A Neoclassical Approach to the Paradox of Thrift*

Preliminary and Incomplete

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September 26, 2013

Abstract

A test of the paradox of thrift is conducted throughout the lens of a business cycle model. To this aim, a simple extension of the neoclassical framework with concave frontier is developed which leads to a dramatic improvement on the prediction of the saving rate. Then, it is possible to isolate periods when saving changes are not a consequence of technology shocks. A VAR identified through these episodes suggests that a 1% increase in the saving rate leads to half a percentage point decrease in output growth.

JEL Classification Codes: E13, E21, E32
Keywords: Business cycle, technology shocks, saving shocks

*I thank Martin Gervais, John Knowles and Paul Klein, Ramon Marimon, Dalibor Stevanovic for helpful comments. All remaining errors are mine.

1 Introduction

What is the effect of an increase in the saving rate? The paradox of thrift, a famous conjecture popularaized by Keynes, but already debated at least since the 18th century,[1] is that an increase in the saving rate will depress economic activity. In its strict form, the paradox states that the effect on output is so strong to even reduce savings, thereby nullifying the original intent. Throughout centuries, this conjecture received support within and beyond academic circles capturing media attention and affecting the policy debate, despite the fact that there is virtually no empirical evidence in, or against its favor. Chamley (2012), Huo and Ríos-Rull (2012) and Rendahl (2012) are examples of the growing recent theoretical investigation spurred by the financial and debt crisis.[2] The main difficulty to empirically test the paradox of thrift hypothesis is that savings are endogenous and strongly positively correlated to economic activity. Unlike another popular conjecture of Keynes–the spending multiplier–which engendered a multitude of empirical studies, the paradox of thrift remained largely uninvestigated because it is hard to identify episodes with exogenous changes in the saving rate.

This paper proposes a methodology to identify periods of times in which saving rate changes were not due to the endogenous reaction to the business cycle. To do so, the paper adopts a polar approach to the keynesian one; a neoclassical business cycle model is used to develop a positive theory of savings through which it is possible to isolate periods when changes in the saving rate are not due to technological shocks. These periods are identified as those when the saving rate moves in opposite directions to the one predicted by the model.[3]

[1] See Mandeville (1924) and Robertson (1892) among others.

[2] A related but different literature studies the effects of financial shocks. Mian and Sufi (2010) and Mian and Sufi (2012) provide empirical evidence that shocks to households balance sheets affect consumption and employment. While financial shocks can lead to an increase in the saving rate, the effects of an increase in the saving rate may not be confined to the ones that follow from financial shocks. In fact, a saving rate increase may well improve financial conditions by increasing the supply of loanable funds, yet having negative aggregate effects.

[3] It is important to notice that the correlation of output growth and the saving rate over the selected sample is strongly positive. This suggests that the procedure does not only select periods where savings and output move in opposite directions.

An S-VAR identified through these observations finds a negative effect of the saving rate on output. Quantitatively, the VAR suggests that a 1% increase in the saving rate leads to half a percentage point decrease in output growth.

To isolate saving rate changes not due to technology shocks, a simple real business cycle model is simulated with the shocks and initial conditions identified from the data. Then, it is possible to compare true and simulated data and identify the periods when the model prediction of the saving rate moves in opposite direction to the data. This simulation procedure is not common in the literature, where the identified shocks are only used to estimate their stochastic processes. Then, simulations consist in drawing from these processes and simulating around the balanced growth path. The alternative exercise proposed here uncovers a puzzle that characterizes business cycle models driven by technology shocks: they are very poor predictors of the saving rate time series. This may prevent the ability of the model to filter periods of time when changes in the saving rate are truly not consequential to technology shocks.[4] A large part of the paper is devoted to address this issue by elaborating a simple extension of the real business cycle model where the frontier between consumption and investment goods can be concave.

Letting the frontier be concave relaxes the common assumption of linear frontier maintained throughout the Dynamic Stochastic General Equilibrium (DSGE) literature. Prominent examples include Greenwood et al. (2000), Cummins and Violante (2002) and Fisher (2006). A concave frontier has important implications for the equilibrium price between consumption and investment goods, commonly used to identify I-shocks. This price equation is key because without identifying investment shocks from the price equation, Justiniano et al. (2008) find that the I-shock should be 4 times more volatile in order to match business cycle fluctuations. This sharp contrast calls for an investigation of this price equation (and consequently on the assumption of linear frontier) which, despite its wide use, remained largely under-investigated. The paper points out that the two shocks identified through a linear frontier are strongly negatively

[4]The paper distinguishes between *neutral* or total factor productivity shocks that hit all sectors of the economy (N-shock) and shocks specific to the productivity of investment goods (I-shock) (Greenwood et al. (1988)), which may be an important driver of the saving rate.

correlated. The paper argues that this fact is related to the poor fit of the saving rate mentioned before, and points to a concave frontier. Indeed, with a concave frontier, the model improves dramatically on the prediction of the saving rate. The paper argues that this way the model improves the ability to distinguish between the saving rate changes that are due to technology shocks from those that are not.[5]

Methodologically, the approach proposed in this paper to identify saving shocks is similar to the one adopted by the literature aimed at measuring fiscal multipliers based on the narrative approach (Romer and Romer (2010)), see Mertens and Ravn (2012) for a comprehensive analysis. There, external information is used to identify fiscal shocks, here this kind of information, which is not available, is replaced by a theory of savings. Then, it is possible to isolate periods in which changes in the saving rate are not due to technological shocks. Even during these periods, an assumption on the contemporaneous effects between saving shocks and output is required to fully identify the VAR. The paper shows that the negative effect of saving shocks on output holds true for a wide range of identification assumptions.

The paper is organized as follows, the next section introduces a simple real business cycle model and discusses the inability to match the saving rate. Section 3 modifies the framework, introducing curvature in the transformation frontier and illustrates the findings. Section 4 compares this model with capital adjustment costs. Section 5 tests the Paradox of Thrift hypothesis, and Section 6 concludes. The Appendix contains data sources, the equilibrium conditions and equivalent de-trended specifications.

[5]An alternative assumption to improve the prediction of the saving rate is to consider capital adjustment costs. The paper compares the effects of a concave frontier with those that come from adding capital adjustment costs. This friction only induces a negligible improvement on the saving rate prediction of the original model with linear frontier and do not affect the identification of the I-shock. Richer models of the business cycle also do not affect the identification of I-shocks; for example Schmitt-Grohe and Uribe (2008), Justiniano et al. (2008) and Justiniano et al. (2009) consider medium-scale models with several frictions that do not affect the investment price equation.

2 A Real Business Cycle Model

Below follows a description of the standard growth model with investment-specific technological change like, for instance, the one adopted in Fisher (2006). The representative household solves the following problem, taking prices as given:

$$\max_{\{c_t, k_{t+1}, n_t\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\log(c_t) - \chi \frac{n_t^{1+1/\nu}}{1+1/\nu} \right) \right]$$
$$s.t. \quad c_t + p_t k_{t+1} = w_t n_t + p_t k_t (1 + r_t - \delta).$$

These preferences are adopted for instance by Ríos-Rull et al. (2009), as they point out ν is the Frisch elasticity of hours, n_t. χ scales the cost of working and it determines the average level of hours. β is the discount factor. c_t is consumption, p_t the price of capital in terms of consumption goods. w_t is the wage rate and r_t the rental price of capital, k_t. δ is the rate at which capital depreciates. Production takes place through a constant returns to scale Cobb-Douglas technology and capital evolves according to the law of motion

$$k_{t+1} - k_t(1-\delta) = V_t A_t k_t^{\alpha} n_t^{1-\alpha} s_t,$$

while non-durable consumption is

$$c_t = (1 - s_t) A_t k_t^{\alpha} n_t^{1-\alpha} \tag{1}$$

where s_t is the fraction of physical production allocated to investment. V_t is the investment shock, which only hits the production devoted to increasing the capital stock. A_t is a neutral shock that hits both sectors in the same way. These two shocks evolve according to the following processes:

$$V_t = \gamma_{0,v} \gamma_{1,v}^t V_{t-1}^{\rho_v} e^{\varepsilon_{v,t}}, \;\; \rho_v \leq 1, \tag{2}$$

$$A_t = \gamma_{0,a} \gamma_{1,a}^t A_{t-1}^{\rho_a} e^{\varepsilon_{a,t}}, \;\; \rho_a \leq 1. \tag{3}$$

where $\varepsilon_{v,t}$ and $\varepsilon_{a,t}$ are independently and identically distributed random variables with standard deviation σ_{ε_v} and σ_{ε_a}. $\gamma_{0,v}, \gamma_{1,v}, \gamma_{0,a}, \gamma_{1,a}$ are positive constants.

