
Weak	$W^{\pm}$, Z	$M_W = 80.4$ GeV/c^2 $M_Z = 91.2$ GeV/c^2	short-range ($\sim\ 10^{-3}$ fm)
Electromagnetic	Photon	0	long-range
Gravity	Graviton	0	long-range

Gravity is by far the weakest interaction. The gravitational force between two protons is about 10^{-9} of the electromagnetic force between them. We shall not discuss gravity further in these lectures. [Experiments designed to investigate the theory of gravity (General Relativity), are astronomical observations - the other end of the magnitude scale from particle physics.]

14.1 Relativistic Approach to Interactions

Electromagnetic Interaction:



Potential of a particle of charge e located at $\mathbf{r}$, due to another charge e' (fixed) at the origin is

$$V(r) \;=\; \frac{ee'}{4\pi\epsilon_0 r}$$

If this charge has initial momentum $\mathbf{p}_i$ and final momentum $\mathbf{p}_f$, its initial and final (time-independent) wavefunctions are given by

$$\Psi_i \;\propto\; e^{i\mathbf{p_i}\cdot\mathbf{r}/\hbar}$$
$$\Psi_f \;\propto\; e^{i\mathbf{p_f}\cdot\mathbf{r}/\hbar}$$

The amplitude for such a transition is

$$\mathcal{A} \;=\; \int \Psi_f^* V(r)\Psi_i d^3\mathbf{r} \;\propto\; ee' \int e^{i(\mathbf{p_i}-\mathbf{p_f})\cdot\mathbf{r}/\hbar}\frac{1}{r}d^3\mathbf{r}.$$

Performing the integral (this is a Fourier transform) we get

$$\mathcal{A} \;\propto\; \frac{ee'}{-|\mathbf{q}|^2},$$

where $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$, is the momentum transferred from the scattered charged particle to the charge at the origin. (We have seen this in the Rutherford scattering formula where the cross-section is proportional to $1/|\mathbf{q}|^4$, so that the amplitude is proportional to $1/|\mathbf{q}|^2$).

For the scattering of a relativistic particle, this expression is modified to

$$\mathcal{A} \;\propto\; \frac{ee'}{(q_0^2 - |\mathbf{q}|^2)},$$

where $q_0 = (E_f - E_i)/c$, with E_i and E_f being the initial and final energy of the scattered particle. (In the non-relativistic limit $(E_f - E_i)/c \ll |\mathbf{p}_f - \mathbf{p}_i|$ so q_0^2 is negligible.)

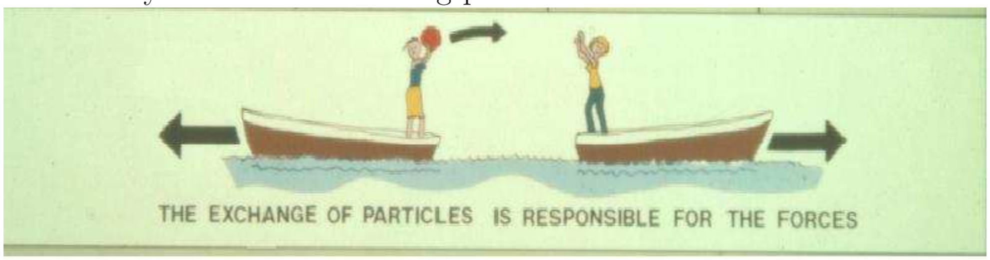
The interpretation of this process is that a photon, which is the 'carrier' of the electromagnetic interactions, with energy $c\,q_0$ and momentum $\mathbf{q}$, is exchanged between the two charged particles. The electric charges e and e' measure the strengths of the coupling of the charged particles to the photon, and the quantity

$$D(q_0, \mathbf{q}) \;=\; \frac{1}{q_0^2 - |\mathbf{q}|^2} \tag{14.1.1}$$

is the amplitude for the propagation of a photon whose energy is $c\,q_0$ and whose momentum is $\mathbf{q}$. This is known as a "propagator".

Recall that $(E_f - E_i)^2 - |\mathbf{p}_f - \mathbf{p}_i|^2 c^2 = c^2(q_0^2 - |\mathbf{q}|^2)$ is invariant under Lorentz transformations and so this propagator is the same in any reference frame.

In relativity, we cannot really talk about a 'potential' since this implies some sort of instantaneous action at a distance. The relativistic approach to all interactions is via the exchange of a "gauge boson" which carries the interaction between the particles that are interacting. The interaction between the interacting particles and the gauge bosons are always local in space and time. Particles can only influence each other at a distance because gauge bosons are emitted by one of the particles, they propagate through space-time, and are then absorbed by the other interacting particle.



There is a classical picture of this. Two people who throw a ball to each other back and forth will experience a repulsive force - there is no action at a distance but only an exchange of 'particles' that carry momentum.

14.2 Virtual particles

For a photon of energy $c\,q_0$ and momentum $\mathbf{q}$ we have the relation, $q_0 = |\mathbf{q}|$, so one would think that the propagator defined in eq.(14.1.1) would diverge.

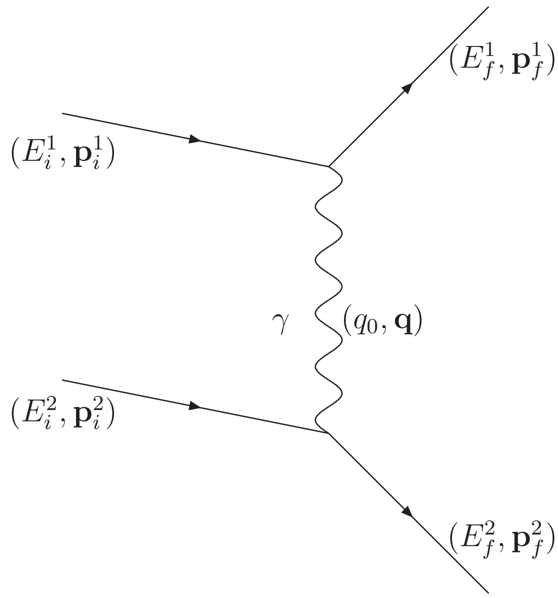
We are rescued by Heisenberg's uncertainty principle that tells us that over a sufficiently short period of time there is an uncertainty in energy. This means that if a particle only exists for a very short time we no longer have the usual relation between energy and momentum

$$E^2 \;=\; |\mathbf{p}|^2 c^2 \,+\, m^2 c^4$$

and in the case of the photon this means $q_0 \neq |\mathbf{q}|$. Such particles, which are exchanged rapidly between other particles, are called "virtual particles" and because their energy and momentum do not obey the relativistic energy-momentum relation they are said to be "off mass-shell".

14.3 Feynman Diagrams

We can 'unpin' the charge at the origin and use these techniques to calculate the scattering amplitude for the scattering of a particle of charge e_1 and incoming energy and momentum, $(E_i^1, \mathbf{p}_i^1)$ against a particle with charge e_2 and incoming energy and momentum $(E_i^2, \mathbf{p}_i^2)$ to form final state energies and momenta $(E_f^1, \mathbf{p}_f^1)$ and $(E_2^2, \mathbf{p}_f^2)$. We represent this process diagrammatically as



This is known as a "Feynman diagram" (or "Feynman graph"). The amplitude for the process is obtained by applying a set of "Feynman rules" for each vertex and internal line. The full set of Feynman rules takes into account the spins of the external and internal particles (gauge bosons, such as photons have spin one) - these are beyond the scope of these lectures.

Some of the Feynman rules in the case of electromagnetic interactions are:

- A factor of the charge at each vertex between a charged particle and a photon.
- Energy and momentum are conserved at each vertex.
- A factor of
$$D(q_0, \mathbf{q}) \; = \; \frac{1}{(q_0^2 - |\mathbf{q}|^2)}$$
for the propagation of an internal gauge boson with energy $c\, q_0$ and momentum $\mathbf{q}$.

In the above example we have (by conservation of energy and momentum at each vertex)

$$q_0 \; = \; (E_f^1 - E_i^1)/c \; = (E_f^2 - E_i^2)/c$$

$$\mathbf{q} \; = \; \mathbf{p}_f^1 - \mathbf{p}_i^1 \; = \mathbf{p}_f^2 - \mathbf{p}_i^2,$$

so that the amplitude is proportional to

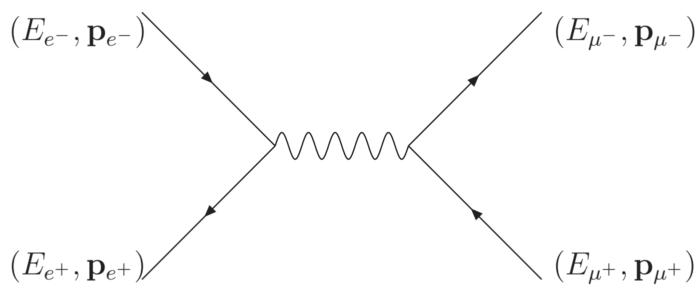
$$\frac{e_1 e_2}{(q_0^2 - |\mathbf{q}|^2)} \; = \; \frac{e_1 e_2}{(E_f^1 - E_i^1)^2/c^2 - |(\mathbf{p}_f^1 - \mathbf{p}_i^1)|^2}.$$

We can turn these diagrams on their side and consider the annihilation of a particle and its antiparticle. For example, the process

$$e^+ + e^- \rightarrow \mu^+ + \mu^-,$$

(the muons, $\mu^\pm$ are more massive copies of the electron or positron, with the same electric charge, e)

The Feynman diagram for this process is



Note the convention that the direction of the arrow on the antiparticles, e^+ and μ^+ are drawn against the direction of motion of the particles.

In this case the energy and momentum carried by the gauge boson are the sum of the initial (or final) energies and momenta

$$q_0 = (E_{e^-} + E_{e^+})/c = (E_{\mu^-} + E_{\mu^+})/c$$

$$\mathbf{q} = (\mathbf{p}_{e^-} + \mathbf{p}_{e^+}) = (\mathbf{p}_{\mu^-} + \mathbf{p}_{\mu^+})$$

so that the amplitude is proportional to

$$\frac{e^2}{(E_{e^-} + E_{e^+})^2/c^2 - |(\mathbf{p}_{e^-} + \mathbf{p}_{e^+})|^2}.$$

In the centre-of-mass frame of the incoming electron-positron pair (e.g. in the lab. frame of LEP) $(\mathbf{p}_{e^-} + \mathbf{p}_{e^+}) = 0$ and $(E_{e^-} + E_{e^+})$ is the total centre-of-mass energy, which we denote as $\sqrt{s}$ so that (absorbing a factor of c^2 into the constant of proportionality) we have

$$\mathcal{A} \propto \frac{e^2}{s}.$$

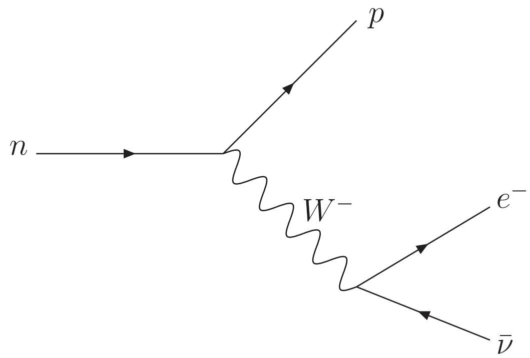
14.4 Weak Interactions

The gauge bosons (interaction carriers) of the weak interactions are $W^\pm$ and Z. the fact that the W-bosons carry electric charge tells us that electric charge can be exchanged in weak interaction processes, which is how we get β-decay.

The Feynman diagram for the process

$$n \rightarrow p + e^- + \bar{\nu}$$

is



(note that once again the direction of the arrow on the antineutrino is opposite to the direction of the antineutrino.)

What is happening is that the neutron emits a W^- and is converted into a proton (we will see later that the neutron and proton are not point particles but are made up of quarks, so that it is actually a d-quark, whose electric charge is $-\frac{1}{3}e$, that is converted into a u-quark, whose charge is $+\frac{2}{3}e$). At the other end (after the W^- has propagated) the W^- decays into an electron and an antineutrino.

The equivalent of the electric charge in weak interactions is a coupling, g_W, which indicates the strength of the coupling of the weakly interacting particles to the W-bosons and is approximately twice the electron charge (the coupling to the neutral Z-boson is almost equal to this value).

The W^- has a mass $M_W = 80.4\,\mathrm{GeV/c^2}$ and for the propagation of a massive particle, the propagator is

$$D^W(q_0, \mathbf{q}) \;=\; \frac{1}{q_0^2 - |\mathbf{q}|^2 - M_W^2 c^2},$$

where again $c\,q_0$ and $\mathbf{q}$ are the energy and momentum difference between the incoming neutron and outgoing proton and are transferred to the electron antineutrino pair. ($c\,q_0$ is the Q-value of the decay). Once again if the W-boson were a real particle we would have $q_0^2 - |\mathbf{q}|^2 - M_W^2 c^2 = 0$ and the propagator would diverge. However, in this process the W-boson propagates for a very short time before decaying into the electron and antineutrino- it is a virtual particle and therefore off mass-shell so that this energy momentum relation is violated by virtue of the uncertainty principle.

The amplitude for this decay is therefore proportional to

$$\frac{g_W^2}{q_0^2 - |\mathbf{q}|^2 - M_W^2 c^2}$$

If we take the non-relativistic limit we may neglect q_0^2 compared with $|\mathbf{q}|^2$ and this amplitude can be viewed as the matrix element of a weak potential, V^{wk}, between the initial (neutron) state with momentum $\mathbf{p}_n$ and final (proton) state momentum $\mathbf{p}_p$, with $\mathbf{q} = \mathbf{p}_p - \mathbf{p}_n$ i.e.

$$\frac{g_W^2}{-|\mathbf{q}|^2 - M_W^2 c^2} \;=\; \int e^{-i\mathbf{p}_p\cdot\mathbf{r}/\hbar} V^{wk}(r) e^{i\mathbf{p}_n\cdot\mathbf{r}/\hbar} d^3\mathbf{r}$$

The potential for which this relation is obeyed is

$$V^{wk}(r) \ = \ \frac{g_W^2}{r} \exp(-M_W c r/\hbar)$$

Such a potential is called a Yukawa potential (originally proposed by Yukawa as a description of the strong interactions mediated by pions - the mass of the W- in the above formula would then be replaced by the pion mass - this picture of the strong interactions is now obsolete).

As well as decreasing as $1/r$ (like the Coulomb potential) this has an exponentially suppressed term for large values of r. The effective force therefore has a range R, where

$$R \sim \frac{\hbar}{M_W c}.$$

At distances much larger than this the potential is rapidly suppressed. This is an extremely short range (about 10^{-3} fm.)

Returning to the amplitude in terms of the energy and momentum transferred, in the case of β-decay the momentum transferred (a few MeV/c) is very small compared with $M_W c$ which is 80.4 GeV/c so we can also neglect the momentum term and approximate the amplitude by

$$\frac{-g_W^2}{M_W^2 c^2}$$

Because of the very large mass of the W-boson this is extremely small and it is the reason that weak interactions are actually so weak.

We see that in general an interaction for which the gauge boson is massive has a range which is inversely proportional to the mass of the gauge boson, whereas interactions for which the gauge boson is massless are long range - meaning that the potential only falls as $1/r$.

14.5 Strong Interactions

The strong interaction is an exception to this rule. The gauge bosons (gluons) are massless and yet the strong interactions have a range of only a fm. The reason for this is due to a phenomenon known as "quark confinement", which will be discussed later. The essential idea is that the couplings of the strongly interacting particles to the gluons, which binds these strongly interacting particles together, grows as the distance between the particles increases - making it impossible to separate the particles to very large distances.

Chapter 15

Classification of Particles

Particle type		Strong interaction	Weak interaction	Electromagnetic interaction	Spin
Leptons		No	Yes	Some	$\frac{1}{2}$
	Mesons	Yes	Yes	Yes	integer
Hadrons	Baryons	Yes	Yes	Yes	half-integer

Interaction carriers

Interaction	Gauge-boson
strong	gluon
weak	$W^{\pm},\ Z$
electromagnetic	photon (γ)

15.1 Leptons

The electron and the neutrino are leptons. They partake in the weak interactions and the electron, being electrically charged, also has electromagnetic interactions. They do *not* interact strongly and are not found inside the nucleus.

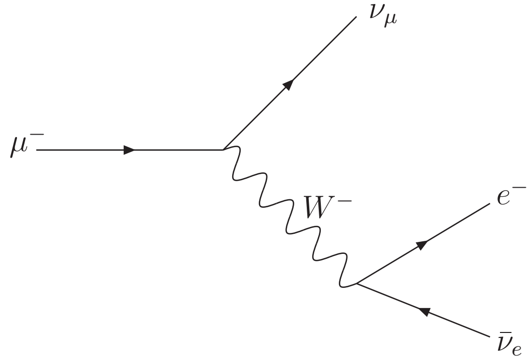
In terms of coupling to gauge bosons, this means that they both couple to $W^{\pm}$- and Z-bosons and the electron couples to the photon. There is no coupling between leptons and gluons.

Nature gives us three copies of each "family" or "generation" of particles. There are, therefore, two particles with similar properties to the electron (electric charge $-e$, spin-$\frac{1}{2}$, weakly interacting but not strongly interacting). These are called the muon (μ) and the tau (τ). Each of these has its own neutrino, ν_μ and ν_τ respectively.

Thus the six leptons are

Leptons			Electric Charge
ν_e	ν_μ	ν_τ	0
e	μ	τ	-1

The electron has a mass of 0.511 Mev/c^2, the muon a mass of 106 Mev/c^2 and the tau a mass of 1.8 Gev/c^2. The heavier charged leptons can decay via the weak interactions into an electron a neutrino and an anti-neutrino. The charged lepton emits a W^- and converts into its own neutrino. The W^- then decays into an electron and an electron-type anti-neutrino - just as in the β-decay of a neutron.



The muon has a lifetime of about 2×10^{-6} s. and the tau about 3×10^{-13} s. (these are regarded as "long-lived" particles!)

In the same way that the electron has an antiparticle (positron) with positive electric charge and the same mass and spin, the μ^-, and τ- also have antiparticles, μ^+ and τ^+, respectively. Likewise ν_μ and ν_τ have antiparticles $\bar{\nu}_\mu$ and $\bar{\nu}_\tau$, respectively.

