

Quasi-Normal Modes of Black-Holes and Branes from Quantum Seiberg-Witten Curves

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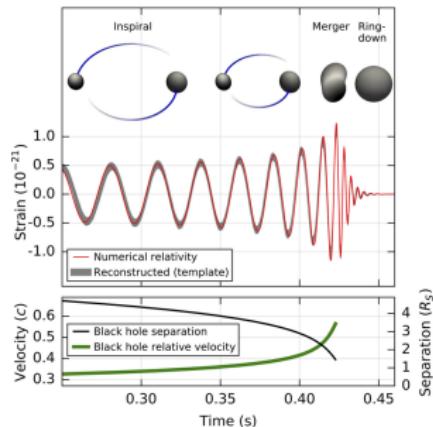
Table of contents

- 1 Gravitational waves, photon-spheres and photon-halos
- 2 Quasi-normal modes
- 3 The Seiberg-Witten/QNMs connection
- 4 Exact solutions at $q = 0$
- 5 Numerical solutions
- 6 Turning BHs and branes inside out their photon-spheres/halos
- 7 Conclusions and outlook

Based on

- Bianchi, Consoli, Grillo, Morales [2105.04245] and [2109.09804]
- Bianchi, Di Russo [2110.09579] and [2111.abcba]

Gravitational Waves from BH mergers



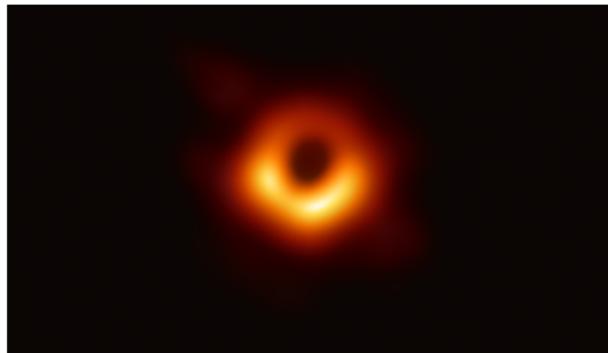
GW signal

- Inspiral ... perturbative (classical gravity from quantum scattering)
- Merger (highly non-linear ... numerical gravity, string fuzz balls)
- Ring-down ... QNMs, ... echoes

Gravitational Wave Detectors

- Ground-based: LIGO, Virgo, KAGRA, ET, CE high frequencies (\sim kHz)
- Space-based: LISA medium frequencies (\sim mHz)
- Pulsar timing arrays: low frequencies (\sim nHz)

Compact objects, photon-spheres and photon-halos



M87 galaxy supermassive compact object
 $M = 6.5 \times 10^{12} M_{\odot}$

Credits: Event Horizon Telescope collaboration et al.

Very likely, this is NOT a black-hole.

For sure it is NOT an Event Horizon

It may be a photon-halo ($r_c(b) > r_H$) or plasma emission (ISCO)

Quasi-normal modes: Schwarzschild black hole

BH metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \quad f(r) = 1 - \frac{2M}{r}$$

Radial equation for massless scalar perturbation $\Phi = \frac{1}{r}\psi(r)Y(\theta, \phi)e^{-i\omega t}$

$$f(r)\frac{d}{dr}\left[f(r)\frac{d}{dr}\psi(r)\right] + [\omega^2 - V(r)]\psi(r) = 0$$

Regge-Wheeler potential, tortoise coordinate $x = \int \frac{dr}{f(r)} = r + 2M \log \frac{r-2M}{Me^{3/2}}$

$$V_{RW}(x) = f(r(x)) \left(\frac{K^2}{r(x)^2} + \frac{2M}{r(x)^3} \right), \quad K^2 = \ell(\ell+1)$$

Boundary conditions for QNMs

- in-going wave at the horizon: $r = 2M$ ($x \rightarrow -\infty$)
- out-going wave at infinity: $r = x \rightarrow \infty$

Quasi-normal modes: exact solutions

Still too complicated, approximate with Pöschl-Teller (PT) potential barrier

$$V_{\text{PT}}(x) = \frac{V_0}{\cosh^2 \alpha x}$$

with

$$\left. \frac{dV_{\text{RW}}}{dx} \right|_0 = 0, \quad V_0 = V_{\text{RW}}(0), \quad \alpha^2 = -(2V_0)^{-1} \left. \frac{d^2 V_{\text{RW}}}{dx^2} \right|_0$$