Firms are competitive and can choose wether to sell production as consumption or capital: given prices they solve the following static problem:

$$\max_{k_t, n_t, s_t \in [0,1]} y_t - w_t n_t - p_t k_t r_t$$
$$s.t.$$

$$y_t = (1 - s_t) A_t k_t^{\alpha} n_t^{1-\alpha} + p_t V_t s_t A_t k_t^{\alpha} n_t^{1-\alpha}. \tag{4}$$

The first order conditions for the firm are as below:

$$\begin{aligned} \alpha A_t k_t^{\alpha-1} n_t^{1-\alpha}(1 - s_t + p_t V_t s_t) &= p_t r_t \\ (1-\alpha) A_t k_t^{\alpha} n_t^{-\alpha}(1 - s_t + p_t V_t s_t) &= w_t \end{aligned}$$

and for an interior s_t

$$p_t = 1/V_t. \tag{5}$$

The price equation (5) reflects the fact that a firm can choose where to allocate its inputs with no costs. Hence, it will be indifferent between producing consumption or investment goods if and only if (5) holds. This strong implication of the model is what is disputed in the present paper. This assumption is innocuous for the growth analysis of the model as in Greenwood et al. (1997) [6] for which the model was originally built, but it matters for the business cycle analysis.

From (1), (4) and (5), $s_t = 1 - \frac{c_t}{y_t}$ holds. Therefore total output simplifies to

$$y_t = A_t k_t^{\alpha} n_t^{1-\alpha}.$$

From this and (5), time series for A and V are identified as follows

$$A_t = \frac{y_t}{k_t^{\alpha} n_t^{1-\alpha}}, \tag{6}$$

$$V_t = 1/p_t. \tag{7}$$

[6] The modification introduced in section 3 leaves the balanced growth path unchanged and therefore it maintains the same growth implications of the original framework as shown in Appendix 2, section A.2.

2.1 Correlation Between Shocks

The data are constructed by extending to 2012 II the data-set in Ríos-Rull et al. (2009). In particular, data on the relative price of investment goods extend those constructed by Gordon (1990), and successively by Cummins and Violante (2002) and Fisher (2006). The dataset starts in 1948 I and the sources are detailed in Appendix A.1. The two shocks are identified through equations (6)–(7). To identify the neutral shock A, α is assumed equal to 0.36 and the results of this section are robust to changes in this parameter.

ADF and Phillips-Perron tests accept the hypothesis of a unit root–stochastic trend–for $\ln(A)$ and for $\ln(V)$. I therefore estimate the regression

$$d\ln(A_t) = \underset{0.0007}{0.0023} - \underset{0.086}{0.292} d\ln(V_t) + \varepsilon_t.$$

The relationship between these two variables is negative and strongly significant. The correlation is strongly negative:

$$corr[d\ln(A), d\ln(V)] = -0.22.$$

Considering sub-samples of this sample gives similar results. I conclude that the two time series for the shocks identified through the usual framework are negatively correlated. This result is consistent with the finding of Schmitt-Grohe and Uribe (2011) that the N-shock and the relative price of investment are cointegrated.

2.2 Calibration

The other dimension where the misspecification is notable is that the model predicts counter-factual savings rates. To asses this, the model is simulated with the shocks identified from the data under a fairly standard parametrization summarized in table 1.

The parameters of the model are $\beta, \alpha, \delta, \gamma_{0,a}, \gamma_{0,v}, \gamma_{1,a}, \gamma_{1,v}, \rho_a, \rho_v, \sigma_{\varepsilon_v}, \sigma_{\varepsilon_v}, \chi, \nu$.

In this model $1-\alpha$ is equal to the labor share and hence α is calibrated equal to 0.36.

ρ_a and ρ_v are restricted to be equal to one as suggested by the unit-root tests. $\gamma_{0,a}, \gamma_{0,v}, \gamma_{1,a}, \gamma_{1,v}, \sigma_{\varepsilon_a}, \sigma_{\varepsilon_v}$ are estimated by running OLS regressions on the logs of the shocks. As is customary, it has implicitly been assumed that there is zero covariance between the innovations.

δ is equal to 0.014, the average depreciation rate of total capital calculated by Cummins and Violante (2002). The discount factor β is equal to 0.99. This parametrization implies an average capital-output ratio of 10.2 an investment-output ratio of 0.26 and an interest rate of 3.5%.

It remains to calibrate the parameters of the supply of labour: the critical one is ν, which represents the Frisch elasticity. As pointed out by King and Rebelo (1999) among others, how much of the business cycle can be explained by technology shocks depends crucially on this parameter. Micro estimates suggest a small number: a recent survey of the micro evidence by Chetty et al. (2011) on the Frisch elasticity points to a value of 0.5 on the intensive margin and of 0.25 on the extensive margin. Macro studies point to a larger role of the extensive margin which theoretically can lead ν up to ∞ even when the intensive margine is zero; see Rogerson (1988) and Hansen (1985). Prescott (2004) considers a value of approximately 3. Ríos-Rull et al. (2009) estimate the very model of this section using Bayesian techniques and find posterior means between $\nu = 0.12$ and $\nu = 1.56$, depending on the variables and the shocks included in the estimation.

In the context of the present application, which aims at measuring the extent to which the model replicates some empirical observations, in particular the observed saving rates, it seems instructive to consider a relatively high level of Frisch elasticity, to give the model the best chance to match the data. Values of 0.75, 1.5 and 3 are considered.

Finally, χ is chosen so that the average number of market hours is 0.33.

2.3 Saving Rate

To compare the saving rate of the model with the one in the data, the model is simulated with the time series of innovations $\varepsilon_{a,t}$, $\varepsilon_{v,t}$ identified from the data and initial conditions for A_0,V_0 and k_0, all coming from the data. To avoid dependence on initial conditions, the model is compared to the data from 1960 III (the 50th period of simulation).

Table 1: Summary of Parametrization

Parameter	Moment to Match	Value
β	interest rate	0.99
δ	direct measurement by Cummins and Violante (2002)	0.014
α	labor share	0.36
ν	micro and macro evidence on Frisch elasticity	0.75, 1.5, 3
χ	average market hours	11.97, 5.64, 3.87
$\gamma_{0,a}$	OLS	1.003
$\gamma_{0,v}$	OLS	1.004
$\gamma_{1,a}$	OLS	1
$\gamma_{1,v}$	OLS	1
ρ_a	ADF and PP tests	1
ρ_v	ADF and PP tests	1
σ_{ε_a}	OLS	0.0069
σ_{ε_v}	OLS	0.0051

The upshot of this experiment is that, although the model performs reasonably well according to the most common statistics used to evaluate the model–standard deviations and covariances presented in tables 2 and 3–the saving rate is very poor. Let $\hat{s}$ be the time series of s predicted by the model, and s the time series realized. The two time series are so different from one another, that the $R^2 = 1 - var(\hat{s} - s)/var(s)$ is even negative: $R^2 = -0.036$ when $\nu = 3$, $R^2 = -0.013$ when $\nu = 1.5$ and $R^2 = 0.004$ when $\nu = 0.75$.

This shortcoming is not easily captured by simply looking at correlations and standard deviations in tables 2 and 3. Indeed, similarly to other RBC models, the only major shortcoming notable from these tables is that the model underpredicts the volatility of hours.