15.2 Hadrons

These are particles that partake in the strong interactions.

Hadrons with integer spins (bosons) are called "mesons", whereas hadrons with half (odd-)integer spins (fermions) are called "baryons".

Well over one hundred of each type have been identified so far. For this reason, hadrons are no longer considered to be elementary particles (leptons probably are) but to be constructed out of elementary spin-$\frac{1}{2}$ particles known as "quarks".

Mesons are bound states of a quark and an anti-quark and can therefore have integer spin. Baryons are bound states of three quarks and can have spin-$\frac{1}{2}$ or spin-$\frac{3}{2}$.

The proton is the only hadron which is absolutely stable (the lifetime is known to be greater than 10^{32} years!). All other hadrons decay eventually into protons, leptons and photons.

Hadrons participate in all interactions since quarks from which they consist participate in all interactions. As for leptons, there are also 3 generations for quarks

Quarks			Electric Charge
u	c	t	+2/3
d	s	b	-1/2

Some hadrons cannot decay via the strong interactions and can only decay weakly (the neutron is such an example). This is because the quarks come in different types called "flavours" and the strong interactions conserve this flavour. For example the meson K^- is a bound state of a s-quark and an $\bar{u}$ anti-quark. The s-quark has a flavour called "strangeness" (for unfortunate historical reasons the s-quark is assigned strangeness -1 - and its antiparticle has strangeness +1). The strong interactions conserve flavor and strangeness in particular, and since the K^- is the lightest meson which contains an s-quark, it cannot decay via the strong interactions.

The weak interactions do not necessarily conserve flavour, so that via the weak interactions the $s-$quark can decay into a $u-$quark, emitting a W^- which decays into and electron and anti-neutrino. The final state meson is a bound state of an u-quark and its anti-quark, which can bind together to form a neutral pion, π^0. Thus the decay

$$K^- \rightarrow \pi^0 + e^- + \bar{\nu}_e$$

can proceed through the weak interactions. The lifetime of the K^- is 1.2×10^{-8} s. (another long-lived particle)

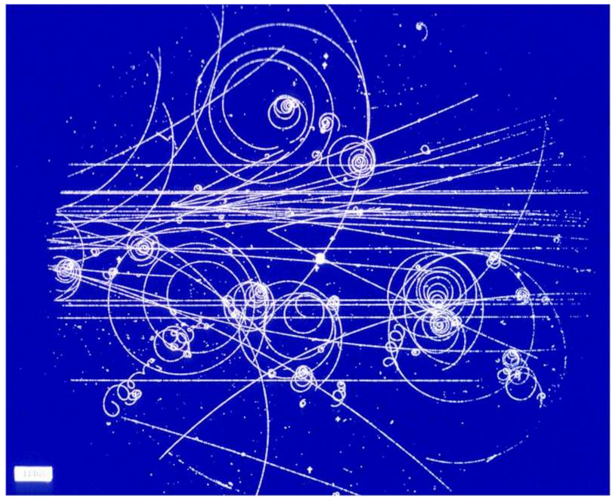
On the other hand there exists a meson K^{*-}, which is also a bound state of an $s-$quark and a $\bar{u}$ anti-quark, but in an excited state and with a mass which is greater than that of a K^- and a π^0 combined. The decay

$$K^{*-} \rightarrow K^- + \pi^0,$$

can indeed proceed via the strong interactions, since flavour is conserved (note that an anti-quark always has the opposite flavour of the corresponding quark - which means that a π^0 is flavour neutral). Typical lifetimes of particles that can decay through strong interactions are 10^{-23} s.

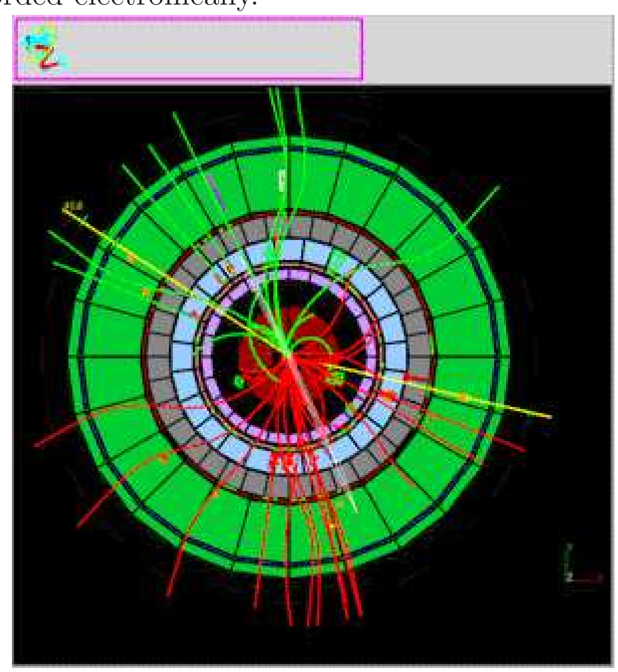
15.3 Detection of "Long-lived" particles

A particle whose lifetime exceeds about 10^{-11} s. and is travelling almost with the speed of light can leave a discernible track in a detector (recall that particles travelling with velocities very close to the velocity of light suffer a considerable time dilation so that the lifetime in the laboratory is larger than the lifetime in the frame of the particle - so that the track left is longer.)



The first particle detectors were bubble chambers. These were filled with saturated vapour. When a charged particle travelled through the vapour small bubbles would condense on it leaving a visible track. The bubble chamber was placed in a magnetic field which caused the path of the charged particle to curve by an amount which depends on its momentum and mass - this enables the particle to be identified (for example electron tracks have a very small radius of curvature) and the momentum of the particle to be measured. 'Vertices' can also be seen - these are caused by a neutral particle, which leaves no track, decaying into two (or more) charged particles.

Modern detectors do not give rise to visible tracks. Some types consist of arrays of electric wires, fibers or silicon layers. When a charged particle approaches a wire it causes an electric discharge which is recorded electronically.



By tracking which of the wires discharge it is possible to reconstruct the paths of the charged particles.

15.4 Detection of Short-lived particles - Resonances

Particles with a lifetime of less than about 10^{-11} s. do not live long enough to leave a track in a detector. They are observed as "resonances" - peaks in production cross-sections or in decay channels when the centre-of-mass energy of the incident particles in a scattering experiment is equal to the mass of the resonance particle (times c^2), or if the centre-of-mass energy of some subset of the final state particles is to the mass of the resonance particle (times c^2).

These peaks have a width Γ, corresponding to the uncertainty in their energy due to the fact that they have a short lifetime, τ. According to Heisenberg's uncertainty relation

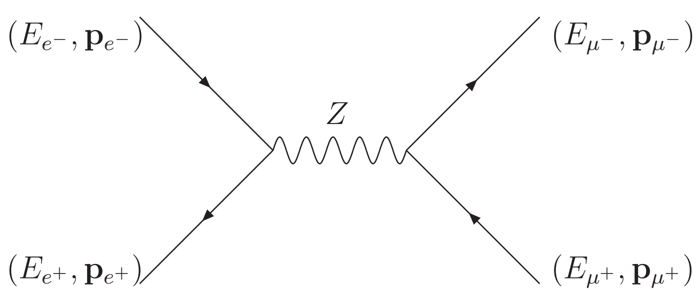
$$\Gamma \; = \; \frac{\hbar}{\tau}$$

For example, consider the Z-boson. This is a neutral particle that couples to all particles that partake in the weak interaction. Thus it can decay into a pair of charged leptons (i.e. a charged lepton and its antiparticle), a pair of neutrinos, or into quarks - ending up as showers of hadrons. It can also be produced by the annihilation of any of the above-mentioned pairs of particles. It was discovered in 1983 in an proton-antiproton scattering, but studied in more detail in electron-positron scattering in which the centre-of-mass energy was tuned to match the rest energy of the Z-boson.

The Feynman graph for a typical production and decay process

$$e^+ \, e^- \; \rightarrow Z \; \rightarrow \mu^+ \, \mu^-$$

is



The amplitude for the exchange of the Z is (up to an overall constant)

$$\frac{1}{(s - M_z^2 c^4)}$$

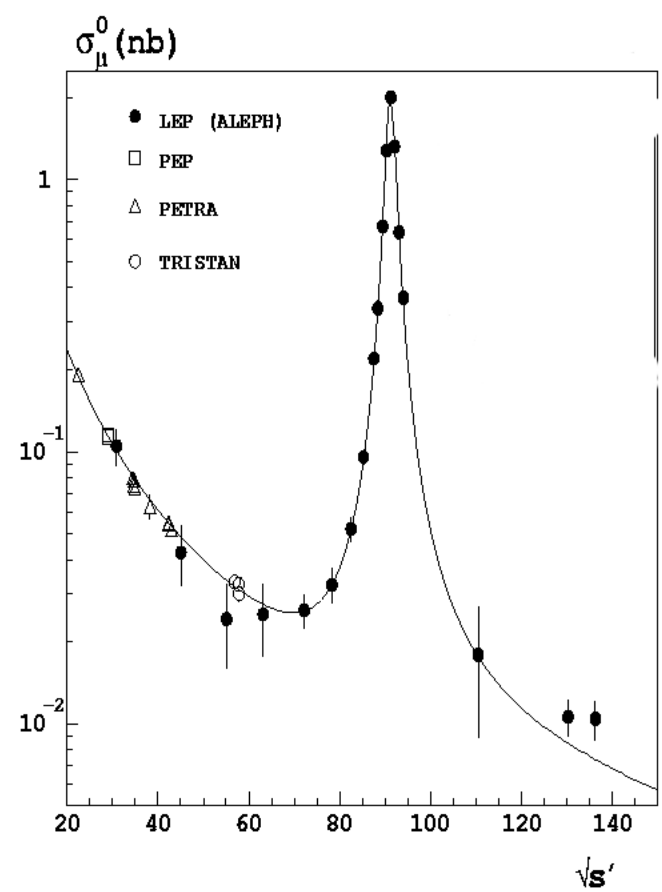
where $s = (E_{e^-} + E_{e^+})^2 - (\mathbf{p}_{e^-} + \mathbf{p}_{e^+})^2 c^2$, is the square of the centre-of-mass energy of the incident electron-positron pair.

We see that if we tune $\sqrt{s}$ to be exactly equal to $M_Z c^2$ this diverges. The above formula neglects the fact that the particle is unstable and has a width, Γ. Far away from the resonant energy this approximation is reasonable, but in the resonant region the above amplitude is modified to the (complex) expression

$$\frac{1}{(s - M_z^2 c^4 + i\Gamma M_z c^2)}$$

The transition probability is the square modulus of the amplitude, so that the scattering cross-section is proportional to

$$\sigma \propto \frac{1}{(s - M_z^2 c^4)^2 + \Gamma^2 M_Z^2 c^4}.$$

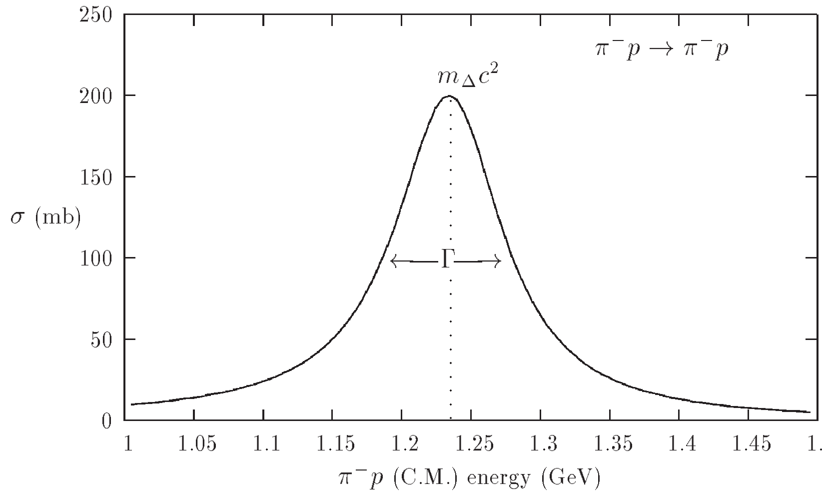


We note that this has a maximum when $\sqrt{s} = M_z c^2$ i.e. when the centre-of-mass energy is exactly equal to the rest-energy of the Z and the cross-section falls to one-half its maximum value when $\sqrt{s} = M_z c^2 \pm \frac{1}{2}\Gamma$

A strongly interacting particle resonance will occur whenever the centre-of-mass of the incident scattering particles is close to the rest energy of a particle that has the same flavour as the sum of the flavours of the incoming particles - this means that the resonant particle can be made up from the same quarks (and anti-quarks) as the two incident particles. For example, there exists a baryon called the Δ^0 which has the same flavour as a neutron (zero electric charge, zero strangeness etc. - although it does have spin-$\frac{3}{2}$). This can be produced in the scattering of a proton against a negative pion.

$$p + \pi^- \;\rightarrow\; \Delta^0 \;\rightarrow p + \pi^-$$

If we plot the cross-section for this scattering in the region of centre-of-mass energy 1 - 1.5 GeV, we get a resonance



We see that the Δ^0 has a mass of 1.23 GeV/c^2 and a width of about 0.1 GeV.

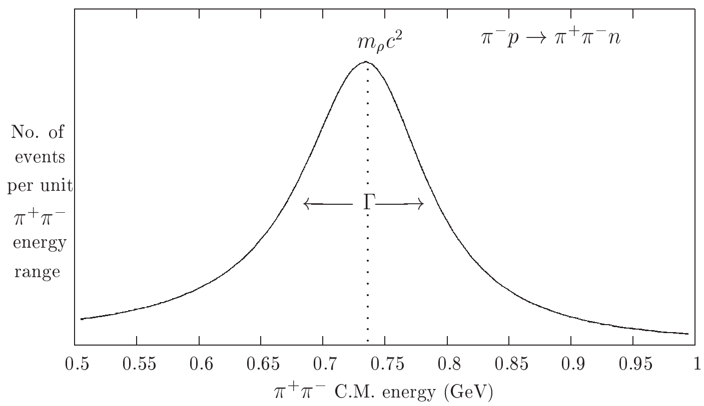
It is not always possible to prepare an initial state state with the correct flavour for the production of a given particle. In such cases one can look for resonances in the decay of the resonant particle when the centre-of-mass energy of the decay products is equal to the rest energy of the resonant particle. For example, consider the process

$$p + \pi^- \;\rightarrow\; \pi^+ + \pi^- + n$$

The pions leave tracks in detectors and their momenta can be measured by observing the radius of curvature of the tracks in a magnetic field. Thus we can contract the Lorentz invariant quantity which is the centre-of-mass energy of the two-pion system

$$E_{\pi\pi} \;=\; \sqrt{(E_{\pi^+} + E_{\pi^-})^2 - (\mathbf{p}_{\pi^+} + \mathbf{p}_{\pi^-})^2 c^2}$$

There is a particle called a ρ^0 which has the same flavour as the $\pi^+\pi^-$ pair and a mass of 740 MeV/c^2 and a width of about 0.1 GeV. If we plot the number of events for a given $E_{\pi\pi}$ against $E_{\pi\pi}$, we observe a resonance.



15.5 Partial Widths

An unstable particle can usually decay into several different possible "channels". The fraction of the decays into a particular channel is called the "branching ratio". For example the branching ratio for a Z to decay into a $\mu^+ \mu^-$ pair, $B_{Z\to\mu\mu}$ is 3.4 %. The width of the resonance in the process

$$e^+ e^- \to Z \to \mu^+ \mu^-$$

is the "partial width", $\Gamma_{Z\to\mu\mu}$ and in this case it is 0.084 GeV. The total width Γ_{tot} is the sum of all the partial widths. The branching ratio is the ratio of the partial width for a particular channel and the total width

$$B_X = \frac{\Gamma_X}{\Gamma_{tot}}.$$

Thus in the case of the Z a measurement of the partial width for the decay channel into a muon pair and a determination of the branching ratio yields the total width

$$\Gamma_{tot} = \frac{\Gamma_{Z\to\mu\mu}}{B_{Z\to\mu\mu}} = 2.5 \text{ GeV}.$$

Chapter 16

Constituent Quark Model

Quarks are fundamental spin-$\frac{1}{2}$ particles from which all hadrons are made up. Baryons consist of three quarks, whereas mesons consist of a quark and an anti-quark. There are six types of quarks called "flavours". The electric charges of the quarks take the value $+\frac{2}{3}$ or $-\frac{1}{3}$ (in units of the magnitude of the electron charge).

Symbol	**Flavour**	**Electric charge (e)**	**Isospin**	$\mathbf{I_3}$	**Mass Gev/c^2**
u	up	$+\frac{2}{3}$	$\frac{1}{2}$	$+\frac{1}{2}$	≈ 0.33
d	down	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	≈ 0.33
c	charm	$+\frac{2}{3}$	0	0	≈ 1.5
s	strange	$-\frac{1}{3}$	0	0	≈ 0.5
t	top	$+\frac{2}{3}$	0	0	≈ 172
b	bottom	$-\frac{1}{3}$	0	0	≈ 4.5

These quarks all have antiparticles which have the same mass but opposite I_3, electric charge and flavour (e.g. anti-strange, anti-charm, etc.)

16.1 Hadrons from u,d quarks and anti-quarks

Baryons:

Baryon	**Quark content**	**Spin**	**Isospin**	$\mathbf{I_3}$	**Mass Mev/c^2**
p	uud	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	938
n	udd	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	940
Δ^{++}	uuu	$\frac{3}{2}$	$\frac{3}{2}$	$+\frac{3}{2}$	1232
Δ^{+}	uud	$\frac{3}{2}$	$\frac{3}{2}$	$+\frac{1}{2}$	1232
Δ^{0}	udd	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	1232
Δ^{-}	ddd	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	1232

- Three spin-$\frac{1}{2}$ quarks can give a total spin of either $\frac{1}{2}$ or $\frac{3}{2}$ and these are the spins of the baryons (for these 'low-mass' particles the orbital angular momentum of the quarks is zero - excited states of quarks with non-zero orbital angular momenta are also possible and in these cases the determination of the spins of the baryons is more complicated).

- The masses of particles with the same isospin (but different I_3) are almost the same - the differences being due to the electromagnetic interactions which distinguish members of the isospin multiplet with different electric charge. If it were possible to 'switch off' the electromagnetic interactions these masses would be exactly equal.

- The baryons which consist of three u-quarks or three d-quarks only occur for spin $\frac{3}{2}$ (we return to this later)

Mesons:

Meson	**Quark content**	**Spin**	**Isospin**	$\mathbf{I_3}$	**Mass Mev/c^2**
π^+	$u\bar{d}$	0	1	$+1$	140
π^0	$\frac{1}{\sqrt{2}}\left(u\bar{u}-d\bar{d}\right)$	0	1	0	135
π^-	$d\bar{u}$	0	1	-1	140
ρ^+	$u\bar{d}$	1	1	$+1$	770
ρ^0	$\frac{1}{\sqrt{2}}\left(u\bar{u}-d\bar{d}\right)$	1	1	0	770
ρ^-	$d\bar{u}$	1	1	-1	770
ω	$\frac{1}{\sqrt{2}}\left(u\bar{u}+d\bar{d}\right)$	1	0	0	782

- A spin-$\frac{1}{2}$ quark and an anti-quark with the same spin can combine (when the orbital angular momentum is zero) to give mesons of spin-0 or spin-1.

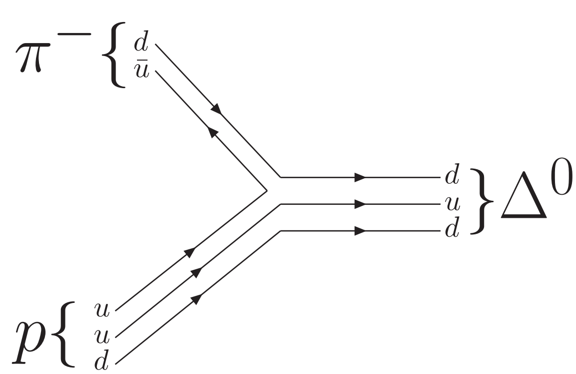
- The neutral mesons are not pure $u\bar{u}$, or $d\bar{d}$ states, but quantum superpositions of these.

- The neutral mesons have $I_3 = 0$. They could be in an isospin $I = 1$ state, (π^0, ρ^0), in which case their masses are similar to those of their charged counterparts, or $I = 0$ (ω) in which case their masses are somewhat different.

The strong interactions conserve flavour. There a d-quark cannot be converted into an s-quark (or vice versa), even though the electric charge is the same.

However, in a scattering process a quark *can* annihilate against an anti-quark of the same flavour, giving some energy which can be converted into mass and used to create a more massive particle. An example of this is

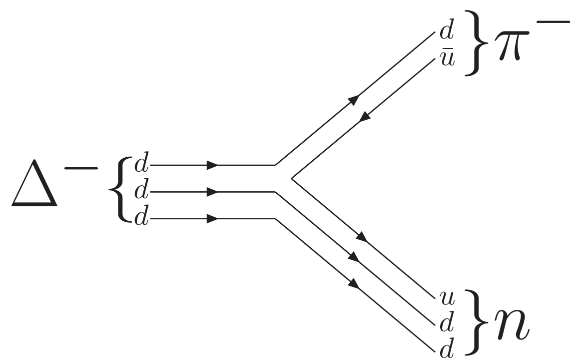
$$\begin{array}{ccccc} \pi^- & + & p & \to & \Delta^0 \\ (d\bar{u}) & & (uud) & & (udd) \end{array}$$



A u-quark from the proton and a $\bar{u}$ anti-quark from the pion have annihilated and the extra energy goes into the extra mass of the Δ^0, which is very short-lived and appears as a resonance in the $\pi^- p$ scattering cross-section.

Likewise in a decay process it is possible for some of the mass of the decaying particle to create a quark and anti-quark pair of the same flavour which go to forming the decay products, e.g.

$$\begin{array}{ccccc} \Delta^- & \rightarrow & \pi^- & + & n \\ (ddd) & & (ddu) & & (d\bar{u}) \end{array}$$



Here a u-quark and $\bar{u}$ anti-quark pair are created when the Δ^- decays and the $\bar{u}$ anti-quark binds with one of the d-quarks in the decaying Δ^- to make a π^-, whereas the u-quark binds with the other two d-quarks in the decaying Δ^- in order to make a neutron.

Quark and anti-quark pair creation is possible in any particle particle scattering process provided there is sufficient energy to create the final state particles. Thus for example it is possible to have the inelastic process

$$\begin{array}{ccccccccccc} p & + & p & \rightarrow & n & + & n & + & \pi^+ & + & \pi^+ \\ (uud) & & (uud) & & (udd) & & (udd) & & (u\bar{d}) & & (u\bar{d}) \end{array}$$

In this process two pairs of d-quarks and $\bar{d}$ anti-quarks are created. The d-quarks bind with the u and d quarks from the incoming protons to form neutrons, whereas the $\bar{d}$ anti-quarks bind with the remaining u-quarks in the incoming protons to form pions. In the centre-of-mass frame in which the total momentum is zero, so that the outgoing particles can be at rest - this is the lowest energy that they can have and is equal to sum of the masses of two neutrons and two pions (times by c^2), which is therefore the lowest total centre-of-mass energy of the incoming protons.

16.2 Hadrons with s-quarks (or $\bar{s}$ anti-quarks)

Baryons:

Baryon	Quark content	Spin	Isospin	I_3	Mass Mev/c^2
Σ^+	uus	$\frac{1}{2}$	1	$+1$	1189
Σ^0	uds	$\frac{1}{2}$	1	0	1193
Σ^-	dds	$\frac{1}{2}$	1	-1	1189
Ξ^0	uss	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	1314
Ξ^-	dss	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1321
Λ	uds	$\frac{1}{2}$	0	0	1115
Σ^{*+}	uus	$\frac{3}{2}$	1	$+1$	1385
Σ^{*0}	uds	$\frac{3}{2}$	1	0	1385
Σ^{*-}	dds	$\frac{3}{2}$	1	-1	1385
Ξ^{*0}	uss	$\frac{3}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	1530
Ξ^{*-}	dss	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1530
Ω^-	sss	$\frac{3}{2}$	0	0	1672

- For historical reasons the s-quark was assigned strangeness equal to -1, so these baryons have strangeness -1, -2 or -3 for one, two, or three strange quarks respectively. (likewise the b-quark has bottom flavour -1, whereas the c-quark has flavour charm=+1, and the t-quark has flavour top=+1)
- As in the case of Δ^- and Δ^{++}, the Ω^- which has three s-quarks (strangeness=-3) has spin-$\frac{3}{2}$.

The Ω^- had not discovered when the Quark Model was invented - its existence was a prediction of the Model. Furthermore its mass was predicted from the observation

$$M_{\Sigma^*} - M_\Delta \approx M_{\Xi^*} - M_{\Sigma^*} \approx 150\,\mathrm{MeV/c^2}$$

giving a predicted value for M_Ω of

$$M_\Omega = M_{\Xi^*} + 150 = 1680\,\mathrm{MeV/c^2}.$$

Mesons:

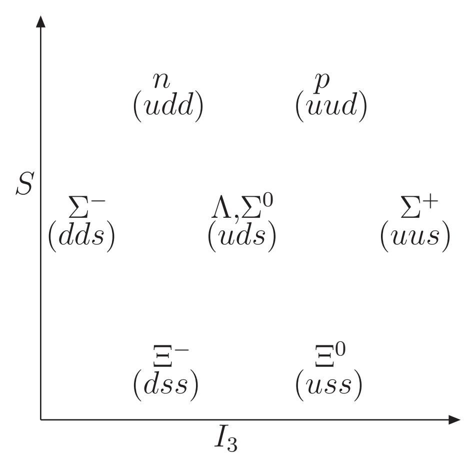
Meson	**Quark content**	**Spin**	**Isospin**	$\mathbf{I_3}$	**Mass Mev/c^2**
K^+	$u\bar{s}$	0	$\frac{1}{2}$	$+\frac{1}{2}$	495
K^0	$d\bar{s}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	495
$\overline{K^0}$	$s\bar{d}$	0	$\frac{1}{2}$	$+\frac{1}{2}$	495
K^-	$s\bar{u}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	495
η	$(u\bar{u},\, d\bar{d},\, s\bar{s})$	0	0	0	547
K^{*+}	$u\bar{s}$	1	$\frac{1}{2}$	$+\frac{1}{2}$	892
K^{*0}	$d\bar{s}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	896
$\overline{K^{*0}}$	$s\bar{d}$	1	$\frac{1}{2}$	$+\frac{1}{2}$	896
K^{*-}	$s\bar{u}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	892
ϕ	$s\bar{s}$	1	0	0	1020

16.3 Eightfold Way:

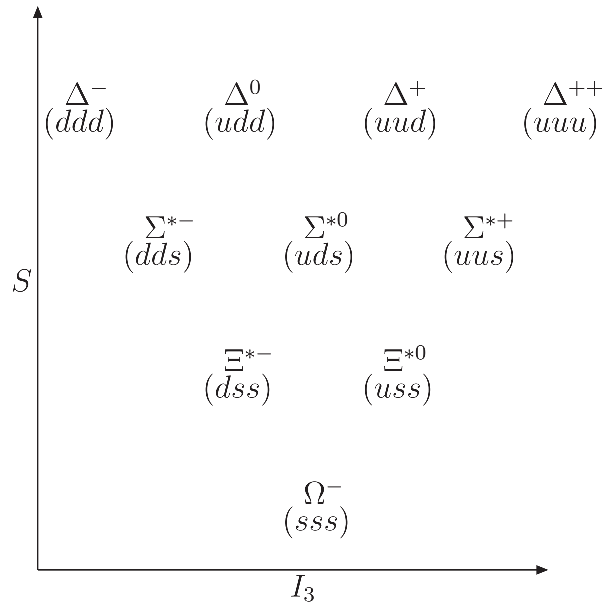
There is a method of classifying hadrons made up from u, d and s quarks and their anti-quarks by plotting particles with the same spin on a plot of strangeness against I_3.

For the lightest mesons and baryons there are eight particles on each plot. For this reason this classification method is known as the "Eightfold Way".

Spin-$\frac{1}{2}$ Baryons:

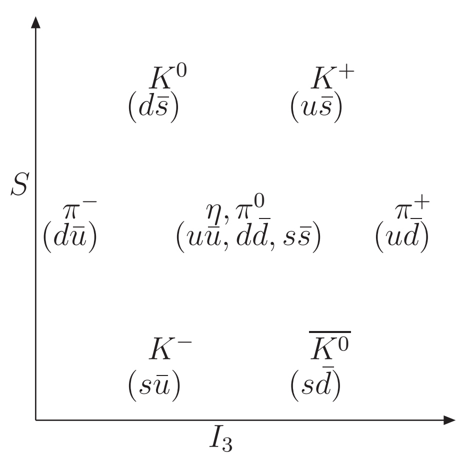


Spin-$\frac{3}{2}$ Baryons:

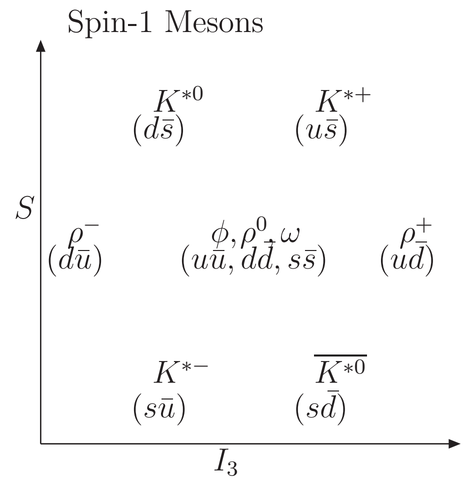


The rows contain the isospin multiplets. However, in the case of the row for $I = 1$, there can also be states with $I = 0$, $I_3 = 0$, so that the point in the middle can have two (or more) entries.

<u>Spin-0 Mesons:</u>



<u>Spin-1 Mesons:</u>



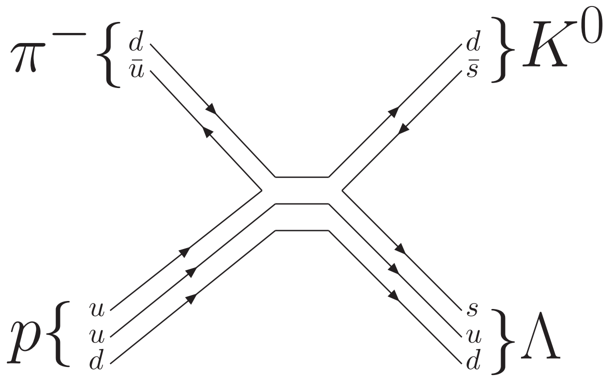
These meson multiplets contain the mesons and their antiparticles (obtained by replacing each quark by its anti-quark and vice versa), whereas the baryon multiplets have separate antiparticle multiplets which are bound states of three anti-quarks.

Some mesons, such as π^0, ρ^0, η are their own antiparticle, because they are bound states of a quark and an anti-quark of the same flavour so that replacing a quark by its anti-quark with the same flavour (and vice versa) produces the same particle. Other charged or neutral particles have separate antiparticles which have opposite electric charge and/or strangeness.

16.4 Associated Production and Decay

In strong interaction processes, quark flavour is conserved. s-quarks cannot be created or destroyed by the strong interactions (they can be created or destroyed by the weak interactions). This means that in a scattering experiment (e.g proton-proton or pion-proton scattering) one can only create a particle containing a strange quark if at the same time there is a particle containing an $\bar{s}$ anti-quark, so that the total strangeness is conserved. An example of such a process is

$$\begin{array}{ccccccc} \pi^- & + & p & \rightarrow & \Lambda & + & K^0 \\ (d\bar{u}) & & (duu) & & (dus) & & (\bar{s}d) \end{array}$$



What happens is that a u-quark annihilates against a $\bar{u}$ anti-quark and an s-quark $\bar{s}$ anti-quark pair has been created. This reaction is only possible above a threshold energy. In the centre-of-mass frame, the lowest total energy of the incoming particles is the sum of the masses of the Λ and the K^0, i.e.

$$\sqrt{s} = E_{CM}^{TOT} = (M_\Lambda + M_{K^0})\, c^2,$$

(here E_{CM}^{TOT} means to the *total* energy of the incoming (or outgoing) particles in the centre-of-mass frame - as the particles are not of the same mass, the individual energies of the particles will be different). In a (proton) fixed target experiment the pions must be accelerated to sufficient energy such that the centre-of-mass energy is greater than this value.

On the other hand the process

$$\begin{array}{ccccccc} \pi^- & + & p & \rightarrow & \Lambda & + & \pi^0 \\ (d\bar{u}) & & (duu) & & (dus) & & (\bar{d}d, \bar{u}u) \end{array}$$

is forbidden because the number of strange quarks in the initial and final states is not the same.

It is possible to scatter charged kaons ($K^\pm$) against nucleons. The K^- contains an s-quark. it is therefore possible to produce strange baryons in this process, such as

$$\begin{array}{ccccccc} K^- & + & n & \rightarrow & \Lambda & + & \pi^- \\ (s\bar{u}) & & (udd) & & (dus) & & (\bar{u}d) \end{array}$$

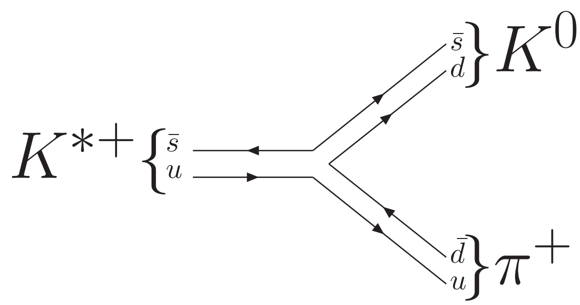
All flavours have been conserved in this reaction. However, $K^+ - n$ scattering will *not* produce a strange baryon because a strange baryon contains s-quarks but no $\bar{s}$ anti-quarks, whereas the K^+ contains a $\bar{s}$ anti-quark, but no s-quark.

Recent evidence has suggested that there is a resonance in $K^+ - n$ scattering at a centre-of-mass energy of 1.5 GeV. This suggests that there is an (unstable) particle with mass 1.5 GeV/c^2. If confirmed this would be a new type of particle called a "pentaquark" since it must be a bound state of four quarks and an anti-quark ($\bar{s}uudd$). Such a particle does *not* fit in with the usual quark model picture of hadrons.

Similarly, in the decays of particles containing s-quarks (or $\bar{s}$ anti-quarks), the decay can proceed via the strong interactions and will be very rapid - leading to a large width - only if the decay products have a total strangeness which is equal to the strangeness of the decaying particle. For such a process to occur, the mass of the decaying particle must be larger than the combined mass of the decay products.

An example is

$$\begin{array}{ccccc} K^{*+} & \rightarrow & K^0 & + & \pi^+ \\ (u\bar{s}) & & (d\bar{s}) & & (u\bar{d}) \end{array}$$



A d-quark and $\bar{d}$ anti-quark pair have been created but the initial and final states both contain an $\bar{s}$ anti-quark - so flavour is conserved.

$$m_{K^*} = 842, \quad m_{K^*} = 495, \quad m_\pi = 135 \ (\mathrm{MeV/c^2})$$

so there is enough energy in the decaying K^* to produce a kaon (K) and a pion, since the mass of the K^* exceeds the sum of the masses of the kaon and pion. This decay therefore can proceed via the strong interactions which means that the K^* has a very short lifetime. It is only seen as a resonance in the centre-of-mass frame of the $K - \pi$ system width a width of 50 MeV.

Likewise the Ξ^* has enough mass to decay into a Ξ plus a pion - the inital and final states both having strangeness -2, and similarly the Σ^* can decay into a Σ plus a pion, or into a Λ plus a pion - conserving strangeness. These decays are therefore very rapid as they proceed though strong interactions.

Most of the lighter strange particles do not have enough energy to decay into other strange particles. They therefore decay through the weak interactions - and therefore have a much longer lifetime. The usually can leave a track in a detector.