The two facts highlighted–the negative correlation between the shocks and the counterfactual saving rate–are taken as a sign of misspecification in the model. Alternatively, one could argue that the saving rate may move for other non technological shocks, not considered here. That notwithstanding, the poor performance highlighted calls for an investigation of this issue. Indeed, the next subsection interprets the negative correlation between the shocks and the bad

Table 2: Standard deviations

	Output	Consumption	Investment	Hours
Data	1.57	0.67	5.21	1.88
Model				
$\nu = 3$	1.03	0.72	2.73	0.47
$\nu = 1.5$	0.97	0.72	2.47	0.34
$\nu = 0.75$	0.92	0.72	2.25	0.22

Table 3: Correlation with output

	Output	Consumption	Investment	Hours
Data	1	0.40	0.95	0.87
Model				
$\nu = 3$	1	0.76	0.88	0.74
$\nu = 1.5$	1	0.78	0.86	0.69
$\nu = 0.75$	1	0.80	0.84	0.64

fit in the saving rate as being suggestive of a concave transformation frontier.

2.4 The Case for Curvature in the Transformation Frontier

Since the model can be expressed in recursive form with state variables A_t, V_t, k_t, assume that the true price equation is of the form

$$p_t = p(A_t, V_t, k_t) \tag{8}$$

and let the total production measured in consumption units be

$$y_t = y(A_t, V_t, k_t). \tag{9}$$

Considering instead the price equation (5) and the aggregate resource constraint (4) would wrongly impute all the increase (decrease) in the relative price to a decrease (increase) in V_t, and all the variation in production not explained by k_t to A_t. If instead $\frac{\partial p}{\partial A_t} > 0$, increases in p_t may be due to increases in A_t, and when this happens, also y_t increases through A_t. With the misspecified policy functions, the increase in the price would be attributed to a decrease in V_t, while instead only an increase in A_t occurred. This leads to the negative correlation between A_t and V_t, which is not a pure negative correlation between the two shocks, but is due to the misspecification of the model.

The misspecification also leads to counter-factual saving rates: when there is an increase in A_t, according to the true policy function (8) p_t grows. When this happens, the original model identifies a decrease in V_t. Because the productivity of investments decreased, the saving rate predicted by the model decreases. If, on the contrary, no I-shock occurred, the increase in A_t would imply an increase in the saving rate. Therefore, a misspecified price equation leads to counter-factual saving rates.

It follows from these considerations that a model used to measure these shocks for the business cycle should be specified in a way such that the time series for the shocks that it predicts, appear to be independent and match as closely as possible the saving rate time series. These are the two facts that will be targeted in the specification and calibration of the model presented in the next section.

3 The Modified Framework

Consider the following modification to the model: a generic firm produces

$$y_t = A_t k_t^a n_t^{1-a}(1-s_t)^{1-\rho} + p_t V_t A_t k_t^a n_t^{1-a} s_t^{1-\rho}, \tag{10}$$

where $\rho \in [0, 1)$. s_t measures the share of inputs allocated to the production of investment goods.

Therefore,

$$c_t = A_t k_t^a n_t^{1-a}(1-s_t)^{1-\rho} \tag{11}$$

$$k_{t+1} - k_t(1-\delta) = V_t A_t k_t^a n_t^{1-a} s_t^{1-\rho}. \tag{12}$$

The firm can produce for both sectors and solves the following problem:

$$\max_{k,n,s\in[0,1]} A_t k_t^a n_t^{1-a}(1-s_t)^{1-\rho} + p_t V_t A_t k_t^a n_t^{1-a} s_t^{1-\rho} - wn - rpk. \tag{13}$$

When $\rho > 0$, the marginal productivity of consumption and of investment goods is decreasing and therefore the firm will always choose to produce both types of goods even when pV differs from one. This very simple specification, capturing curvature in somewhat reduced form, has the advantage of being closely related to the original framework from which it departs, thereby being able to isolate the role of curvature from any other possible change that can be made. In particular, this technology preserves the assumption of constant returns to scale, so the problem remains consistent with perfect competition where the size and number of firms does not matter and firms take prices as given.

The equilibrium conditions that correspond to a competitive equilibrium are reported in appendix A.2 and are essentially unchanged with respect to the usual framework, except for the resource constraints above and for the price equation, which comes from the optimal choice of s_t:

$$p_t V_t = \frac{(1-s_t)^{-\rho}}{s_t^{-\rho}}. \tag{14}$$

This price equation shows that the change in the relative price is not only due to a change in V_t, but it also depends on the change in s_t, i.e. on the change in the relative demand for the two goods. This in turns depends on both the shocks and on capital. The reason for this is that the production possibility frontier is concave as illustrated in Figure 1.

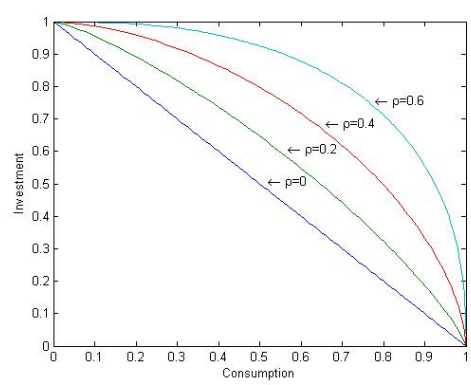

Figure 1: Production possibility frontier

When $\rho = 0$ the price equation (and the whole model) boils down to the usual framework. The next section pins down ρ.

3.1 Estimating ρ

As mentioned, two strategies are employed. The first is to pick ρ such that the shocks identified are uncorrelated. Similarly to the original model, the shocks can be identified from the production equation(10) and the price equation (14) as

$$V = \left(\frac{1-s}{s}\right)^{-\rho} \frac{1}{p} \tag{15}$$

$$A = \frac{y}{k_t^{\alpha} n_t^{1-\alpha} \left[(1-s)^{1-\rho} + pVs^{1-\rho}\right]}. \tag{16}$$

As becomes clear from observing the two equations above, to identify the shocks, it is first necessary to identify s. From the resource constraints (11) and (12) it follows that

$$\frac{(1-s)^{1-\rho}}{pVs^{1-\rho}} = \frac{c}{y-c}, \tag{17}$$

Substituting into equation (14) one gets

$$s = 1 - \frac{c}{y}. \tag{18}$$

s is the saving rate which can be taken from the data.
With s, k, y, p and n at hand, at each ρ correspond time series for A and V through (15) and (16) and a correlation $\text{corr}(\varepsilon_a, \varepsilon_v)$ from the estimation of the processes (2) and (3). Figure 2 shows this correlation as a function of ρ.

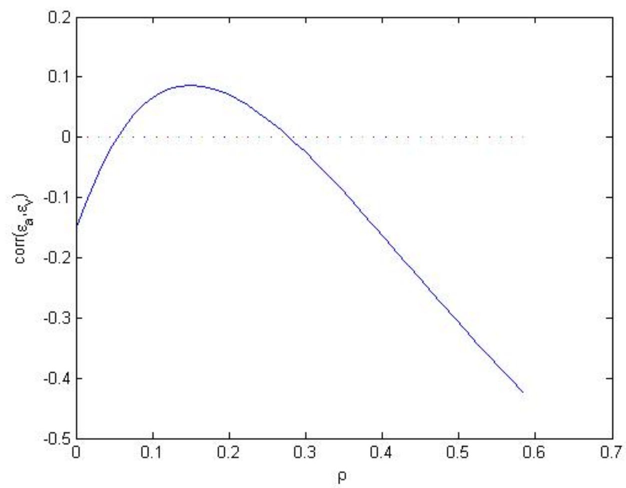

Figure 2: Correlation between the shocks'innovations as a function of ρ.