Combining associated production and decay one can have an event such as

$$\pi^+ + n \ \rightarrow \ K^{*+} \ + \ \Lambda \ \rightarrow \ K^0 + \pi^+ + \Lambda.$$

The observed particles are the K^0, π^+, Λ but if we look at the energies and momenta of the K^0 and π^+ and construct the quantity

$$P_{K\pi}^2 \ = \left(E_{K^0} + E_{\pi^+}\right)^2 / c^2 - \left(\mathbf{p}_{K^0} + \mathbf{p}_{\pi^+}\right)^2,$$

we would get a resonance peak at

$$P_{K\pi} \ = \ 842 \, \mathrm{MeV/c},$$

indicating that at such momenta a K^* particle is produced for a very short time.

16.5 Heavy Flavours

When the quark model was invented only $u-$, $d-$ and s-quarks were postulated and all known hadrons could be built out of these three quarks and their anti-quarks. Since then three new quarks, c, b and t gave been discovered. They are much more massive than the $u-$, $d-$ and s-quarks, so they were not discovered until sufficiently large accelerators had been built and were in use. In the same way that there are hadrons containing one or more $s-$quarks (or $\bar{s}$ anti-quarks), there are hadrons which contain these heavy quarks. So far, only hadrons containing one c-quark or one b-quark (or their antiparticles) have been observed. It is believed that a hadron which contained a t-quark would be too unstable to form a bound state.

There are also bound states of $c\,\bar{c}$ and $b\bar{b}$. These are neutral - like the ϕ meson which is a bound state of $s\,\bar{s}$. These heavy quarks were first observed by observing these neutral bound states.

16.6 Quark Colour

There is a difficulty within the quark model when applied to baryons. This can be seen if we look at the Δ^{++} or Δ^{-} or Ω^{-}, which are bound states of three quarks of the same flavour. For these low-mass states the orbital angular momentum is zero and so the spatial parts of the wavefunctions for these baryons is symmetric under interchange of the position of two of these (identical flavour) quarks.

We know that the total wavefunction for a baryon must be antisymmetric as baryons have half-odd-integer spin, so the spin part of the wave function should be antisymmetric. On the other hand these baryons have spin-$\frac{3}{2}$ which means that the spin part of the wavefunction is symmetric (for example the $S_z = +\frac{3}{2}$ state is the state in which all three quarks have $s_z = +\frac{1}{2}$ and this is clearly symmetric under the interchange of two spins).

This is solved by assuming that quarks come in three possible "colour" states - R, G or B. The antisymmetry of the baryon wavefunction is restored by the assumption that the baryon wavefunction is antisymmetric under the interchange of two colours. If a baryon is composed of three quarks with flavours f_1, f_2 and f_3 the these should also have a colour index, e.g. f_1^R, f_1^G or f_1^B etc. The colour antisymmetric wavefunction is written

$$\frac{1}{\sqrt{6}}\left(|f_1^R f_2^G f_3^B\rangle + |f_1^B f_2^R f_3^G\rangle + |f_1^G f_2^B f_3^R\rangle - |f_1^B f_2^G f_3^R\rangle - |f_1^R f_2^B f_3^G\rangle - |f_1^G f_2^R f_3^B\rangle\right)$$

We can see that this changes sign if we interchange any two colours. This means that in order to have a totally antisymmetric wavefunction (including the colour part), the spin and spatial part must be *symmetric* so that a particle in which all three quarks have the same flavour (and zero orbital angular momentum) must be symmetric under the interchange of any two of the spins - and this is the spin-$\frac{3}{2}$ state.

A state of three different colours which is antisymmetric under the interchange of any two of the colours is called a "colour singlet" state - we can think of it as a colourless state.

The quarks themselves are a colour triplet - meaning that they can be in any one of three colour states.

It is assumed that all physically observable particles (i.e all hadrons) are colour singlets (colourless particles). This means that it is not possible to isolate individual quarks and observe them. indeed no individual quark has ever been observed. This is called "quark confinement" and it is the explanation of why the strong interactions are short-range, despite the fact that the gluons, which are the strong-interaction carriers, are massless - you can't pull a quark too far away from the other quarks or antiquarks in the hadron to which it is bound.

For mesons we also require that the quarks and anti-quarks bind in such a way that the meson is a colour singlet. in the case of a quark and ant-quark bound state this means that the wavefunction is a superposition of R with $\bar{R}$, G with $\bar{G}$, and B with $\bar{B}$. Thus, for example, the wavefunction for the π^+ is written

$$|\pi^+\rangle \;\; = \;\; \frac{1}{\sqrt{3}}\left(|u^R\overline{d^R}\rangle + |u^G\overline{d^G}\rangle + |u^B\overline{d^B}\rangle\right)$$

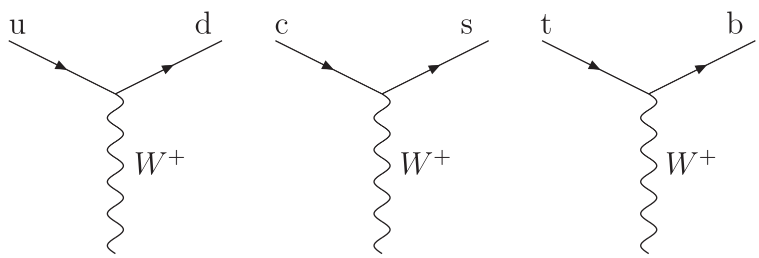
The colourless property is achieved by requiring that a quark of a given colour binds with an antiquark which is the antiparticle of a quark of the same colour - then we have to 'average' over all the colours by taking a superposition of all three possible colour pairs.

Chapter 17

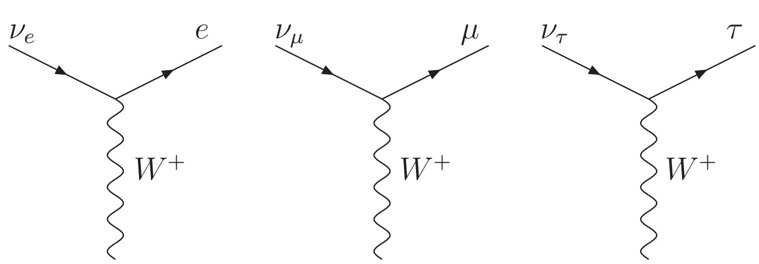
Weak Interactions

The weak interactions are mediated by $W^{\pm}$ or (neutral) Z exchange. In the case of $W^{\pm}$, this means that the flavours of the quarks interacting with the gauge boson can change.

$W^{\pm}$ couples to quark pairs (u, d). (c, s), (t, b) with vertices

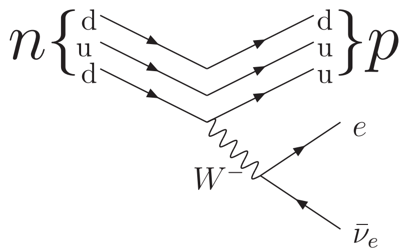


as well as to leptons (ν_e, e). (ν_μ, μ), (ν_τ, τ) with vertices



Note that in these interactions both quark number (baryon number) and lepton number are conserved.

It is this process that is responsible for β-decay. Neutron decays into a proton because a d-quark in the neutron converts into a u-quark emitting a W^- which then decays into an electron and anti-neutrino.



The amplitude for such a decay is proportional to

$$\frac{g_W^2}{(q^2 - M_W^2 c^2)},$$

where g_W is the strength of the coupling of the W^- to the quarks or leptons and $q^2 = E_q^2/c^2 - |\mathbf{q}|^2$, where $\mathbf{q}$ is the momentum transferred between the neutron and proton and E_q is the energy transferred. This momentum is of order 1 MeV/c and so we can neglect it in comparison with $M_W c$ which is 80.4 GeV/c. Thus the amplitude is proportional to

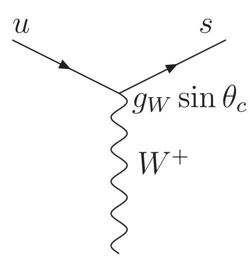
$$\frac{g_W^2}{M_W^2 c^2}.$$

The coupling g_W is not so small. In fact it is twice as large as the electron charge e. Weak interactions are weak because of the large mass term in the denominator.

At modern high energy accelerators, it is possible to produce weak interaction processes in which $|\mathbf{q}| \sim M_W c$ or even $|\mathbf{q}| \gg M_W c$. In such cases weak interactions are larger than electromagnetic interactions and almost comparable with strong interactions.

17.1 Cabibbo Theory

Particles containing strange quarks, e.g. $K^\pm$, K^0, Λ etc. cannot decay into non-strange hadrons via the strong interactions, which have to conserve flavour, but they can decay via the weak interactions. This is possible because $W^\pm$ not only couples a u-quark to a d-quark but can also (with a weaker coupling) couple a u-quark to an s-quark so we have a vertex



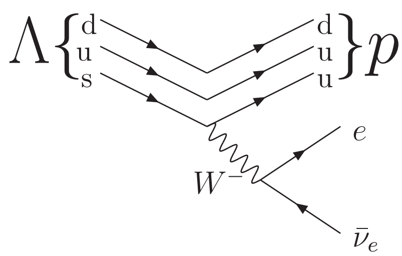
with coupling $g_W \sin\theta_C$, whereas the $u - d - W$ coupling is actually $g_w \cos\theta_C$. θ_C is called the “Cabibbo angle” and its numerical value is $\sin\theta_C \approx 0.22$.

This coupling allows a strange hadron to decay into non-strange hadrons and (sometimes) leptons.

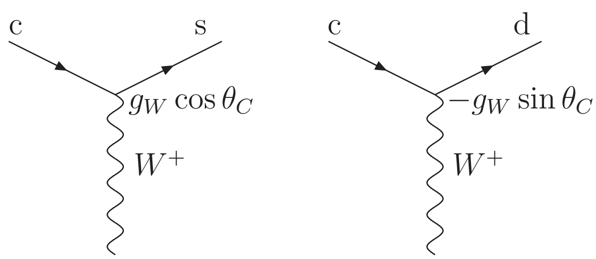
Thus, for example the decay

$$\Lambda \;\rightarrow\; p + e^- + \bar{\nu}_e$$

occurs when an s-quark converts into a u quark and emits a W^- which then decays into an electron and anti-neutrino. The Feynman graph is



Likewise, the c-quark has a coupling to the s-quark with coupling $g_W \cos\theta_C$ and a coupling to a d-quark with coupling $-g_W \sin\theta_C$.



This implies that charm hadrons are more likely to decay into hadrons with strangeness, because the coupling between a c-quark and a s-quark is larger than between a c-quark and a d-quark.

We can piece this together in a matrix form as follows

$$g_W \begin{pmatrix} d & s \end{pmatrix} \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} u \\ c \end{pmatrix}$$

This 2×2 matrix is called the "Cabbibo matrix". It is described in terms of a single parameter, the Cabibbo angle.

Since we know that there are, in fact, three generations of quarks this matrix is extended to a general 3×3 matrix as follows

$$g_W \begin{pmatrix} d & s & b \end{pmatrix} \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix}$$

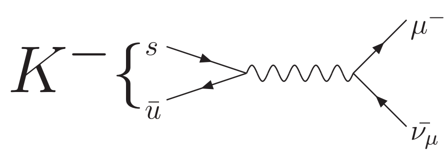
The 3×3 matrix is called the "CKM" (Cabibbo, Kobayashi, Maskawa) matrix. Quantum-mechanical constraints lead to the conclusion that of the nine elements there are only four independent parameters. Comparing the CKM matrix with the Cabibbo matrix we see that to a very good approximation, $V_{ud} \approx V_{cs} \approx \cos\theta_C$ and $V_{us} \approx -V_{cd} \approx \sin\theta_C$.

17.2 Leptonic, Semi-leptonic and Non-Leptonic Weak Decays

Because the $W^{\pm}$ couples either to quarks or to leptons, decays of strange mesons can either be leptonic, meaning that the final state consists only of leptons, semi-leptonic, meaning that the final state consists of both hadrons and leptons, or non-leptonic, meaning that the final state consists only of hadrons. For strange baryons only semi-leptonic and non-leptonic decays are possible because baryon number is strictly conserved - so there must be a baryon in the final state. Lepton number is also strictly conserved which means that a charged lepton is always accompanied by its anti-neutrino (or vice versa) in the final state.

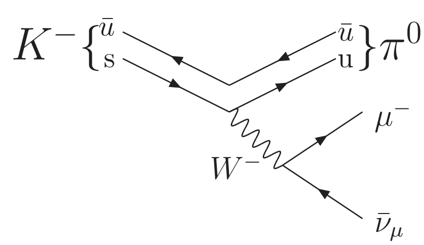
For mesons, examples are:

Leptonic decay $K^- \rightarrow \mu^- + \bar{\nu}_\mu$

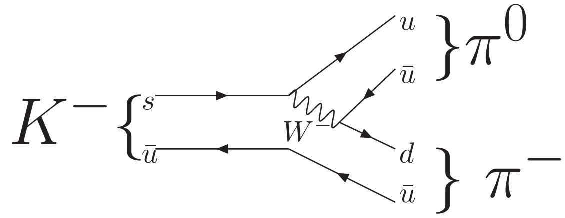


As well as converting an s-quark into a u-quark to emit a W^-, it is also possible to create a W^- from the annihilation of an s-quark with a $\bar{u}$ anti-quark.

Semi-leptonic decay $K^- \rightarrow \mu^- + \bar{\nu}_\mu + \pi^0$



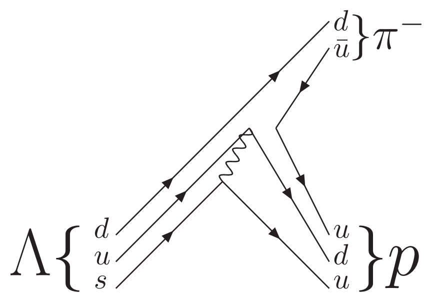
Non-leptonic decay $K^- \rightarrow \pi^0 + \pi^-$



Note that $m_K > 2m_\pi$ which is why this non-leptonic decay mode is energetically allowed.

In the case of baryons, we have already seen an example of a semi-leptonic decay, $\Lambda \rightarrow p\, e^-\ \bar{\nu}_e$. An example of a non-leptonic decay is

$$\Lambda \ \rightarrow \ p\, \pi^-$$



A W^- is exchanged between the s-quark and the u-quark in the Λ, converting them into a u-quark and a d-quark respectively. A $u - \bar{u}$ quark-antiquark pair is created in the process in order to make up the final state hadrons of a proton and a negative pion.

17.3 Flavour Selection Rules in Weak Interactions

Since in the exchange of a single $W^\pm$ an s-quark can be converted into a non-strange quark, it is highly unlikely that two strange quarks would be converted into non-strange quarks in the same decay process. We therefore have a selection rule for weak decay processes

$$\Delta S \ = \ \pm 1$$

Therefore, hadrons with strangeness -2 which decay weakly must first decay into a hadron with strangeness -1 (which in turn decays into non-strange hadrons). Thus, for example, we have

$$\Xi^0 \ \rightarrow \ \Lambda + \pi^0$$

The same selection rules apply for changes in other flavours (charm, bottom).

17.4 Parity Violation

The parity violation observed in β-decay arises because the $W^{\pm}$ tends to couple to quarks or leptons, which are left-handed (negative helicity), i.e. states in which the component of spin in their direction of motion is $-\frac{1}{2}\hbar$.

$W^{\pm}$ always couple to left-handed neutrinos. For quarks and massive leptons the $W^{\pm}$ *can* couple to positive helicity (right-handed) states, but the coupling is suppressed by a factor

$$\frac{mc^2}{E},$$

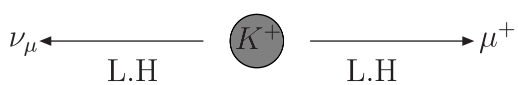
where m is the particle mass and E is its energy. The suppression is much larger for relativistically moving particles

In the case of nuclear β-decay, the nucleus is moving non-relativistically, but the electron typically has energy of a few MeV (and a mass of 0.511 MeV/c^2), so there is a significant suppression of the coupling to right-handed electrons. This is what was observed in the experiment by C.S. Wu on ^{60}Co.

For the coupling of $W^{\pm}$ to anti-quarks or anti-leptons, the helicity is reversed -i.e. the $W^{\pm}$ always couples to positive helicity anti-neutrinos and usually to positive helicity e^+, μ^+, τ^+ or to antiquarks, with a suppressed coupling to left-handed antileptons or anti-quarks.

A striking example of the consequence of this preferred helicity coupling can be seen in the leptonic decay of K^+.

$$K^+ \ \rightarrow \ \mu^+ + \nu_\mu$$



In the rest frame of the K^+ the momentum is zero, so the μ^+ and the ν_μ must move in opposite directions. The K^+ has zero spin, so by conservation of angular momentum, the two decay particles must have opposite spin component in any one chosen direction (e.g. the direction of the μ^+. This means that they have the *same* helicity. This means that the $W^{\pm}$ couples to the left-helicity anti-muon, μ^+ and such a coupling is suppressed by

$$\frac{m_\mu c^2}{E_\mu}$$

If we look at the decay mode

$$K^+ \ \rightarrow \ e^+ + \nu_e,$$

the same argument would lead to a suppression (of the decay amplitude) of

$$\frac{m_e c^2}{E_e}.$$

Since $m_e \ll m_\mu$ we expect the decay into a positron to be heavily suppresses. In fact we expect the ratio of the partial widths

$$\frac{\Gamma(K^+ \to \mu^+ \nu_\mu)}{\Gamma(K^+ \to e^+ \nu_e)} \ = \ \frac{m_\mu^2}{m_e^2} \ \approx \ 4 \times 10^4$$

This coincides very closely to the experimentally observed ratio.

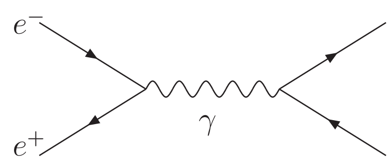
17.5 Z-boson interactions

As well as exchange of $W^\pm$ in which flavour is changed, the weak interactions are also mediated by a neutral gauge-boson, Z. This couples to both quarks and leptons but does not change flavour.

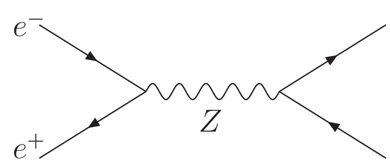
In that sense the interactions of the Z are similar to that of the photon, but there are some important differences.

- The Z couples to neutrinos whereas the photon does not (neutrinos have zero electric charge).
- The Z has a mass of 91.1 Gev/c^2, so the interactions are short range - like the interactions of the $W^\pm$.
- The Z also has a coupling of different strength to left-handed (negative helicity) and right-handed (positive helicity) quarks and leptons and so these interactions also violate parity.

Nevertheless, in any process where there can be photon exchange, there can also be Z exchange. In terms of Feynman diagrams for $e^+ e^-$ scattering into any pair of final state particles, we have



but also



The first diagram (photon exchange) has a propagator

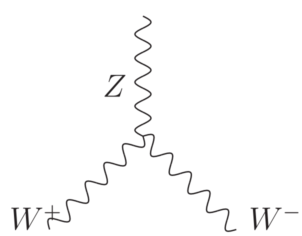
$$1/s,$$

where $\sqrt{s}$ is the centre-of-mass energy, whereas the second diagram (Z exchange) has a propagator

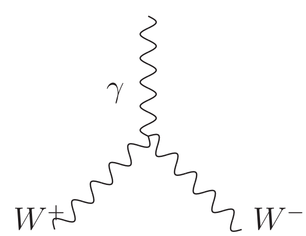
$$\frac{1}{s - M_Z^2 c^4}.$$

For relatively low centre-of-mass energies for which $\sqrt{s} \ll M_Z c^2$, the second diagram may be neglected and the second diagram gives a negligible contribution. But as $\sqrt{s}$ grows to become comparable (or greater than) $M_Z c^2$ both of these diagrams are equally important.

The Z and photon can both couple to $W^{\pm}$, so we get interaction vertices

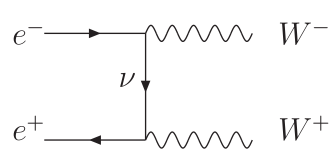


and

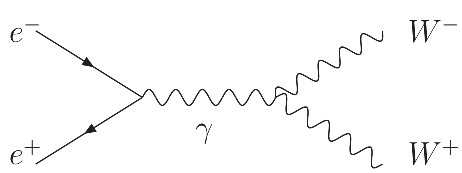


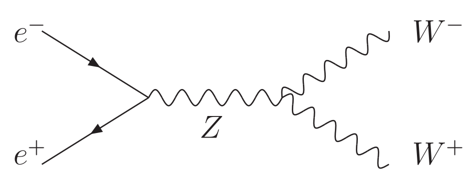
The interaction between the photon and $W^{\pm}$ is not surprising since the $W^{\pm}$ are charged and we would expect them to interact with photons, with coupling e. The interaction of $W^{\pm}$ with the Z is similar but has a different coupling.

The coupling of the Z and photon to the $W^{\pm}$ was confirmed at the LEPII experiment at CERN where it was possible to accelerate electrons and positrons to sufficient energies to produce a W^{+} and a W^{-} in the final state. From the coupling of the W to electron and neutrino the Feynman diagram for this process is

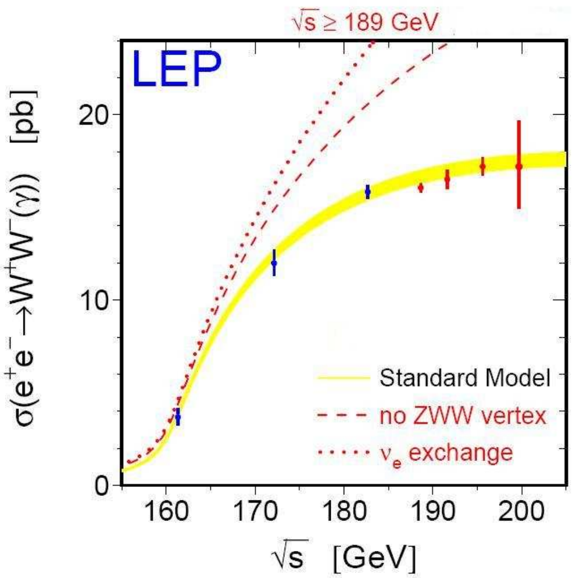


but because of the coupling of the Z and photon to $W^{\pm}$ we also have diagrams





The data from LEPII clearly show that these graphs have to be taken into account



It turns out that the Standard Model of weak and electromagnetic ("electroweak") interactions, developed in the 1960's by Glashow, Weinberg, and Salam, gives a relation between the weak coupling g_W, the (magnitude of the) electron charge, e and the masses of the Z and $W^{\pm}$

$$\frac{M_W}{M_Z} \;=\; \cos\theta_W$$

where θ_W is known as the weak mixing angle.

$$e \;=\; g_W \sin\theta_W \;=\; g_W \sqrt{1 - \frac{M_W^2}{M_Z^2}}$$

This enables us to make an order of magnitude estimate of the rates for weak processes at low energies.

At energies $\ll M_W c^2$, the amplitude for a $W^{\pm}$ exchange process is proportional to

$$\frac{g_W^2}{4\pi\epsilon_0 M_W^2 c^4},$$

so that the rate is proportional to

$$\left(\frac{g_W^2}{4\pi\epsilon_0 M_W^2 c^4}\right)^2.$$

Now for a weak decay rate we want dimensions of inverse time, so we need to multiply this by something with dimensions of the fourth power of energy divided by time. The only quantity proportional to the energy is the Q value of the decay, Q_β and to get inverse time we can divide by $\hbar$ so we get an estimate

$$\text{Rate} \;\sim\; \left(\frac{g_W^2}{4\pi\epsilon_0 \hbar c M_W^2 c^4}\right)^2 . \frac{Q_\beta^5}{\hbar}$$

The pre-factor is actually quite small. For example, for muon decay $Q_\beta \approx m_\mu c^2$, and the muon decay rate is actually

$$\frac{1}{\tau_\mu} = \frac{1}{768\pi^3}\left(\frac{g_W^2}{4\pi\epsilon_0\hbar c}\right)^2 \frac{m_\mu^4}{M_W^4}\frac{m_\mu c^2}{\hbar}.$$

We know

$$\frac{g_W^2}{4\pi\epsilon_0\hbar c} = \frac{e^2}{4\pi\epsilon_0\hbar c\sin^2\theta_W} = \frac{\alpha}{\sin^2\theta_W}$$

and

$$\sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}.$$

Therefore from the measured masses of the W and Z we can determine the muon lifetime.

17.6 The Higgs mechanism

There is one further particle predicted by the Standard Model of electroweak interactions which has not yet been discovered.

This arises from the mechanism, discovered by P.Higgs, by which particles acquire their mass. The basic idea is that there exists a field, ϕ called the "Higgs field" which has a constant non-zero value everywhere in space. This constant value is called the "vacuum expectation value", $\langle\phi\rangle$.

In the absence of this field it is assumed that all particles would be massless and would travel with velocity c. But because of their interaction with the background Higgs field they are slowed down - thereby acquiring a mass, M

$$M = \frac{1}{2}\frac{g_H}{\sqrt{\epsilon_0\hbar c}}\langle\phi\rangle,$$

where g_H is the coupling of the particle to the Higgs field (the denominator factor $\sqrt{\epsilon_0\hbar c}$ gives it the correct dimensions.) This mechanism is part of the Standard Model.

The Higgs field couples to $W^\pm$ with coupling g_W so that

$$M_W = \frac{1}{2}\frac{g_W}{\sqrt{\epsilon_0\hbar c}}\langle\phi\rangle.$$

Inserting $g_W = e/\sin\theta_W$ with $\cos\theta_W = M_W/M_Z$ and $M_W = 80.4\,\mathrm{GeV/c^2}$, and $M_Z = 91.2\,\mathrm{GeV/c^2}$, we get the value of the vacuum expectation value

$$\langle\phi\rangle = 250\,\mathrm{GeV/c^2}$$

Other particles couple to the Higgs field with couplings that are proportional to their mass.

In the same way that there are quanta of the electromagnetic field which are particles (photons), so there must be quanta of the Higgs field. These are called “Higgs particles”. They must necessarily exist if the Higgs mechanism for generating masses for particles is to be consistent with quantum physics.

As it was mentioned in the introduction, the Higgs boson was discovered on the 4th of July 2012 by ATLAS and CMS collaborations at the LHC, completing the set of particles of the Standard Model. With a high confidence level this particle is confirmed to have the following properties:

1. It has a spin zero. This is consistent with the theoretical predictions since the vacuum expectation value has to be invariant under Lorentz transformations - so that it is the same in all frames of reference.

2. Higgs boson couples to $W^{\pm}$ and Z (which are consequently massive).

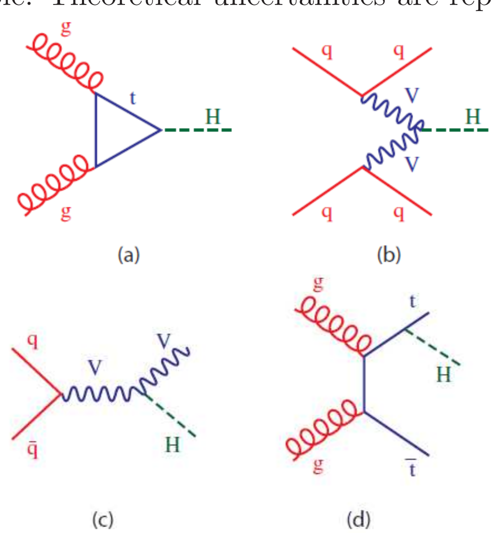
3. It does *not* directly couple to photons (which are massless) so it is uncharged.

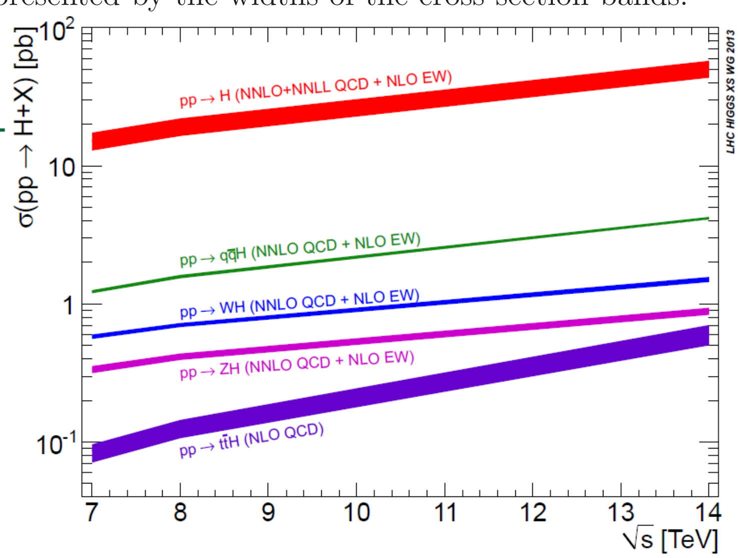
4. it does not not couple directly to gluons (which are massless) and so it does not take part in the strong interactions.

5. Its coupling to massive particles is proportional to the particle mass.

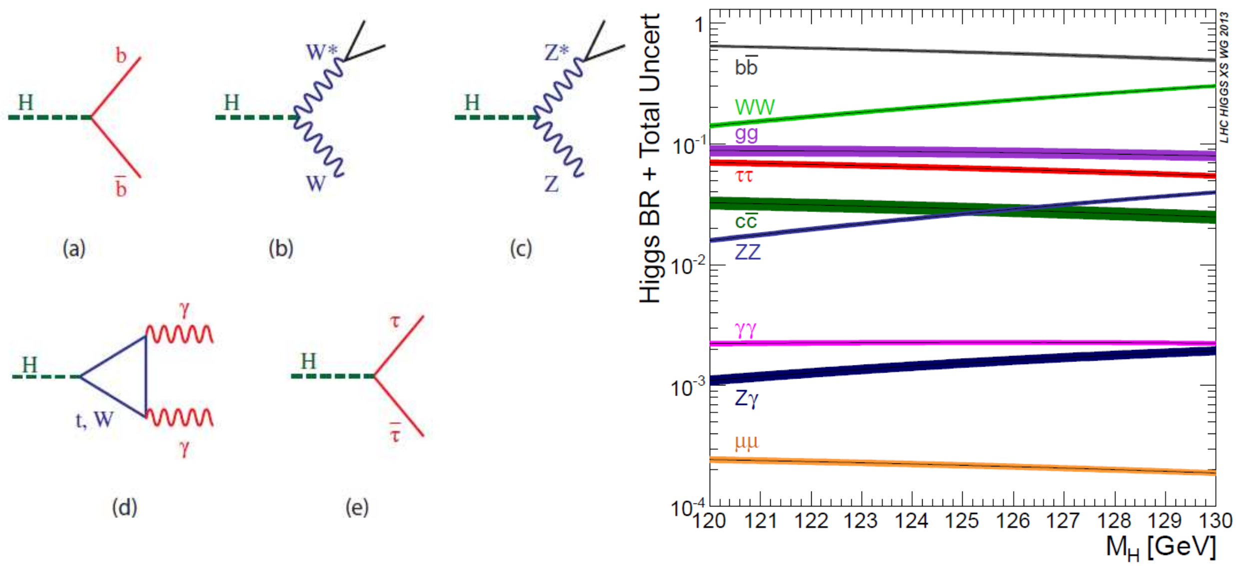
6. Its mass is measured to be about 125 GeV/c^2.

Diagrams for production mechanisms of the Higgs boson at the LHC are shown below (left) together with the respective cross sections (right). They include: (a) gluon fusion, (b) weak-boson fusion, (c) Higgs-strahlung (or associated production with a gauge boson) and (d) associated production with top quarks processes. Note, that the first process is the loop-induce one: while Higgs boson does not interact directly with massless gluons, it actually can interact with gluons via virtual massive quarks (e.g. top-quarks which the strongest coupling to the Higgs boson) in the triangle loop diagram. Actually gluon fusion is the main production process of the Higgs boson, while the weak-boson fusion plays the next to leading role. Theoretical uncertainties are represented by the widths of the cross section bands.





Higgs boson decay is dominated by the most massive particles allowed by its mass because its coupling to particles is proportional to the mass. The t-quark mass is 175 GeV/c^2 so it cannot decay into a $t - \bar{t}$ pair. The next most massive quark is the b-quark so Higgs boson predominantly decays into a $b - \bar{b}$ pair, shown in diagram (a) below. Higgs boson is not sufficiently massive to decay into real W^+W^- or two real Z particles, however it can decay to one real and another virtual W or Z boson (W* and Z*) followed by their decay into fermion-antifermion pair as shown in diagrams (b) and (c). As in case of gluon fusion production process, Higgs boson can also decay into photon pair via its interactions with virtual top quark and W-boson as shown in diagram (d). Higgs boson also decays to $\tau^+\tau^-$ pair, the dominant leptonic decay channel since τ-lepton is the most massive amongst the leptons. The respective branching ratios for Higgs boson decay channels are shown in the right frame of the figure below as a function of the Higgs boson mass.



Higgs boson discovery was based not on the process with the highest production and decay rates, which would be naively the $gg \to H \to b\bar{b}$ process. Actually it was was based on the processes with optimal signal-to-background ratio and the highest signal significance. In particular, one of the most significant and cleanest signatures comes from $H \to \gamma\gamma$ decay for which Standard Model background is relatively low. Another very important and significant signature is based on $H \to ZZ^* \to$ 4leptons decay which also provide clean 4-lepton signature.

Chapter 18

Electromagnetic Interactions

18.1 Electromagnetic Decays

There are a few cases of particles which could decay via the strong interactions without violating flavour conservation, but where the masses of the initial and final state particles are such that this decay is not energetically allowed.

For example the Σ^{*0} (mass 1385 MeV/c^2) can decay into a Λ (mass 1115 MeV/c^2) and a π^0 (mass 135 MeV/c^2). The quark content of the Σ^{*0} and Λ are the same and the π^0 consists of a superposition of quark-antiquark pairs of the same flavour. As required in strong interaction processes the isospin is conserved - the Σ^{*0} has isospin I=1, the Λ has isospin $I = 0$ and the pion has isospin $I = 1$.

On the other hand, the Σ^0 whose mass is 1189 MeV/c^2 does not have enough energy to decay into a Λ and a pion.

In such cases the decay can proceed via the electromagnetic interactions producing one or more photons in the final state. The dominant decay mode of the Σ^0 is

$$\Sigma^0 \ \rightarrow \ \Lambda + \gamma$$

The quark content of the Σ^0 and the Λ are the same, but one of the charged quarks emits a photon in the process. Note that in this decay the isospin is *not* conserved - the initial state has isospin I=1, whereas the final state has isospin $I = 0$. Electromagnetic interactions do not conserve isospin.

Because the electromagnetic coupling constant, e, is much smaller than the strong coupling constant the rates for such decays are usually much smaller than the rates for decays which can proceed via the strong interactions.The lifetime of the Σ^0 is 10^{-10} seconds, whereas the Σ^{*0} has a width of 36 MeV, corresponding to a lifetime of about 10^{-23} seconds.

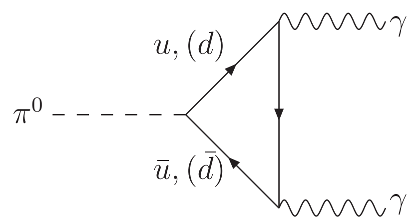
Another important example of electromagnetic decay is the decay of the π^0 into two photons.

$$\pi^0 \ \rightarrow \gamma + \gamma.$$

Note that to produce only one photon would not be possible by conservation of energy and momentum. For a π^0 decaying from rest momentum is conserved because the two photons have identical frequency and move in opposite directions. The π^0 is actually a superposition of a $u - \bar{u}$ quark-antiquark pair and a $d - \bar{d}$ quark-antiquark pair

$$|\pi^0\rangle \;=\; \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle \;-\; |d\bar{d}\rangle\right)$$

In either of these states, the quark can annihilate against the antiquark of identical flavour to produce two photons. In terms of Feynman diagrams we have



Using Quantum Field Theory, we can work out the decay rate for this process and summing over the u and d contributions we get an estimate for the π^0 lifetime of

$$\tau \;=\; 7.6 \times 10^{-16}\, s,$$

whereas the measured value if 8.4×10^{-17} seconds.