As it can be observed, the correlation is concave, and it crosses zero twice. It should be clear that starting with a linear frontier ($\rho = 0$), an increase in ρ reduces the correlation for the arguments in section 3. The picture also shows that for ρ sufficiently high, the correlation is decreasing in ρ. This can be rationalized as follows: after a positive I-shock, inter-temporal optimization calls for an increase in the saving rate s. The increase in s implies that the marginal productivity of consumption goods $(1-\rho)A_t k_t^{\alpha} n_t^{1-\alpha}(1-s_t)^{-\rho}$ increases. The marginal productivity of investments, measured in consumption goods $p_t V_t (1-\rho) A_t k_t^{\alpha} n_t^{1-\alpha} s_t^{-\rho}$ also has to increase, since the two marginal productivities must be equal in equilibrium. This calls for an increase in pV. Compared with the original framework, the price reacts less to a change in the investment shock, making the product pV procyclical. Unlike what happens in the original framework, the fact that pV increases even after an investment shock, makes aggregate productivity increase. With too much curvature, this effect may be exacerbated, total output is predicted to increase more than in the data and a negative N-shock is identified. Thus, the correlation between the two shocks is negative if ρ is too high.

The second strategy to pin down ρ is to maximize the R^2 of the saving rate predicted by the model given the shocks identified.[7] This is done through a grid search over ρ and for each value of ρ, by doing the following: 1. given the other parameters, back out the two shocks time series through (15) and (16); 2. Estimate the parameters of the shocks' processes. 3. Solve the model.[8] 4. Simulate given the shocks identified, and compute the R^2 after discarding the first 50 observations.

Figure 3 plots the R^2 as a function of ρ.

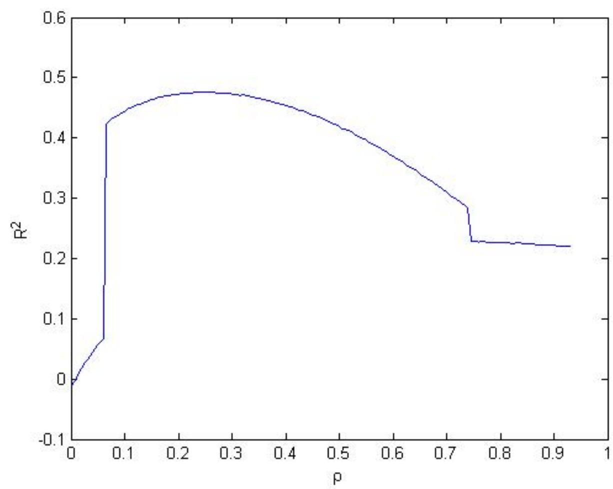

Figure 3: R^2 as a function of ρ.

There is a kink when ρ is approximately 0.06. At that point the N-shock becomes stationary, this leads to a much higher portion of variance explained. With Frisch elasticity $\nu = 1.5$, the value of ρ that gives the highest R^2 is $\rho = 0.243$. R^2 of 0.476 is a substantial increase in the portion of variance of the saving rate explained by this model compared to the original framework where the variance explained is essentially zero. With $\nu = 0.75$ this procedure leads to $\rho = .233$ with $R^2 = .470$. With $\nu = 3$, $\rho = .252$ and $R^2 = .481$.

[7]This is equivalent to minimize the squared sum of residuals $s - \hat{s}$.

[8]As explained below, with higher values of ρ the N-shock is stationary. When this is the case, the model is solved assuming a stationary process for the N-shock and maintaining a non-stationary process for the I-shock.

Strikingly, these estimates for ρ are very close to the highest value obtained with the other procedure. In fact the properties of the shock processes are essentially unchanged when ρ is found by maximizing the saving rate or with the highest value obtained through the correlation procedure. This suggests that among the two values estimated through the correlation procedure, the highest value may be favored. To clear out any doubt, a GMM procedure is ran where the moments above–the correlation between the residuals $\text{corr}(\varepsilon_a, \varepsilon_v)$ and the sum of squares $(s - \hat{s})$–are combined. Not surprisingly, this procedure gives a value between the one that maximizes the R^2 and the highest value obtained through the correlation procedure. These estimations are summarized in table 3.1. The table also reports the standard deviation of the parameter estimated and the p-value of the J test for over-identification, which does not reject the null that the model is correctly specified. Given the asymptotic normality of the GMM estimator, standard errors suggest that ρ is significantly larger than zero.

Table 4: Curvature Parameter ρ

Method	Estimate	St.Dev	J test (p-value)
1^{st} strategy	0.279	-	-
$\nu = 0.75$			
2^{nd} strategy	0.233	-	-
GMM	0.266	0.065	0.424
$\nu = 1.5$			
2^{nd} strategy	0.243	-	-
GMM	0.265	0.047	0.428
$\nu = 3$			
2^{nd} strategy	0.252	-	-
GMM	0.267	0.049	0.432

Results are not very sensitive to the stand taken on ν. Thus, $\nu = 1.5$ is considered hereafter unless otherwise specified.

Figure 4 compares the saving rate from the data to those of the model with $\rho = 0.265$ and $\rho = 0$. As is evident, curvature induces a substantial improvement.

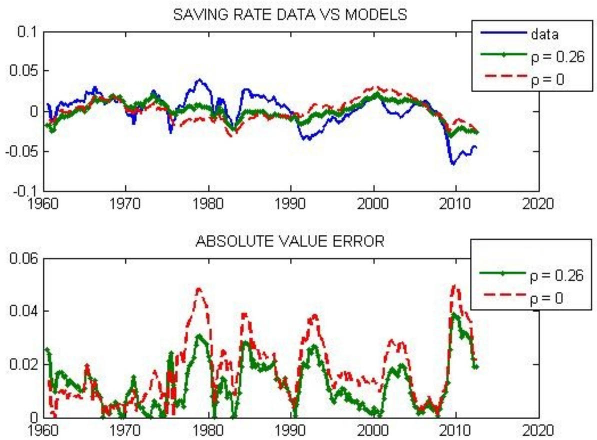

Figure 4: Saving rate: deviation from the mean

3.2 Results

Given the value of ρ estimated with GMM, the parameter values for the shock processes are summarized in table 3.2. A first result is that while the investment shock remains a unit root as with $\rho = 0$, the neutral one is now trend stationary. ADF tests, with various lags, reject the hypothesis of a unit root for the neutral shock with p-values that range between 1% and 9%. The Philip-Perron test rejects the hypothesis of a unit root with p-values always below 5%.[9]

As with the baseline framework, the process for the N-Shock does not show a significant trend: all the growth is captured by the I-Shock. Appendix A.3 derives the equivalent stationary conditions when there is a trend stochastic shock and a stationary one. The transformed stationary model proves the

[9]Whether the business cycle is about stationary fluctuations around a deterministic trend, or is due to a stochastic trend has been debated since the paper by Nelson and Plosser (1982). This is important because the amplification to permanent shocks are typically weaker than the reactions to a transitory shock. There is a simple intuition for this result: when there is a permanent shock, productivity grows but so does expected wealth. Therefore, the expected marginal utility of consumption decreases, lowering the boost in the saving rate and in the labour supply. In most of the previous studies the two shocks were either considered both stationary, as for instance in Smets and Wouters (2007), or both unit roots as in Fisher (2006) and Schmitt-Grohe and Uribe (2011) among others, or the N-shock was assumed to be a unit root and the I-shock stationary (Justiniano et al. (2008)).

existence of a Balanced Growth Path and allows for a recursive formulation. From this it becomes clear that the model has the same long-run implications as the original framework: the expected growth rates of all the variables are unchanged.

Table 5: Other Parameter Values

$\gamma_{0,a}$	$\gamma_{1,a}$	ρ_a	σ_{ε_a}	$\gamma_{0,v}$	$\gamma_{1,v}$	ρ_v	σ_{ε_v}
1.017	1.000	0.983	0.006	1.000	1.005	1.000	0.012

Tables 6 and 7 report standard deviations and correlations with output.