What has gone wrong is that we have forgotten about colour. In the calculation of the decay *amplitude* we must not only sum the contribution from the above Feynman diagram over u and d quarks but also over the three possible colours that these quarks can have. This gives us a further factor of 3 in the decay amplitude and so a factor of 9 in the decay rate.

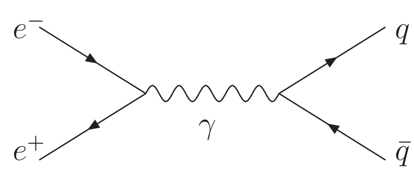
18.2 Electron-positron Annihilation

Another striking piece of evidence that quarks come in three colours comes from the study of the process

$$e^+ + e^- \;\rightarrow\; \text{hadrons}$$

(summed over all possible hadrons in the final state)

At the level of quarks, the Feynman diagram for this process is

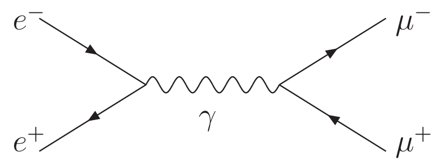


(provided $\sqrt{s} \ll M_Z c^2$ so that the Z-exchange diagram can be neglected.)

For the process

$$e^+ + e^- \;\to\; \mu^+ + \mu^-,$$

the Feynman diagram is



The only difference between these two graphs is the coupling of the final state quarks or final state muons to the photon, i.e. the electric charges of the quarks and the muons.

This means that for a quark of flavour i with electric charge Q_i (in units of e) the ratio of the amplitudes is

$$\frac{\mathcal{A}(e^+e^- \to q_i\bar{q}_i)}{\mathcal{A}(e^+e^- \to \mu^+\mu^-)} \;=\; Q_i.$$

In order to calculate the ratio of total cross-sections we square the amplitude and sum over all possible final state quarks that can be produced, so that

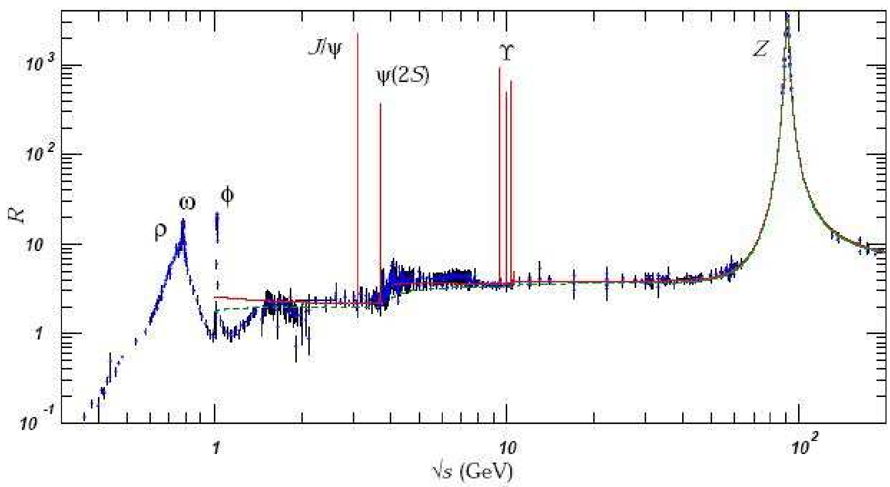
$$R \;=\; \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \;=\; \sum_i Q_i^2.$$

How many quarks we sum over depends on the centre-of-mass energy $\sqrt{s}$. If $\sqrt{s} < 2m_c c^2$, then only u, d and s quarks can be produced in the final state and we have

$$R \;=\; \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \;=\; 3\left(Q_u^2 + Q_d^2 + Q_s^2\right) \;=\; 3\left(\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2\right) \;=\; 2.$$

The factor of 3 is needed because we can produce final state quarks in any of the three colours states (in principle these would be distinguishable at the quark level - so we multiply the cross-section by 3 and not the amplitude.)

In the region $2m_c c^2 < \sqrt{s} < 2m_b c^2$ we can also produce a c- (charm) quark and this has to be added to the cross-section. Likewise, for $\sqrt{s} > 2m_b c^2$ we also have to include the production of b-quarks. Thus we expect this ratio, R, to have jumps as we cross thresholds in incoming energy which allow the production of more massive quarks.



Fragmentation

What we are interested in is the total cross-section for $e^+ e^-$ to annihilate to produce hadrons, whereas what we have calculated is the total cross-section to all quarks.

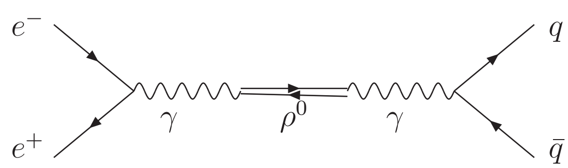
What happens is that the quarks, which cannot be observed directly, interact with gluons in a complicated way and are converted into sets of ordinary hadrons. This process is called "fragmentation". Its mechanism is not understood but several computer simulations have been developed which mimic this process fairly well.

In the centre-of-mass frame, the final state quark and antiquark are moving in opposite directions. What usually happens is that the process of fragmentation acting on the quark and antiquark separately leads to two narrow jets of particles moving in opposite directions.

Resonances

As well as the almost constant value for the ratio, R, between energy thresholds (with jumps near each threshold), the quantity, R, is populated with resonances wherever $\sqrt{s}/c^2$ is equal to the mass of a neutral, spin one, particle that can couple directly to a photon.

The spin has to be the same as the spin of the photon (spin 1), as there is a direct coupling between the photon and the resonant particle which must conserve angular momentum.



At low energies these are mesons such as ρ^0 which consist of a quark and antiquark of the same flavour (or superpositions like in the case of π^0). When $\sqrt{s}/c^2 = m_\rho$ the ρ propagator

$$\frac{1}{(s - m_\rho^2 c^4 + i m_\rho \Gamma_\rho c^2)}$$

gives rise to a resonance.

The thresholds for the production of more massive quarks are also indicated by thresholds. When $\sqrt{s} \approx 2m_c c^2$ it is possible to create a resonance of a particle called J/Ψ which is a bound state of a c-quarks and a $\bar{c}$ antiquark, with mass 3.1 GeV/c^2 - there are further resonances corresponding to excited states with the same quark content.

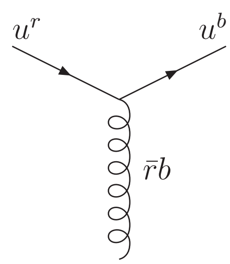
Likewise at the threshold $\sqrt{s} = 2m_b c^2$, there is a resonance called the Υ, which is a bound state of a b-quark and a $\bar{b}$ antiquark, with mass 9.5 GeV/c^2 - and also some further resonances corresponding to excited states.

Chapter 19

Quantum Chromodynamics (QCD)

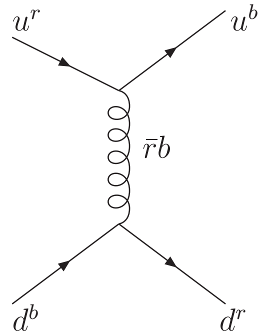
19.1 Gluons and Colour

In the same way that in weak interactions, the weak gauge bosons $W^{\pm}$ can effect changes of flavour when they interact with quarks, in the case of strong interactions the strong gauge bosons (gluons) can effect changes of colour of the quarks (but conserve flavour). Thus we get interaction vertices of the form



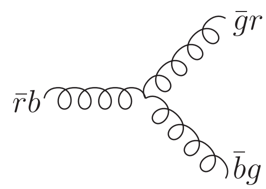
which converts a red quark into a blue quark. The fact that the flavour is unchanged is the reason why flavour is conserved in strong interactions. There are 6 such colour changing gluons and in addition two colour neutral gluons (like the Z in weak interactions) which do not change colour, making a total of 8 gluons (the reason that there are 8 gluons comes from a group theory analysis of the theory of colour - outside the scope of these lectures),

Strong interaction processes consist of quarks exchanging gluons and (usually) changing colour, e.g



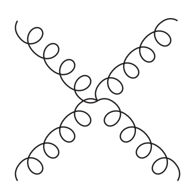
This theory of strong interactions, developed in the 1970's is called Quantum Chromodynamics (QCD).

In the same way that the Z can couple to the $W^{\pm}$, so gluons can couple to each other with vertices such as



(total colour must be conserved)

There are also vertices between four gluons of the form



In the case of weak and electromagnetic interactions, the strengths of the couplings of the gauge bosons to quarks and leptons, controlled by the electron charge, e, and the weak coupling, g_W, are sufficiently small

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137}, \qquad \alpha_W = \frac{g_w^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{30}$$

so that we can calculate the rates for weak and electromagnetic processes using perturbation theory, with higher order corrections being of order α, for electromagnetic processes, and α_W for weak processes. However, for the strong interactions inside a nucleus, the coupling constant is too large for this to be possible. It is for this reason that we cannot, even in principle, calculate the energy levels of nuclei.

19.2 Running Coupling

However, QCD is not entirely useless.

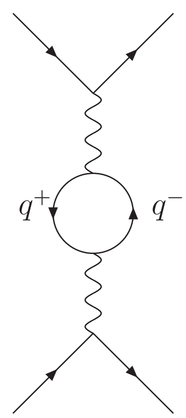
It turns out that at sufficiently high energy/momentum scales, Q, the effective strong coupling becomes small.

It is convenient to work in terms of α_s where

$$\alpha_s \;=\; \frac{g_s^2}{4\pi\epsilon_0\hbar c}.$$

where g_s is the coupling of the gluons to quarks or the coupling of gluons to each other.

The reason that this becomes small is 'negative screening'. When an electric charge is probed by another charge, the virtual photon exchanged between them can sometimes create a pair of charged particles (a particle and its antiparticle), which exist for a short while before annihilating each other again. Diagrammatically we would represent this as



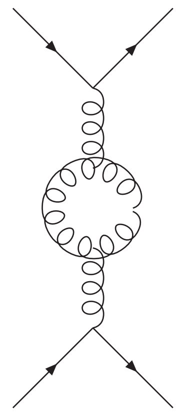
The effect is to surround the probed charge by a cloud of charged particles which act as a screen - reducing the effective measured charge. As the energy/momentum scale increases and the probe penetrates further into this screen the measured charge increase.

When we write

$$\alpha \;=\; \frac{1}{137}$$

this refers to low energy/momentum measurements of the electron charge. At a momentum scale $Q \sim M_Z c$ the value is closer to 1/129.

In the case of QCD, we can also have processes in which a cloud of gluons can be produced by the exchanged virtual gluon, because gluons interact with each other (unlike photons). Thus we have diagrams like



The effect of this is negative screening and it turns out that at large momenta the effective coupling decreases.

Mathematically we describe the momentum scale dependence (running) of the coupling in terms of a function known as the β-function, defined as

$$\beta = \frac{d\alpha(Q)}{d\ln Q^2}$$

For electromagnetism, β is positive so that α increases with increasing Q.

But for QCD, we have an expansion of β as a power series in α_s,

$$\beta = \alpha_s^2\beta_0 + \mathcal{O}(\alpha_s^3).$$

where

$$\beta_0 = -\frac{1}{4\pi}\left(11 - \frac{2}{3}n_f\right)$$

Here n_f means the number of active flavours and is used in the same way as in the calculation of R in $e^+e^- \to$hadrons.
For $Q < 2m_c c,\ n_f = 3$,
for $2m_c c < Q < 2m_b c,\ n_f = 4$
for $Q > 2m_b c,\ n_f = 5$.
In this expression for β the $-11/(4\pi)$ comes from the interaction of the gluons with each other producing a gluon cloud which decreases the effective coupling with increasing Q, whereas the term proportional to n_f (with a positive sign) comes from the creation of quark-antiquark pairs by the virtual gluon and is similar to the producing of charged pairs in electromagnetism - so its effect on the running coupling has the same sign as in electromagnetism.

The solution to this differential equation (neglecting the higher order terms in α_s) is

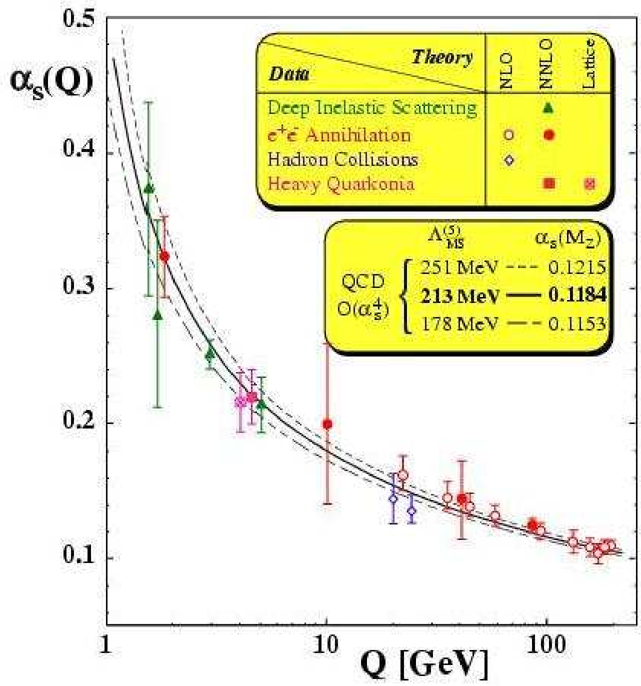
$$\alpha_s(Q) = \frac{\alpha_s(\mu)}{(1 - \beta_0\alpha_s(\mu)\ln(Q^2/\mu^2))}$$

where $\alpha_s(\mu)$ is the value of α_s at some reference momentum scale (it serves as the integration constant for the differential equation). Usually this is taken to be $\mu = M_Z c$, since the value of α_s was measured very accurately at LEPI at this scale and its value was found to be

$$\alpha_s(M_Z c) = 0.12,$$

which is not too large.

Experimental measurements of α_s over a large range of energy/momentum scales agrees well with this formula.



We see that for Q greater than a few GeV, $\alpha_s(Q)$ is small enough that we would expect a calculation using perturbation theory to be fairly reliable. Below these energy/momentum scales (e.g. inside the nucleus) we cannot use perturbation theory and QCD is not very helpful.

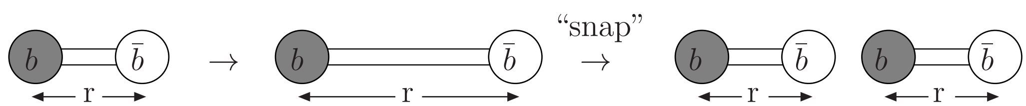
The property of QCD that the effective coupling decreases with increasing energy/momentum is called "asymptotic freedom".

19.3 Quark Confinement

We have seen that the weak interactions are short-range because the gauge bosons $W^\pm$ and Z are massive and so the weak potential is of the Yukawa type with an exponential fall-off with distance $\exp(-M_W cr/\hbar)$.

In the case of QCD the gluons are massless, so we might expect the strong interactions to be long-range (as in the case of electromagnetic interactions mediated by massless photons), whereas we know that the strong interactions have a range of a few fermi.

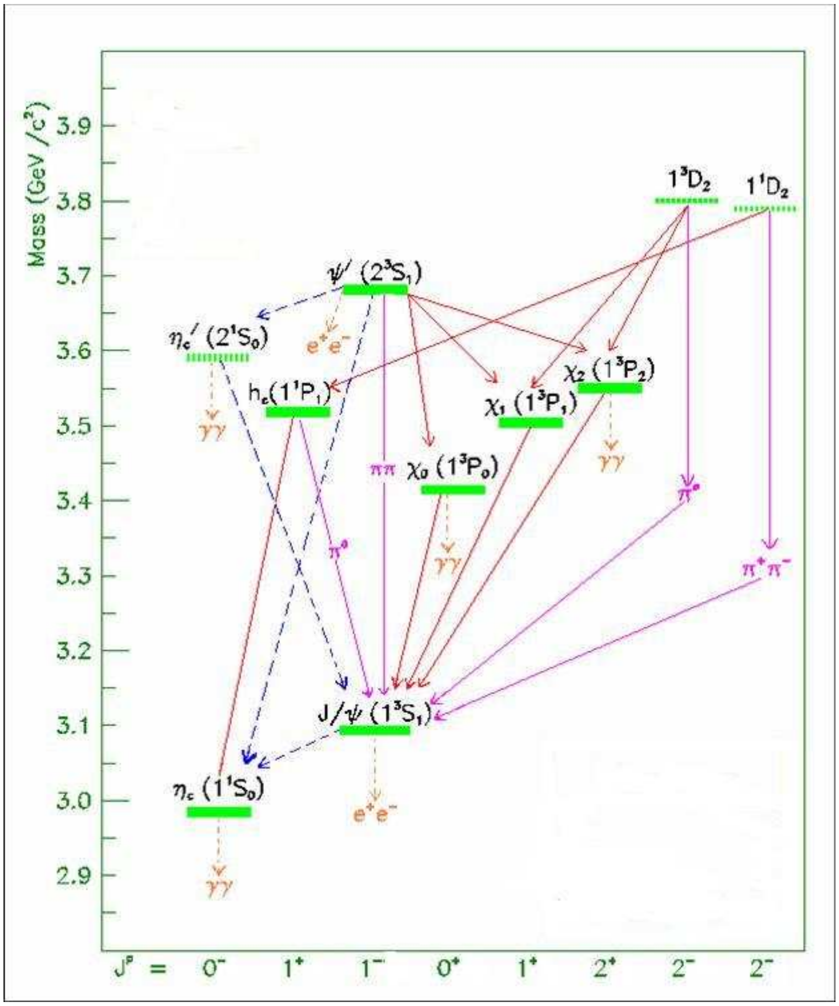
The answer to this puzzle is the converse of asymptotic freedom. At large momentum, where we are probing short distances, the effective coupling decreases. Conversely at large quark separations the effective coupling increases and the binding between them gets stronger.



It is not possible to isolate a single quark or gluon. Consider a meson, which is a quark-antiquark state of the opposite colour (e.g. red and anti-red) bound together by a 'string' of gluons. As we try to pull the quark and antiquark apart, the tension in the string increases and eventually the string will 'snap' producing a quark a the end of the part of the string containing the antiquark (of opposite colour) and likewise and antiquark of opposite colour at the end of the part of the string containing the quark. So we end up with two mesons, both of which are colour singlets (colourless), but we do not succeed in isolating a single quark or antiquark.

The only hadron states that we can observe are colourless (colour singlet) states - either mesons which are superpositions of quark-antiquark pairs of opposite colours, or baryons which consist of three quarks but which are antisymmetric under the interchange of any two quark colours. This is known as "quark confinement". Its exact mechanism is not understood, but numerical studies in QCD confirm that this confinement does indeed take place.

19.4 Quark-antiquark Potential and Heavy Quark Bound States



At momentum scales $Q \sim 2m_c c$, the running coupling is $\alpha_s \sim 0.3$ which is small enough to use perturbation theory to obtain energy levels for the J/Ψ (also known as "charmonium"), by solving the Schrödinger equation using a potential which contains a term that represents the confinement,

$$V(r) \;=\; -\frac{4}{3}\frac{\alpha_s}{r} \,+\, kr.$$

The first term is the usual Coulomb-like potential which dominates at short distances where the potential can be viewed as the exchange of massless gluons - the factor of 4/3 is associated with the number of quark colours and the number of gluons. The term kr increases with increasing separation and represents confinement. By fitting the data, k is found to be about 0.85 Gev fm^{-1} (1.3×10^5 N. !).

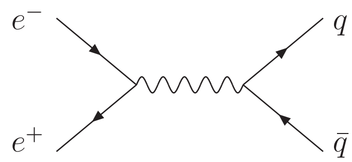
Using this potential and making corrections for relativistic effects and spin-orbit coupling, the spectrum of the $c - \bar{c}$ system (J/Ψ and its excited states) and also the $b - \bar{b}$ system (Υ and its excited states) can be obtained to a high degree of accuracy.

19.5 Three Jets in Electron-positron Annihilation

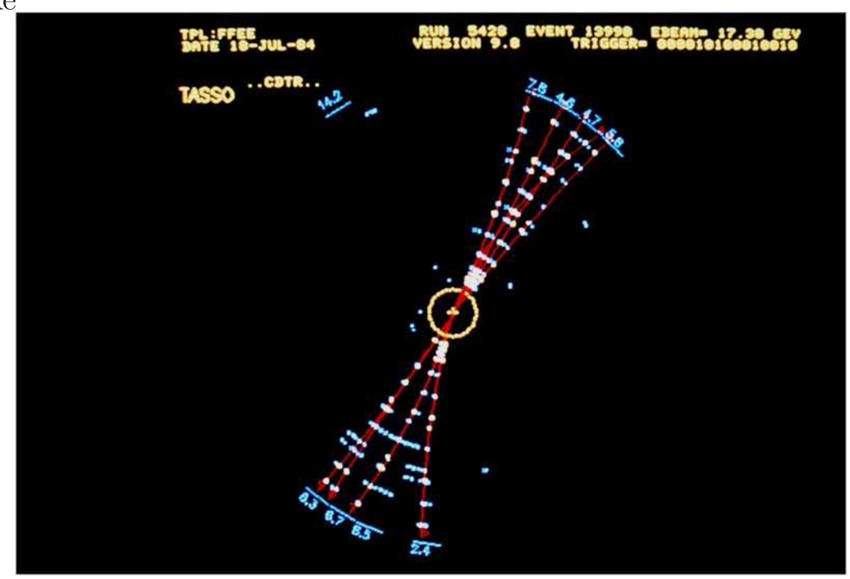
When we were considering the process

$$e^+ + e^- \;\rightarrow\; \text{hadrons}$$

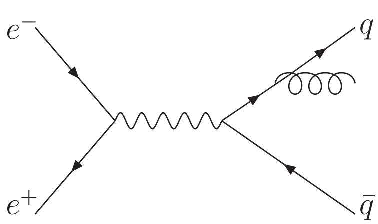
the Feynman diagram considered was

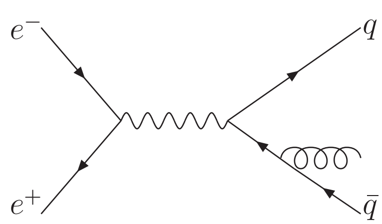


The final state quarks fragment to produce two jets of hadrons moving in opposite directions. Such two-jet events were observed in the $e^+ e^-$ experiment at DESY in the 1970's. A typical event looks like



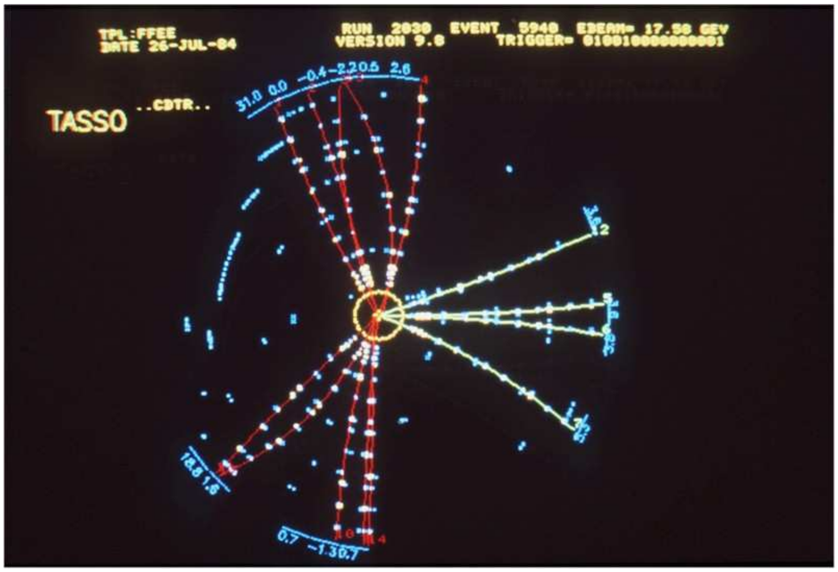
Because quarks interact with gluons one can also have Feynman diagrams





which have a quark, an antiquark, and a gluon in the final state. The gluon also fragments into a hadron jet and so we get three jets of particles.

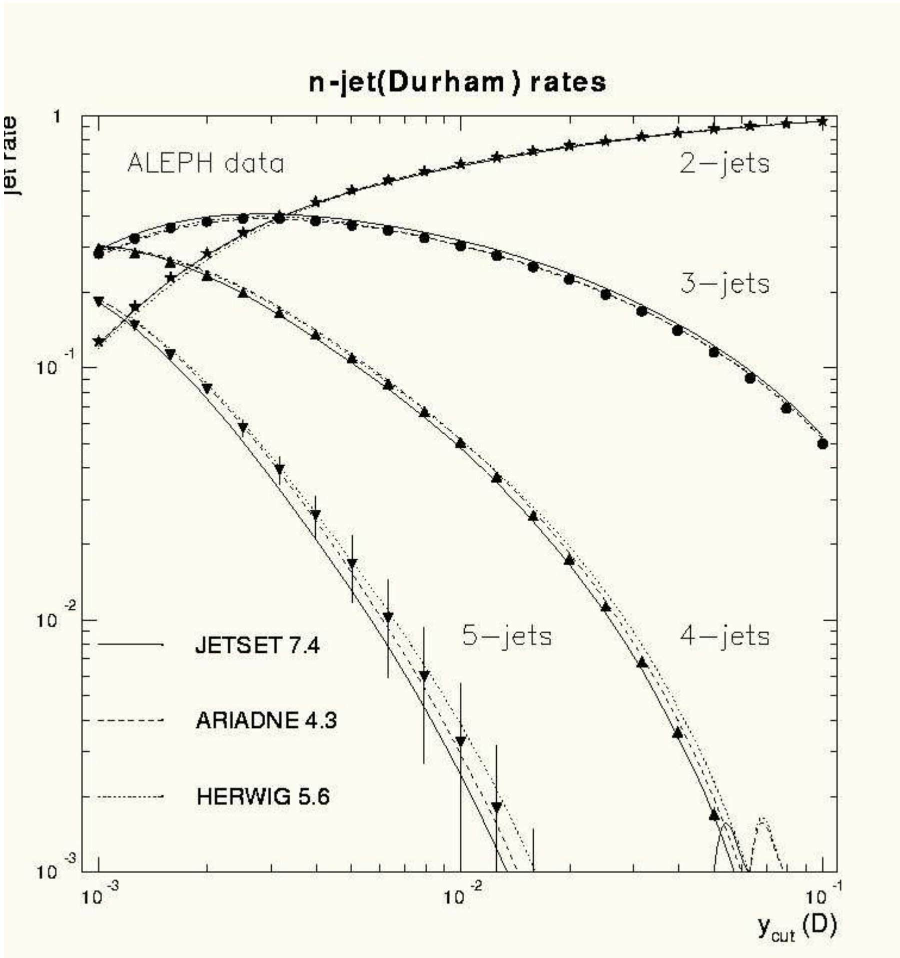
The first such events we observed at DESY in 1979



Since gluons cannot be isolated and observed directly, this was taken as the first piece of evidence that gluons existed and coupled to quarks.

Four and five jet events have now also been observed. At LEP energies (100 - 200 GeV) the running coupling is small ($\alpha_s \sim 0.1$). For this reason three jet events are rarer than two jet events because the amplitude for the process contains a factor of the gluon coupling g_s and so the rate is expected to be suppressed relative to the two jet rate by a factor of order $\alpha_s \propto g_s^2$. Likewise four and five jets events are even rarer

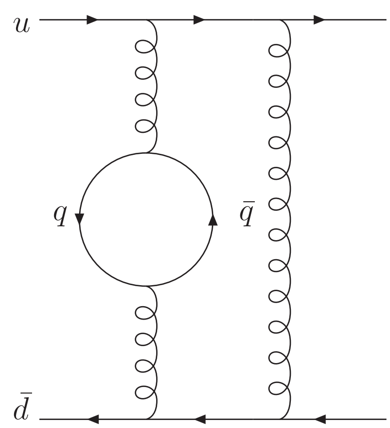
The exact definition of a jet depends on how big an opening angle constitutes a single jet. This is parametrised by a kinematic variable called y_{cut} (this is a measure of the maximum fraction of the total energy that can be contained in a single jet). Because of the small running coupling, perturbative QCD can be used to calculate the number of jets as a function of this y_{cut}



Some correction to the result from pure perturbative QCD has to be made for the process of fragmentation. The different curves shown are for different models used to simulate this fragmentation process. Nevertheless the agreement between the data and QCD theory is impressive.

19.6 Sea Quarks and Gluon content of Hadrons

The quarks (and antiquarks) inside hadrons are bound together by exchanging gluons. Thus, as well as having the quarks inside hadrons there will be gluons. These gluons can in turn create quark-antiquark pairs (which exist for a very short time and then annihilate). Diagrammatically the 'inside' of a π^- may look like



Thus, inside the hadron, we have the main quarks, called valence quarks which determine the quantum numbers (flavour) of the hadron and in addition a cloud of quark-antiquark pairs created by the gluons exchanged between the valence quarks. These extra quark-antiquark pairs are called "sea quarks".

19.7 Parton Distribution Functions

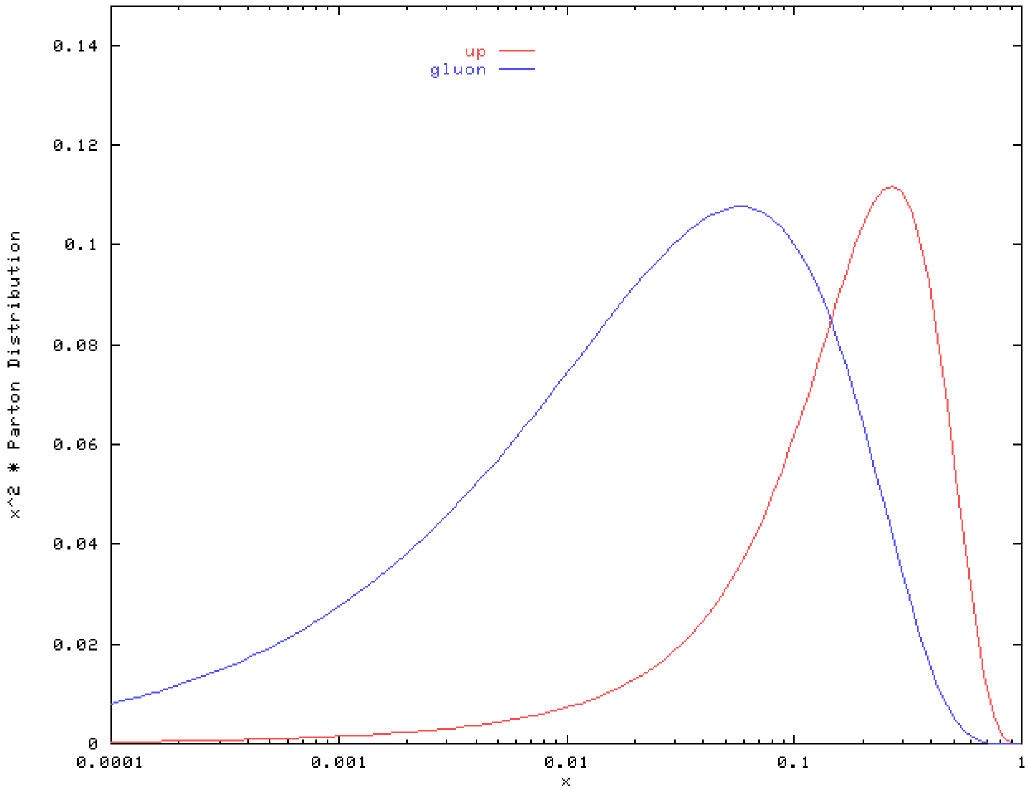
Quarks, antiquarks and gluons are collectively known as "partons". If we consider a relativistically moving hadron ($\approx pc$), some fraction, x (known as "Bjorken-x") will be carried by a parton of each possible type. The probability that a fraction, x, of the momentum of the hadron (say a proton) is carried by a parton of type i is called the "parton distribution function" and is written as

$$f_h^i(x)$$

where i can mean a gluon, quark, or antiquark of any given flavour.

It is not possible to calculate these parton distribution functions in QCD, but they can be inferred by examining experimental data. Once they are known QCD can be used to predict scattering cross-sections for other processes.

An example of these parton distributions (as a function of x) is

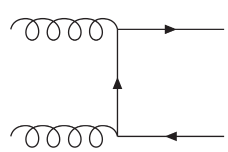


19.8 Factorization

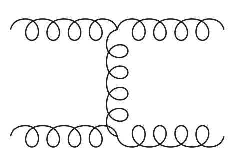
Perturbative QCD can be used to calculate cross-sections at the parton level, provided that the energy/momentum scale of the process, Q is large enough so that $\alpha_S(Q)$ is sufficiently small.

For, example we can calculate the cross-section for the processes

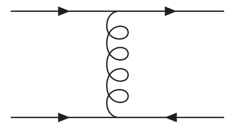
- $$g, +g \rightarrow q + \bar{q}$$



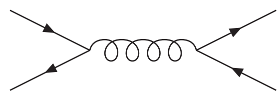
- $$g, +g \rightarrow g + g$$



- $$q, \ + q \ \rightarrow \ q + q$$



- $$q + \bar{q} \ \rightarrow \ q + \bar{q}$$



etc.

Denote the calculated differential cross-section for two partons of type i and j to go into two other partons with momentum p_T transverse to the direction of the incoming partons by

$$\frac{d\hat{\sigma}(\hat{s})}{dp_T},$$

where $\sqrt{\hat{s}}$ is the centre-of-mass energy of the incoming partons.

What we are really interested in is a process in which the initial states are not partons (which cannot be isolated in a laboratory owing to confinement) but initial state hadrons such as a proton and an antiproton.

In order to obtain the differential cross-section for proton-antiproton scattering into two jets of final state hadrons with transverse momentum, p_T we can invoke the factorisation theorem.

If we pull a parton of type i from one of the incoming protons, with a fraction x_1 of the momentum of the parent proton, and a parton of type j from the antiproton, with a fraction x_2 of its momentum, then (in the case of relativistically moving particles whose energy E and momentum $\mathbf{p}$ are related by $E \approx |\mathbf{p}|c$) the centre-of-mass energy of the two partons is given by

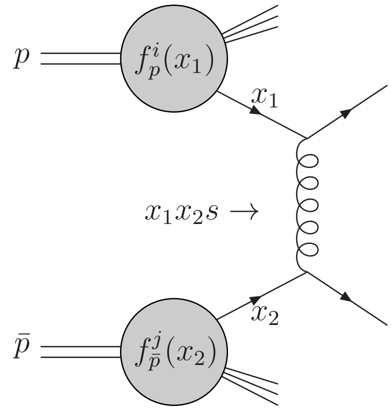
$$\hat{s} \ = \ x_1 x_2 s$$

(where $\sqrt{s}$ is the centre-of-mass energy of the incoming proton and antiproton).

Factorisation tells us that if $f_p^i(x_1)$ and $f_{\bar{p}}^j(x_2)$ are the parton distribution functions for partons i and j, then the contribution to the proton-proton differential cross-section due to this particular parton scattering is

$$\int_0^1 dx_1 \int_0^1 dx_2 f_p^i(x_1) f_{\bar{p}}^j(x_2) \frac{d\hat{\sigma}(x_1 x_2 s)}{dp_T}$$

For example if the parton level scattering is quark-quark scattering we can represent this contribution as



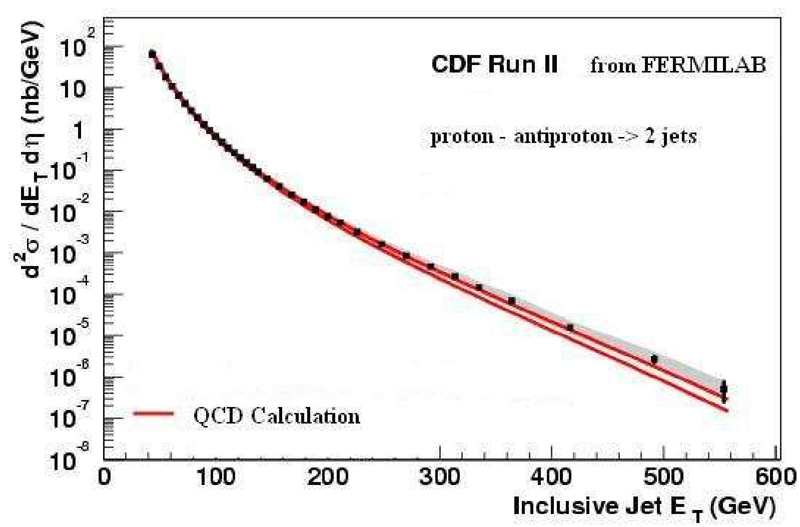
Now the total differential cross-section for proton-antiproton scattering is obtained by summing over all possible parton types that can be pulled out of the incoming protons (quarks, antiquarks, gluons).