Table 6: Standard deviations ($\nu = 1.5$)

	Output	Consumption	Investment	Hours
Data	1.57	0.67	5.21	1.88
Model				
$\rho = 0$	0.97	0.72	2.47	0.34
$\rho = 0.265$	0.92	0.64	2.54	0.36

Table 7: Correlation with output ($\nu = 1.5$)

	Output	Consumption	Investment	Hours
Data	1	0.40	0.95	0.87
Model				
$\rho = 0$	1	0.78	0.86	0.69
$\rho = 0.265$	1	0.75	0.87	0.74

It is possible to revisit the age old question originated by Kydland and Prescott (1982), of how much of the business cycle is accounted for by technology shocks. With curvature, technology shocks account for 59% of the aggregate fluctuations in output, slightly less than with a linear frontier. However, the volatility of hours and their correlation with output are slightly higher.

The volatility of consumption, which was too high with linear frontier, is now lower. The one of investment, too low with linear frontier, is now higher. These results, closely related to the better fit of the saving rate, are best understood in the light of the impulse response functions to the two shocks.

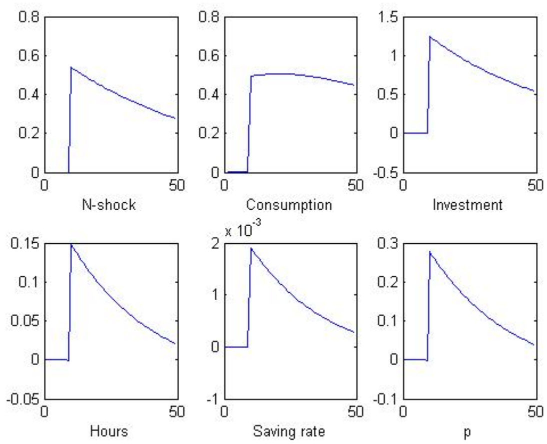

Figure 5: Impulse response function to a N-shock.
All variables other than the saving rate are expressed in percent change, from steady state. The saving rate panel shows the change in level from steady state.

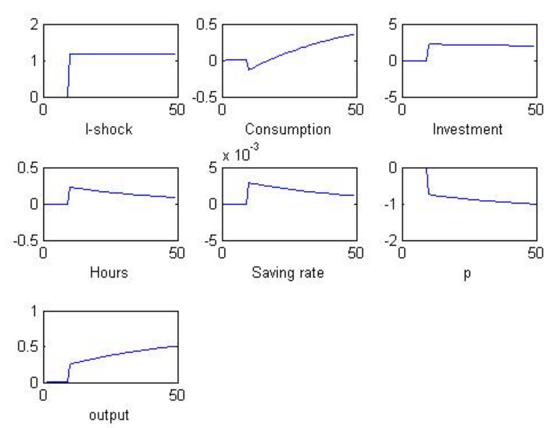

Figure 6: Impulse response function to a I-shock.
All variables other than the saving rate are expressed in percent change, from steady state. The saving rate panel shows the change in level from steady state.

1. As shown in Figure 5, after a positive neutral shock, households want to increase the investment rate in order to smooth consumption. With a concave frontier, firms are reluctant to accommodate this excess demand of investment goods and the price has to increase to induce them to adjust supply.
   This highlights the fact that the change in the relative price of goods is not all due to the I-shock and how it could be misleading to identify the investment shock in the usual way.
   The fact that p increases after an N-shock implies that consumption responds more to the shock relative to the linear framework; the increase in the relative price induces agents to increase consumption, preventing the saving rate to increase as mach as in the linear framework, where the investment price does not depend on the N-shock. This is a feature typical of two-goods models with imperfect input reallocation, which turned out to imply a high equity premium as has been shown by Boldrin et al. (2001).

2. As shown in Figure 6, after an I-shock consumption decreases. However, the decrease is demeaned by the smaller (compared to the linear framework) decrease in the price that follows the investment shock. Thus pV increases, contributing to the increase in GDP measured in consumption goods. Given this increase in GDP, it is possible to increase the saving rate without an abrupt decrease in consumption.

The extent to which this dynamics is an improvement relative to the linear framework can be appreciated by comparing it with the impulse responses from the linear framework, once the negative correlation is taken into account. To this aim, a Choleski decomposition is applied to the covariance matrix between the innovations, so that an innovation to the I-shock can affect the N-shock. As shown in Figure 7, after a positive I-shock output *decreases* for a prolonged period of time. This is because the I-shock also leads to a negative effect on the N-shock. Furthermore, if the negative effect on the N-shock is large enough, an I-shock also induces a *decrease* in the saving rate and hours as shown in

Figure 8. These figures highlight how the typical propagation of I-shocks in the model with linear frontier is hard to rationalize and calls for a concave frontier. One reason why these effects have been overseen might be that typically the correlation between the shocks is ignored. One exception is Schmitt-Grohe and Uribe (2011) who assume a co-integrated process for the two shocks.

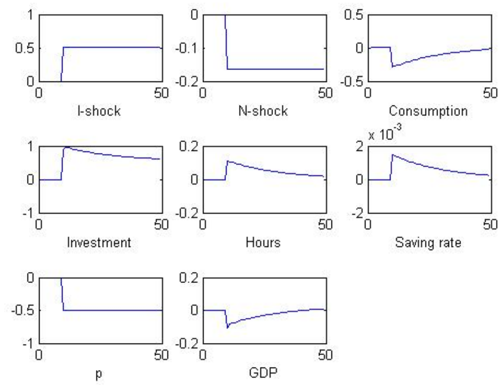

Figure 7: Impulse response function to a I-shock with $\rho = 0$.

All variables other than the saving rate are expressed in percent change, from steady state. The saving rate panel shows the change in level from steady state.

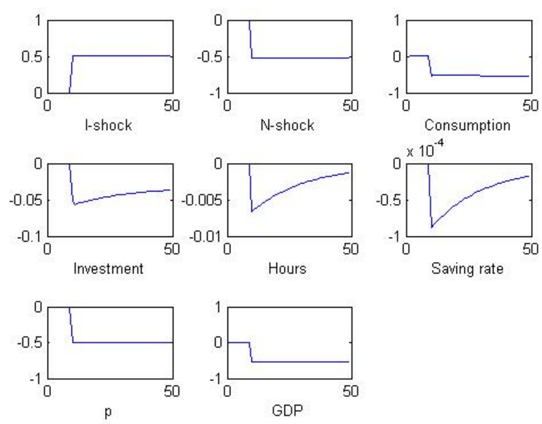

Figure 8: Impulse response function to a I-shock with $\rho = 0$ and strong negative effect on N-shock.

All variables other than the saving rate are expressed in percent change, from steady state. The saving rate panel shows the change in level from steady state.

Finally, it is possible to get a sense of the relative importance of the two shocks. Running the model with either only N-shocks or I-shocks, it is found that most of the variability of output comes from the N-shock, while most of the one of hours comes from the I-shock. In particular, running the model with N-shocks only, 66% of the standard deviation of output is explained but only 6% of the one of hours.[10] Running the model only with I-shocks only, 20% of the standard deviation of output is explained and 18% of the one of hours. The fact that hours are more sensitive to the I-shock has also been found by Ríos-Rull et al. (2009).

4 Capital Adjustment Costs

This section assesses the extent to which capital adjustment costs can be an alternative way to improve on the dimensions considered in this paper.

The household is faced with the following capital accumulation equation:

$$k_{t+1} = k_t(1-\delta) + \psi\left(\frac{i_t}{k_t}\right) k_t \tag{19}$$

where $i_t = V_t A_t k_t^{\alpha} n_t^{1-\alpha} s_t$ is investment goods and $\psi(\cdot)$ is an increasing and concave function. Following Jermann (1998) the adjustment cost function is

$$\psi\left(\frac{i_t}{k_t}\right) = \frac{a1}{1-\zeta}\left(\frac{i_t}{k_t}\right)^{1-\zeta} + a2, \ \ \zeta > 0. \tag{20}$$

The budget constraint is

$$c_t + p_{k,t} i_t = w_t n_t + k_t R_t. \tag{21}$$

The equilibrium conditions are reported in Appendix A.2. The first finding is that since the firm problem is unchanged, the relative price equation that identifies I-shocks is not affected by capital adjustment costs.[11] Therefore, capital adjustment costs do not help making the shocks orthogonal to each other.