Thus we finally obtain an expression for the proton-proton differential cross-section

$$\frac{d\sigma_{pp}(s)}{dp_T} \; = \; \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2 f_p^i(x_1) f_{\bar{p}}^j(x_2) \frac{d\hat{\sigma}(x_1 x_2 s)}{dp_T},$$

where the sum over i, j means sum over all possible partons.

QCD calculations based on this factorisation theorem agree well with experiment.



Only a single parton from each hadron takes part in the parton scattering process. The other partons in the incoming hadrons finally fragment into hadrons, which are moving almost in the same direction as the incoming protons (and are usually not observed because they get lost in the beam-pipe of the accelerator).

Chapter 20

Parity, Charge Conjugation and CP

20.1 Intrinsic Parity

In the same way that nuclear states have parity, so hadrons, which are bound states of quarks (and antiquarks) have parity. This is called intrinsic parity, η and under a parity inversion the wavefunction for a hadron acquires a factor η

$$P\Psi_{\{P\}}(\mathbf{r}) \ == \ \Psi_{\{P\}}(-\mathbf{r}) \ = \ \eta_{\{P\}}\Psi_{\{P\}}(\mathbf{r}).$$

η can take the values ± 1 noting that applying the parity operator twice must bring us back to the original state.

The lighter baryons (for which there is zero orbital angular momentum) have positive intrinsic parity.

On the other hand, antiquarks have the opposite parity from quarks. This means that the light antibaryons have negative parity. It also means that the light mesons, such as pions and kaons, which are bound states of a quark and an antiquark have negative intrinsic parity. The lightest spin-one mesons, such as the ρ-meson, also have zero orbital angular momentum and thus they too have negative intrinsic parity - they have spin-one because of the alignment of the spins of the (valence) quark and antiquark.

For more massive (higher energy) particles, the quarks can be in non-zero orbital angular momentum states so that both baryons and mesons with higher masses can have either parity.

Parity is always conserved in strong interaction processes. A consequence of this is the decay

$$\rho^0 \ \rightarrow \ \pi^+ + \pi^-$$

Since ρ mesons have spin-one and pions have spin zero the final pion state must have $l = 1$. The ρ has negative intrinsic parity and so do the two pions. The orbital angular momentum $l = 1$ means that the parity of the final state is

$$\eta_\pi^2(-1)^1 \ = \ -1,$$

so that parity is conserved. On the other hand, two π^0's cannot be in an $l = 1$ state. The reason for this is that pions are bosons and so the wavefunction for two identical pions must be symmetric under interchange, whereas the wavefunction for an $l = 1$ state is antisymmetric if we interchange the two pions. This means that the decay mode

$$\rho^0 \rightarrow \pi^0 + \pi^0$$

is forbidden.

If we look at the non-leptonic weak decay of a K^+ into pions (weak because strangeness is not conserved) we find that

$$K^+ \rightarrow \pi^+ + \pi^0$$

and

$$K^+ \rightarrow \pi^+ + \pi^0 + \pi^0, \quad \text{or} \quad \pi^+ + \pi^+ + \pi^-$$

both occur. Since the K^+ has negative parity and zero spin - so that the final state cannot have any orbital angular momentum, the final two pion state has even parity, whereas the final three pion state has odd parity.

This is a demonstration that the weak interactions do not conserve parity - and this was in fact observed before C.S. Wu's experiment on the β-decay of ^{60}Co.

20.2 Charge Conjugation

This is the operation of replacing particles by their antiparticles

$$C\Psi_{\{P\}} = \Psi_{\{\overline{P}\}}$$

e.g.

$$C\Psi_{\pi^+} = \Psi_{\pi^-}$$

$$C\Psi_p = \Psi_{\overline{p}}$$

Some mesons are their own antiparticles such as a π^0 or the J/Ψ (a quark-antiquark pair of the same flavour). In this case we have a charge conjugation quantum number η^C

$$C\Psi_{\pi^0} = \eta^C \Psi_{\pi^0},$$

where η^C can take the values ± 1 - again noting that the application of the charge conjugation operator twice must bring us back to the original state.

A photon has charge conjugation $\eta^C = -1$. This is because under charge conjugation electric charges switch sign and therefore so do electric and magnetic fields. We know that the π^0 can decay into two photons via electromagnetic interaction, which are invariant under charge conjugation

$$\pi^0 \rightarrow \gamma + \gamma$$

This forces the charge conjugation of π^0 to be $\eta^C = +1$.

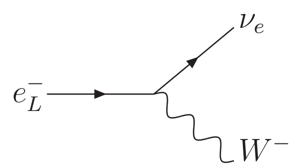
The spectra of charmonium ($c - \bar{c}$ bound states), or bottomonium ($b - \bar{b}$ bound states) contain both positive and negative charge conjugation states.

20.3 CP

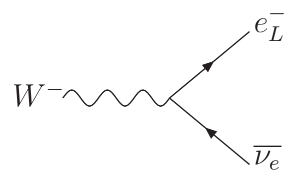
Like parity, charge conjugation is conserved by the strong and electromagnetic interactions but *not* by the weak interactions,

On the other hand, the weak interactions are (almost) invariant under the combined operations of charge conjugation *and* parity inversion, known as "CP".

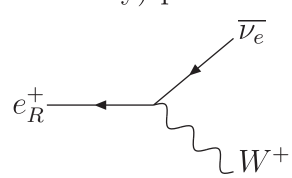
Thus the weak interactions will allow a (highly relativistic) left-handed (negative helicity) electron to convert into a neutrino emitting a W^-



(or alternatively a W^- will decay into a left-handed electron and an antineutrino)



Similarly a right-handed (positive helicity) positron can convert into an antineutrino



If it were possible to repeat the experiment of C.S. Wu using the antiparticle of ^{60}Co, $\overline{^{60}\mathrm{Co}}$ which decays into $\overline{^{60}\mathrm{Ni}}$ emitting positrons and neutrinos, one would find that the positrons tended to be emitted in the same direction as the spin of the antinucleus (whereas in the original experiment they tended to be emitted in the opposite direction from the spin of the nucleus).

20.4 $K^0 - \overline{K^0}$ Oscillations

The invariance of the weak interactions under CP has consequences for the K^0 and $\overline{K^0}$ particles (and also for B^0 and $\overline{B^0}$ mesons currently being studied at the BaBar collaboration at SLAC.)

$$P\Psi_{K^0} = -\Psi_{K^0}$$

and

$$C\Psi_{K^0} = \Psi_{\overline{K^0}}$$

so that

$$CP\Psi_{K^0} = -\Psi_{\overline{K^0}}.$$

This means that the 'particles' K^0 and $\overline{K^0}$ are not eigenstates of CP. But if CP is conserved, then the energy eigenstates (i.e. masses) must also be eigenstates of CP (CP commutes with the Hamiltonian). These eigenstates of CP are

$$\Psi_{K_L} = \frac{1}{\sqrt{2}}\left(\Psi_{K^0} + \Psi_{\overline{K^0}}\right), \quad \mathrm{CP} = -1$$

and

$$\Psi_{K_s} = \frac{1}{\sqrt{2}}\left(\Psi_{K^0} - \Psi_{\overline{K^0}}\right), \quad \mathrm{CP} = +1$$

where L and S stand for long and short for reasons we shall see. These mass eigenstates are therefore not pure K^0 or $\overline{K^0}$ states, but quantum superpositions of the two.

The allowed non-leptonic decays of these states are

$$K_L \to \pi^0 + \pi^0 + \pi^0, \quad \text{or} \quad \pi^0 + \pi^+ + \pi^-,$$

because there is no orbital angular momentum as the kaons and pions both have spin zero and we require three pions to make a $CP = -1$ state because the pions have negative parity. Likewise we have

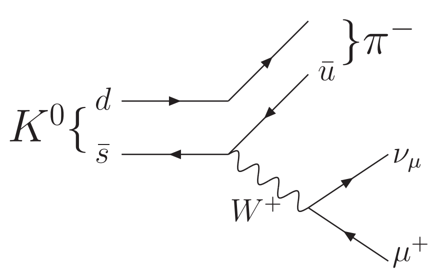
$$K_S \to \pi^0 + \pi^0, \quad \text{or} \quad \pi^+ + \pi^-,$$

because two pions give us a $CP = +1$ state.

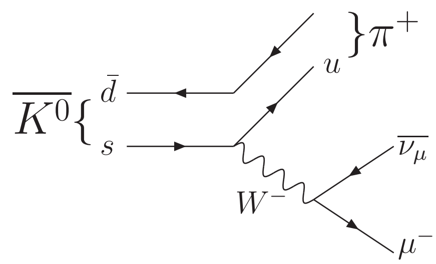
The lifetime of the K_S is shorter than that of the K_L ($\tau_S = 10^{-10}$ s. compared with $\tau_L = 10^{-8}$ s), because the Q-value for the decay into only two pions is larger than that for a decay into three pions ($m_K - 2m_\pi > m_K - 3m_\pi$).

On the other hand, we can distinguish a K^0 ($\bar{s} - d$ bound state) from a $\overline{K^0}$ ($s - \bar{d}$ bound state) by their semi-lepton decay modes

$$K^0 \to \pi^- + \mu^+ + \nu_\mu$$



$$\overline{K^0} \;\rightarrow\; \pi^+ \,+\, \mu^- \,+\, \overline{\nu_\mu}$$



If, at time $t = 0$, we have a pure K^0, this is a superposition of the K_L and K_S states

$$\Psi_{K^0}(t = 0) \;=\; \frac{1}{\sqrt{2}}\left(\Psi_{K_L} \,+\, \Psi_{K_S}\right).$$

The K_L and K_S have slightly different masses

$$\frac{\Delta m}{m} \;= 7 \times 10^{-15}$$

K_L and K_S therefore have different energies, which means that their wavefunctions have different frequencies.

Applying the Schrödinger equation to obtain the time dependence of the wavefunction, whgich at time $t = 0$ represents a pure K^0 state, we obtain a wavefunction which contains oscillations between the wavefunction for a K^0 and the wavefuntion for a $\overline{K^0}$, so that if at some later time t the particle decays semi-leptonically the probabilities $P(K^0)$ or $P(\overline{K^0})$ of observing a K^0 decay (decay products $(\mu^+,\, \pi^-\, \nu_\mu)$) or $\overline{K^0}$ decay (decay products $(\mu^-,\, \pi^+\, \overline{\nu_\mu})$) are of the form

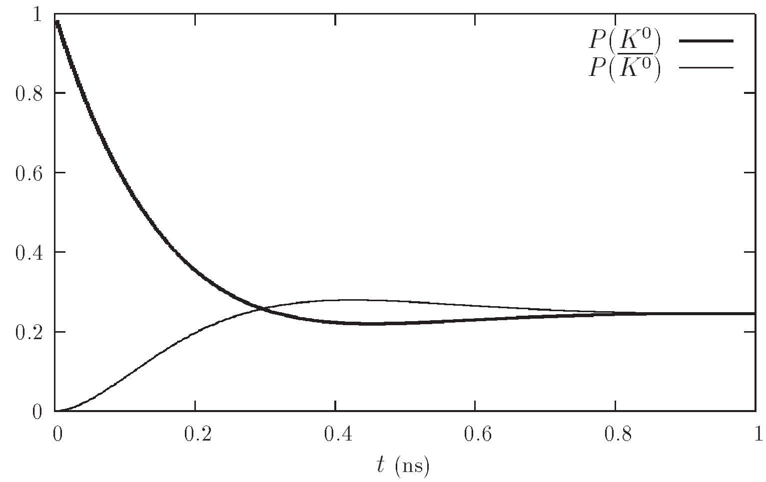
$$P(K^0) \;= A(t) \,+\, B(t) \cos\left(\Delta m c^2 t/\hbar\right)$$

$$P(\overline{K^0}) \;= A(t) \,-\, B(t) \cos\left(\Delta m c^2 t/\hbar\right),$$

where $\Delta m = m_{K_L} - m_{K_S}$.

In other words, as time progresses there are oscillations between the K^0 and $\overline{K^0}$ states (details of the calculation are shown in the Appendix).

This oscillation has been observed experimentally.



This is a striking example of the effects of quantum interference.

In 1964, Christensen et.al. observed a few decays of K_L into two pions. Such a decay, in which the $(CP = -1)$ K_L decays into a $CP = +1$ final state, indicated that CP invariance was violated to a very small extent by the weak interactions.

20.5 Summary of Conservation laws

- **Baryon number:** baryons=+1, antibaryons=-1, mesons, leptons=0.
- **Lepton number:**
  - electron number: e^-, $\nu_e = 1$, e^+, $\overline{\nu_e} = -1$
  - muon number: μ^-, $\nu_\mu = 1$, μ^+, $\overline{\nu_\mu} = -1$
  - τ number: τ^-, $\nu_\tau = 1$, τ^+, $\overline{\nu_\tau} = -1$

	Strong Interactions	**Electromagnetic Interactions**	**Weak Interactions**
Baryon number	yes	yes	yes
Lepton number (all)	yes	yes	yes
Angular momentum	yes	yes	yes
Isospin	yes	no	no
Flavour	yes	yes	no
Parity	yes	yes	no
Charge conjugation	yes	yes	no
CP	yes	yes	almost

Appendix
Neutral Kaon Oscillations

If at time $t = 0$, we prepare a state which is pure K^0 (e.g. a decay product of a strongly decaying particle with strangeness +1), then in terms of the wavefunctions for K_L and K_S the wavefucntion at time $t = 0$ is

$$\Psi(t=0) \;=\; \frac{1}{\sqrt{2}}\left(\Psi_{K_L}(t=0) \,+\, \Psi_{K_S}(t=0)\right).$$

The wavefunctions for K_L and K_S therefore have different oscillatory time dependences

$$\exp\left(-im_{K_L}c^2t/\hbar\right) \quad \text{and} \quad \exp\left(-im_{K_S}c^2t/\hbar\right)$$

as well as exponentially decaying time dependent factors

$$\exp\left(-t/(2\tau_L)\right) \quad \text{and} \quad \exp\left(-t/(2\tau_S)\right),$$

indicating that the probabilities of the particles surviving at time t are

$$\exp\left(-t/\tau_L\right) \quad \text{and} \quad \exp\left(-t/\tau_S\right) \quad \text{respectively.}$$

The time dependence of Ψ_{K_L} and Ψ_{K_S} are given by

$$\begin{aligned}
\Psi_{K_L}(t) &= \Psi_{K_L}(t=0)\exp\left(-im_{K_L}c^2t/\hbar\right)\exp\left(-t/(2\tau_L)\right)\\
\Psi_{K_S}(t) &= \Psi_{K_S}(t=0)\exp\left(-im_{K_S}c^2t/\hbar\right)\exp\left(-t/(2\tau_S)\right)
\end{aligned}$$

Therefore the above wavefunction at time t may be written

$$\Psi(t) \;=\; \frac{1}{\sqrt{2}}\left\{\Psi_{K_L}\exp\left(-im_{K_L}c^2t/\hbar - t/(2\tau_L)\right) + \Psi_{K_S}\exp\left(-im_{K_S}c^2t/\hbar - t/(2\tau_S)\right)\right\}$$

Writing this out in terms of wavefunctions for K^0 and $\overline{K^0}$ we get

$$\begin{aligned}
\Psi(t) &= \frac{1}{2}\left\{\Psi_{K^0}\left[\exp\left(-im_{K_L}c^2t/\hbar - t/(2\tau_L)\right) + \exp\left(-im_{K_S}c^2t/\hbar - t/(2\tau_S)\right)\right]\right.\\
&+ \left.\Psi_{\overline{K^0}}\left[\exp\left(-im_{K_L}c^2t/\hbar - t/(2\tau_L)\right) - \exp\left(-im_{K_S}c^2t/\hbar - t/(2\tau_S)\right)\right]\right\}
\end{aligned}$$

The modulus squared of the coefficients of Ψ_{K^0} and $\overline{\Psi_{K^0}}$ are the probabilities that at time t the particle is a K^0 or $\overline{K^0}$, respectively. These probabilities are

$$P(K^0) \;=\; \frac{1}{4}\left[\exp\left(-t/\tau_L\right) + \exp\left(-t/\tau_S\right) + 2\exp\left(-t(\tau_L+\tau_S)/(\tau_L\tau_S)\right)\cos\left(\Delta mc^2t/\hbar\right)\right]$$

and

$$P(\overline{K^0}) \;=\; \frac{1}{4}\left[\exp\left(-t/\tau_L\right) + \exp\left(-t/\tau_S\right) - 2\exp\left(-t(\tau_L+\tau_S)/(\tau_L\tau_S)\right)\cos\left(\Delta mc^2t/\hbar\right)\right]$$

where $\Delta m = m_{K_L} - m_{K_S}$.

Chapter 21

Epilogue

In the second part of this course, I have tried to give you a 'whirlwind tour' of some of the concepts of modern particle physics.

These concepts have not been easy to assimilate. This is because many of the ideas involved are radically new and differ from one's previous ideas (for example the fact that couplings run and are not constant), and not because they have required a great deal of mathematics. In fact, I have relied very much on a qualitative picture rather than mathematical descriptions. The main reason for this is that in order to carry out proper quantitative calculations in particles physics where it is necessary to account both for Special Relativity and Quantum Mechanics, required the use of Quantum Field Theory, which is taught to Particle Physics graduate students.

Despite the difficulty, it is my hope that you have been stimulated by the fact that much of the material concerns current research activities. So far, nearly all of the core physics that you have studied was developed up to the middle of the last century. Many of the theoretical concepts and experimental results discussed in these last lectures were discovered in your lifetime or shortly before.

For those of you who will stay here next year and share some of my excitement at this rapidly developing subject, I recommend the fourth year option on Particle Physics in which some of the topics discussed in these lectures are developed in more detail.