[10] In the linear case N-shocks account for 56% of the standard deviation of output and 7% of the one of hours. Curvature amplifies the effects of N-shocks.

[11] Assuming that adjustment costs are borne by the firms would be equivalent.

Following Jermann (1998), $\zeta = 0.23$. It determines the elasticity of investments to Tobin Q. $a1$ and $a2$ are such that the model has the same deterministic steady state as in the case with no adjustment costs.

Simulating the model with the shocks identified, gives an R^2 of 0.048. This is a modest improvement relative to the linear case with no adjustment costs, and it is clearly worse than with a concave transformation frontier.

Tables 8 and 9 compare the business cycle statistics of the model with adjustment costs, to the one with a concave transformation frontier. Adjustment costs dump the volatility of investments, which was already too low, and increase the one of consumption beyond the one in the data. This is intuitive given that with adjustment costs, capital has to be kept smooth, so consumption has to adjust.[12] This is also reflected in a too high correlation of consumption with output. The standard deviation of output is also slightly lower while the one of hours is less than half the one with a concave frontier.

From these findings, it seems clear that adjustment costs do not provide a substitute to a concave frontier for the dimensions this paper is concerned with.

Table 8: Standard deviations ($\nu = 1.5$)

	Output	Consumption	Investment	Hours
Data	1.57	0.67	5.21	1.88
Models				
Adjustment costs	0.90	0.77	1.48	0.16
$\rho = 0.265$	0.92	0.64	2.54	0.36

[12]This is because this friction introduces *inter-temporal* adjustment costs. Instead, concavity in the transformation frontier is a concept that is closer to the *intra-temporal* adjustment costs between capital goods considered by Huffman and Wynne (1999).

Table 9: Correlation with output ($\nu = 1.5$)

	Output	Consumption	Investment	Hours
Data	1	0.40	0.95	0.87
Models				
Adjustment costs	1	0.96	0.91	0.60
$\rho = 0.265$	1	0.75	0.87	0.74

5 The Paradox of Thrift

With a sensible theory of how savings react to technological shocks, it is possible to identify periods in which savings have not responded to technological shocks and asses the implication of these movements in savings on the business cycle. These periods are taken to be those when the true data series moves in opposite direction to the one predicted by the model given the identified technology shocks.[13] On this sub-sample, I estimate a VAR of order one with the saving rate and output growth.[14]

To identify the impulse responses, I assume a Choleski decomposition, where saving rates can affect output growth on impact but output growth can affect savings only with a lag. The extent to which this assumption is extreme is lessened by the fact that the sample excludes periods when the saving rate reacts to technology shocks.

The mentioned sampling procedure selects 20 observations out of the 210 considered (1960.III-2012.II).[15] Thus, the probability that the saving rate moves in the opposite direction than the one predicted by the model is 9.5%. Of these 20 observations, 5 occur while output growth is below average and 15 while

[13]Alternatively, one could use the model to fully identify saving shocks as residuals. The procedure adopted here is preferred because it minimizes the extent to which it relies on the theoretical model.

[14]The choice of only one lag is dictated by the small sample size. However, even on the entire sample, Schwartz's Bayesian information criterion, and the Hannan and Quinn information criterion suggest to use one lag, the Akaike's information criterion suggests two lags. Estimating a VAR 2 does not change the direction of the impulse responses.

[15]The first observations of the sample are excluded to avoid that the simulations depend on initial conditions. Including these observations in the VAR does essentially not affect the results.

output growth is above average. Thus the procedure does not over-sample recessions. It is also worth noticing that the correlation between the saving rate and output growth over the selected sub-sample is 0.39. It is actually higher than the one on the entire sample: 0.28. Therefore, the procedure does not simply select periods where the two variables move in opposite directions. The periods selected are reported in Table 10.[16]

Table 10: Selected Sample

60.IV	64.I	64.III	66.I	70.III	70.IV	77.I	79.IV	84.IV	85.IV
86.II	90.II	91.III	92.II	92.IV	98.III	01.III	02.I	03.I	09.IV

Figure 9 left panels show the impulse responses to a one standard deviation innovation in savings. An increase in the saving rate induces a contraction in output growth. The VAR estimated over this sample has very small innovations, with a standard deviation of 0.22% (i.e. a one standard deviations shock induces the saving rate to go from 15 to 15.22%). To get a sense of the magnitude, it is instructive to consider a 1% increase in the saving rate. Such shock would induce a 0.56% decline in output growth (for instance, if the saving rate goes from 15 to 16%, output drops from 2 to 1.44%). The negative effect lasts for 3 quarters. Thereafter, there is a mild but very long positive effect on output growth.[17]

[16]Starting from the beginning of the sample, five more periods would be selected: 51.IV, 53.II, 57.I, 57.IV, 58.II.

[17]It is also possible to estimate the VAR on the entire sample, and only use the sub-sample for the identification of the impulse responses. This procedure does not qualitatively affect the results.

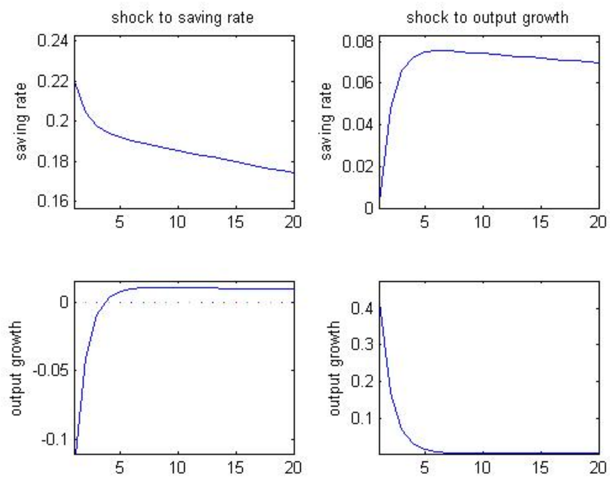

Figure 9: Impulse response function of an S-VAR.

The Figure also shows the response to an output shock (other than technological). Savings react positively from one period after the shock. Presumably, the impact effect on savings may be positive too, contrarily to the decomposition assumption. To check for robustness, I achieve identification by assuming alternative responses of savings to an output shock.

5.1 Alternative Residual Decompositions

Calling η the residuals from the VAR, the structural shocks ϵ can be identified via a matrix M:

$$\eta = M\epsilon. \tag{22}$$

Normalizing the shocks ϵ to have unitary variance, the matrix M has to be such that

$$MM' = \Sigma, \tag{23}$$

where Σ is the covariance matrix of η. Because Σ is symmetric, equation (23) pins down three out of the four parameters in M. The Choleski decomposition puts $M_{1,2} = 0$. Figure 10 shows $M_{2,1}$, the impact response of income to a unitary saving shock, for all possible alternative assumptions for $M_{1,2}$. The range for $M_{1,2}$ is $[-0.002, 0.002]$ because outside this range, M is imaginary.

As the figure shows, the impact response of output to a saving shock is negative for $M_{1,2} > -0.0005$. And for positive, more plausible values of $M_{1,2}$ the effect on income is negative and even stronger than the one of the Choleski decomposition.

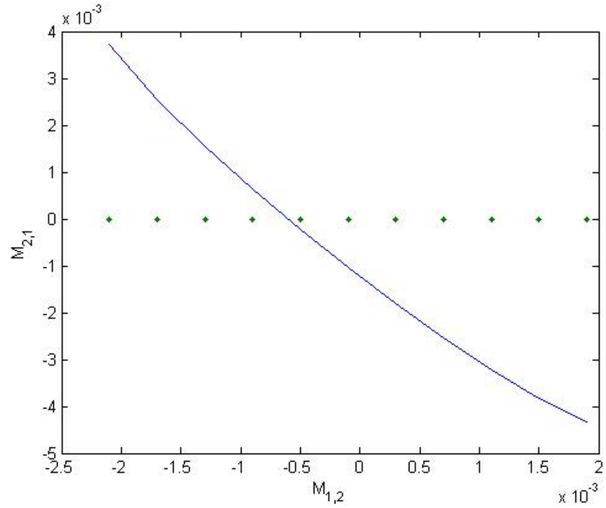

Figure 10: Impact response of income to a saving shock for alternative identification assumptions.

I conclude that this analysis confirms the conjecture that a savings shock depresses economic activity in the short run. Notice that the sample selection procedure is important for this result: the impulse response of output to a saving shock is positive when a Choleski decomposition is used to decompose residuals over the entire sample.

6 Conclusions

To test the paradox of Thrift, a neoclassical business cycle model is used to identify periods when saving rate changes are not consequential to technology shocks.

A by-product of this methodology is to show that the basic framework adopted by the DSGE literature predicts counter-factual saving rates and negatively correlated neutral and investment shocks. I argue that this counter-factual observation emanates from the assumed linearity of the transformation frontier between consumption and investment goods.

A simple extension of the original framework is developed that allows for the transformation frontier to be concave. With a concave frontier, the model improves dramatically on the prediction of the saving rate, dimension that cannot be significatively improved with alternative mechanisms such as capital adjustment costs.

Given the promising results and the simple modeling approach, introducing curvature in the transformation frontier into the large-scale models adopted by the DSGE literature may be a fruitful avenue to pursue. Another interesting direction for future research is to investigate the underlying factors that lead to an aggregate concave frontier.

With a theory of savings that fits the data, it is possible to isolate periods in which changes in the saving rate are not due to technological shocks and test the paradox of thrift hypothesis: that a positive saving shock contracts economic activity. A VAR estimated through this sample suggests that a 1% increase in the saving rate (for instance, from 15 to 16%) leads to a 0.56% decrease in output growth. After a few periods, the effect on output growth is positive. Given these findings, research should be devoted to study the mechanism through which an increase in savings depresses economic activity.

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A Appendix

A.1 Data

The data set extends to 2012 II, the data set of Ríos-Rull et al. (2009), see their online appendix for the construction of a price index for consumption, a quality-adjusted price index for investment, quality-adjusted investment and capital.

A.1.1 Raw Data Series

Bureau of Labor Statistics (BLS)

Hours, ID PRS85006033

Civilian Noninstitutional Population, ID LNU00000000

National Income and Product Accounts (NIPA-BEA)

Nominal Gross National Product, Table 1.7.5
Price Indexes for Private Fixed Investment by Type, Table 5.3.4
Private Fixed Investment by Type, Table 5.3.5
Gross Domestic Product, Table 1.1.5
Government Consumption Expenditures and Gross Investment, Table 3.9.5
Personal Consumption Expenditures by Major Type of Product, Tables 2.3.5 (Nominal) and 2.3.3 (Quantity Index)

Cummins and Violante (2002)

Annual Quality-Adjusted Price Index for Investment in Equipment
Annual Quality-Adjusted Depreciation Rates for Total Capital

A.2 Balanced Growth Path with trend-stochastic shocks

The equilibrium conditions are

$$\lambda_t = \beta E_t \left\{ \frac{1}{c_{t+1}} R_{t+1} + \lambda_{t+1} \left(1 - \delta + \psi\left(\frac{i_{t+1}}{k_{t+1}}\right) - \psi'\left(\frac{i_{t+1}}{k_{t+1}}\right) \frac{i_{t+1}}{k_{t+1}} \right) \right) \tag{24}$$

$$\frac{1}{c_t} p_t = \lambda_t \psi'\left(\frac{i_t}{k_t}\right) \tag{25}$$

$$\frac{w_t}{c_t} = \xi n_t^{1/\nu} \tag{26}$$

$$k_{t+1} = (1-\delta) k_t + \psi\left(\frac{i_t}{k_t}\right) k_t \tag{27}$$

$$p_t V_t = \frac{(1-s_t)^{-\rho}}{s_t^{-\rho}} \tag{28}$$

$$i_t = V_t A_t k_t^{\alpha} n_t^{1-\alpha} s_t^{1-\rho} \tag{29}$$

$$c_t = A_t k_t^{\alpha} n_t^{1-\alpha} (1-s_t)^{1-\rho} \tag{30}$$

$$w_t = (1-\alpha) A_t k_t^{\alpha} n_t^{-\alpha} \left[(1-s_t)^{1-\rho} + p_t V_t s_t^{1-\rho} \right] \tag{31}$$

$$R_t = \alpha A_t k_t^{\alpha-1} n_t^{1-\alpha} \left[(1-s_t)^{1-\rho} + p_t V_t s_t^{1-\rho}\right]. \tag{32}$$

The budget constraint

$$c_t + p_t i_t = w_t n_t + k_t R_t \tag{33}$$

is implied by Walras' law.

Let $z_t = A_t^{\frac{1}{1-\alpha}} V_t^{\frac{\alpha}{1-\alpha}}$. Consider the auxiliary variables $\tilde{c}_t = \frac{c_t}{z_{t-1}}$, $\tilde{k}_t = \frac{k_t}{z_{t-1}V_{t-1}}$, $\tilde{p}_t = p_t V_{t-1}$, $\tilde{i}_t = \frac{i_t}{z_{t-1}V_{t-1}}$, $\tilde{\lambda}_t = \lambda_t z_{t-1} V_{t-1}$, $\tilde{w}_t = \frac{w_t}{z_{t-1}}$, $\tilde{R}_t = R_t V_{t-1}$. Substituting these expressions into equations (24)–(32), one obtains the following equations, which are stationary in the auxiliary variables:[18]

$$\tilde{\lambda}_t = \beta E_t \left\{ \frac{z_{t-1}V_{t-1}}{z_t V_t} \left(\frac{\tilde{R}_{t+1}}{\tilde{c}_{t+1}} + \tilde{\lambda}_{t+1} \left(1 - \delta + \psi\left(\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}}\right) - \psi'\left(\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}}\right) \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) \right) \right\} \tag{34}$$

$$\frac{1}{\tilde{c}_t}\tilde{p}_t = \tilde{\lambda}_t \psi'\left(\frac{\tilde{i}_t}{\tilde{k}_t}\right) \tag{35}$$

$$\frac{\tilde{w}_t}{\tilde{c}_t} = \xi n_t^{1/\nu} \tag{36}$$

$$\tilde{k}_{t+1}\frac{z_t V_t}{z_{t-1}V_{t-1}} = (1-\delta)\tilde{k}_t + \psi\left(\frac{\tilde{i}_t}{\tilde{k}_t}\right)\tilde{k}_t \tag{37}$$

$$\tilde{p}_t \frac{V_t}{V_{t-1}} = \frac{(1-s_t)^{-\rho}}{s_t^{-\rho}} \tag{38}$$

$$\tilde{i}_t \frac{(z_{t-1}V_{t-1})^{1-\alpha}}{A_t V_t} = \tilde{k}_t^{\alpha} n_t^{1-\alpha} s_t^{1-\rho} \tag{39}$$

$$\tilde{c}_t \frac{z_{t-1}^{1-\alpha} V_{t-1}^{-\alpha}}{A_t} = \tilde{k}_t^{\alpha} n_t^{1-\alpha} (1-s_t)^{1-\rho} \tag{40}$$

$$\tilde{w}_t \frac{z_{t-1}^{1-\alpha} V_{t-1}^{-\alpha}}{A_t} = (1-\alpha)\tilde{k}_t^{\alpha} n_t^{-\alpha} \left[(1-s_t)^{1-\rho} + \tilde{p}_t \frac{V_t}{V_{t-1}} s_t^{1-\rho}\right] \tag{41}$$

$$\tilde{R}_t \frac{(z_{t-1}V_{t-1})^{1-\alpha}}{A_t V_{t-1}} = \alpha \tilde{k}_t^{\alpha-1} n_t^{1-\alpha} \left[(1-s_t)^{1-\rho} + \tilde{p}_t \frac{V_t}{V_{t-1}} s_t^{1-\rho}\right]. \tag{42}$$

[18] Since the auxiliary variables are independent of ρ, it follows that the trends in the original variables are not affected by ρ.

These equations imply that the stationary budget constraint is

$$\tilde{c}_t + \tilde{p}_t \tilde{i}_t = \tilde{w}_t n_t + \tilde{k}_t \tilde{R}_t. \tag{43}$$

When the productivity processes are trend stochastic (ρ_a and ρ_v equal to one), the productivity processes (2) and (3) reduce to

$$A_t = \gamma_a A_{t-1} e^{\varepsilon_{a,t}} \tag{44}$$

and

$$V_t = \gamma_v V_{t-1} e^{\varepsilon_{v,t}}. \tag{45}$$

This is because the growth factors $\frac{A_t}{A_{t-1}}$ and $\frac{V_t}{V_{t-1}}$ are stationary and thus the trend factors $\gamma_{1,a}$ and $\gamma_{1,v}$ are equal to one. Then, equations (34)–(42) further simplify to

$$\tilde{\lambda}_t = \beta E_t \left\{ \left(\gamma_a \gamma_v e^{\varepsilon_{a,t}+\varepsilon_{v,t}}\right)^{\frac{1}{\alpha-1}} \left(\frac{\tilde{R}_{t+1}}{\tilde{c}_{t+1}} + \tilde{\lambda}_{t+1} \left(1 - \delta + \psi\left(\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}}\right) - \psi'\left(\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}}\right)\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) \right) \right\} \tag{46}$$

$$\frac{1}{\tilde{c}_t}\tilde{p}_t = \tilde{\lambda}_t \psi'\left(\frac{\tilde{i}_t}{\tilde{k}_t}\right) \tag{47}$$

$$\frac{\tilde{w}_t}{\tilde{c}_t} = \xi n_t^{1/\nu} \tag{48}$$

$$\tilde{k}_{t+1}\left(\gamma_a \gamma_v e^{\varepsilon_{a,t}+\varepsilon_{v,t}}\right)^{\frac{1}{1-\alpha}} = (1-\delta)\tilde{k}_t + \psi\left(\frac{\tilde{i}_t}{\tilde{k}_t}\right)\tilde{k}_t \tag{49}$$

$$\tilde{p}_t \gamma_v e^{\varepsilon_{v,t}} = \frac{(1-s_t)^{-\rho}}{s_t^{-\rho}} \tag{50}$$

$$\tilde{i}_t = \left(\gamma_a \gamma_v e^{\varepsilon_{a,t}+\varepsilon_{v,t}}\right) \tilde{k}_t^{\alpha} n_t^{1-\alpha} s_t^{1-\rho} \tag{51}$$

$$\tilde{c}_t = \left(\gamma_a e^{\varepsilon_{a,t}}\right) \tilde{k}_t^{\alpha} n_t^{1-\alpha} (1-s_t)^{1-\rho} \tag{52}$$

$$\tilde{w}_t = \left(\gamma_a e^{\varepsilon_{a,t}}\right)(1-\alpha)\tilde{k}_t^{\alpha} n_t^{-\alpha}\left[(1-s_t)^{1-\rho} + \tilde{p}_t\left(\gamma_v e^{\varepsilon_{v,t}}\right) s_t^{1-\rho}\right] \tag{53}$$

$$\tilde{R}_t = \left(\gamma_a e^{\varepsilon_{a,t}}\right)\alpha \tilde{k}_t^{\alpha-1} n_t^{1-\alpha}\left[(1-s_t)^{1-\rho} + \tilde{p}_t\left(\gamma_v e^{\varepsilon_{v,t}}\right) s_t^{1-\rho}\right]. \tag{54}$$

A.3 Balanced Growth Path with a trend-stationary N-shock and a trend-stochastic I-shock

Identifying the shocks through this framework with curvature, the neutral shock appears to be trend-stationary ($\rho_a < 1$), while the investment one has a stochastic trend.

To detrend the equilibrium condition in this case where the N-shock is stationary and the I-shock is not, let $z_t = \tilde{A}_t^{\frac{1}{1-\alpha}} V_t^{\frac{\alpha}{1-\alpha}}$, where

$$\tilde{A}_t = \gamma_a^{\frac{t+1}{1-\rho_a}}. \tag{55}$$

Putting $\tilde{\gamma}_a = \gamma_a^{\frac{1}{1-\rho_a}}$ and substituting (55) into equations (34)–(42) and putting $\tilde{a}_t = \frac{A_t}{\tilde{A}_{t-1}}$ one gets the equations

$$\tilde{\lambda}_t = \beta E_t \left\{ \left(\tilde{\gamma}_a \gamma_v e^{\varepsilon_{v,t}}\right)^{\frac{1}{\alpha-1}} \left(\frac{\tilde{R}_{t+1}}{\tilde{c}_{t+1}} + \tilde{\lambda}_{t+1} \left(1 - \delta + \psi\left(\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}}\right) - \psi'\left(\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}}\right) \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) \right) \right\} \tag{56}$$

$$\frac{1}{\tilde{c}_t}\tilde{p}_t = \tilde{\lambda}_t \psi'\left(\frac{\tilde{i}_t}{\tilde{k}_t}\right) \tag{57}$$

$$\frac{\tilde{w}_t}{\tilde{c}_t} = \xi n_t^{1/\nu} \tag{58}$$

$$\tilde{k}_{t+1}\left(\tilde{\gamma}_a \gamma_v e^{\varepsilon_{v,t}}\right)^{\frac{1}{1-\alpha}} = (1-\delta)\tilde{k}_t + \psi\left(\frac{\tilde{i}_t}{\tilde{k}_t}\right)\tilde{k}_t \tag{59}$$

$$\tilde{p}_t \gamma_v e^{\varepsilon_{v,t}} = \frac{(1-s_t)^{-\rho}}{s_t^{-\rho}} \tag{60}$$

$$\tilde{i}_t = \left(\tilde{a}_t \gamma_v e^{\varepsilon_{v,t}}\right) \tilde{k}_t^{\alpha} n_t^{1-\alpha} s_t^{1-\rho} \tag{61}$$

$$\tilde{c}_t = \tilde{a}_t \tilde{k}_t^{\alpha} n_t^{1-\alpha} (1-s_t)^{1-\rho} \tag{62}$$

$$\tilde{w}_t = \tilde{a}_t (1-\alpha) \tilde{k}_t^{\alpha} n_t^{-\alpha} \left[(1-s_t)^{1-\rho} + \tilde{p}_t \left(\gamma_v e^{\varepsilon_{v,t}}\right) s_t^{1-\rho} \right] \tag{63}$$

$$\tilde{R}_t = \tilde{a}_t \alpha \tilde{k}_t^{\alpha-1} n_t^{1-\alpha} \left[(1-s_t)^{1-\rho} + \tilde{p}_t \left(\gamma_v e^{\varepsilon_{v,t}}\right) s_t^{1-\rho} \right]. \tag{64}$$

From the definition of $\tilde{a}_t$ and from (3)–(55), it follows that the stochastic process for $\tilde{a}_t$ is

$$\ln(\tilde{a}_t) = \ln(\gamma_0) - \frac{\rho_a}{1-\rho_a}\ln(\gamma_a) + \rho_a \ln(\tilde{a}_{t-1}) + \varepsilon_{a,t}.\text{[19]} \tag{65}$$

[19] $\frac{A_t}{\tilde{A}_{t-1}} = \frac{\gamma_0 A_{t-1}^{\rho_a} \gamma_a^t e^{\varepsilon_{a,t}}}{\gamma_a^{\frac{t}{1-\rho_a}}} = \gamma_0 \left(\frac{A_{t-1}}{\tilde{A}_{t-2}}\right)^{\rho_a} \gamma_a^{\frac{-\rho_a}{1-\rho_a}} e^{\varepsilon_{a,t}}$.