Setting

$$\xi = [1 + \exp(-2\alpha x)]^{-1}, \quad \psi(\xi) = [\xi(1 - \xi)]^{-\frac{i\omega}{2\alpha}} y(\xi)$$

get hypergeometric equation with

$$a = \frac{\alpha + \sqrt{\alpha^2 - 4V_0} - 2i\omega}{2\alpha}, \quad b = \frac{\alpha - \sqrt{\alpha^2 - 4V_0} - 2i\omega}{2\alpha}, \quad c = 1 - \frac{i\omega}{\alpha}$$

matching asymptotics with correct b.c.'s: $1/\Gamma(a) = 0$ i.e. $a = -n$ so that

$$\omega_{\text{PT}} = \sqrt{V_0 - \frac{\alpha^2}{4}} - i\alpha \left(n + \frac{1}{2} \right) \quad \text{Im}\omega_{\text{PT}} \sim -\lambda < 0$$

Scalar field: Eikonal limit

In the Eikonal limit $K \approx \ell \gg 1$

$$r_0 = 3M \left[1 - \frac{1}{9K^2} \right], \quad V_0 = \frac{K^2}{27M^2} \left[1 + \frac{2}{3K^2} \right], \quad \alpha = \frac{1}{3\sqrt{3}M} \left[1 + \frac{1}{9K^2} \right]$$

and

$$\omega_{\text{PT}} = \frac{K}{3\sqrt{3}M} \left[1 + \frac{5}{24K^2} \right] - \frac{i}{3\sqrt{3}M} \left[1 + \frac{1}{9K^2} \right] \left(n + \frac{1}{2} \right)$$

Alternatively, neglecting derivatives of ‘momenta’

$$H\Phi = 0 = g^{\mu\nu} P_\mu P_\nu \Phi$$

and setting $\Phi = R(r)Y(\theta, \phi)e^{-i\omega t}$, get radial equation

$$R''(r) + Q_r R(r) = 0, \quad Q_r = \frac{1}{f(r)^2} \left(\omega^2 - \frac{K^2 f(r)}{r^2} \right)$$

WKB approximation, Bohr-Sommerfeld quantization

WKB ansatz

$$R(r) = \frac{1}{\sqrt[4]{Q_r}} e^{\pm i \int \sqrt{Q_r} dr}$$

matching condition:

$$\int_{r_-}^{r_+} \sqrt{Q_r} dr = \pi \left(n + \frac{1}{2} \right)$$

expanding around Maximum of $Q_r \approx$ photon-sphere (!!!)

$$Q'_r(r_0, \omega_n) = 0, \quad \frac{Q_r(r_0, \omega_n)}{\sqrt{2Q''_r(r_0, \omega_n)}} = -i \left(n + \frac{1}{2} \right)$$

assuming small imaginary parts: $r_0 = r_c + ir_{\text{Im}}$ and $\omega_n = \omega_c + i\omega_{\text{Im}}$

$$Q_r(r_c, \omega_c) = \partial_r Q_r(r_c, \omega_c) = 0, \quad r_c = 3M, \quad \omega_c = \frac{K}{3\sqrt{3}M}$$

and

$$\omega_{\text{Im}} = -2\lambda \left(n + \frac{1}{2} \right), \quad r_{\text{Im}} = \frac{(n + \frac{1}{2}) \partial_{r,\omega}^2 Q_r(r_c, \omega_c)}{\lambda \partial_\omega Q_r(r_c, \omega_c)^2}, \quad \lambda = \frac{\sqrt{\partial_r^2 Q_r(r_c, \omega_c)}}{\sqrt{2} \partial_\omega Q_r(r_c, \omega_c)}$$

Geodesic interpretation

For large $K \approx \ell$

$$r_0 = 3M - \frac{4Mi}{K} \left(n + \frac{1}{2} \right), \quad \omega_n = \frac{K}{3\sqrt{3}M} - \frac{i}{3\sqrt{3}M} \left(n + \frac{1}{2} \right)$$

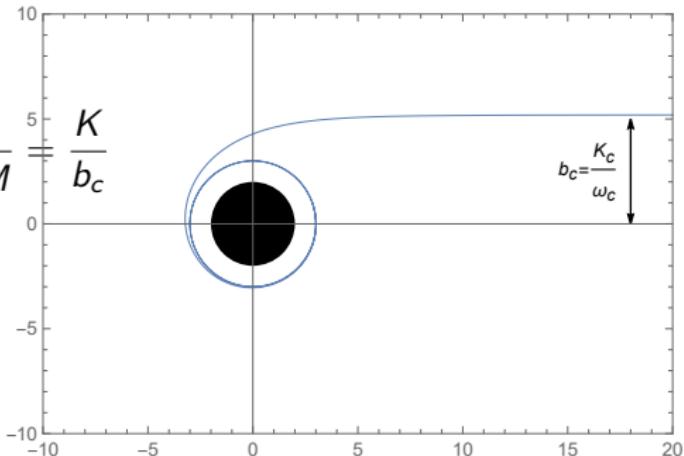
Photon-sphere QNMs \sim prompt Ring-down modes

$$Q_r(r_c, \omega_c) = Q'_r(r_c, \omega_c) = 0$$

$$r_c = 3M > 2M = r_H, \quad \omega_c = \frac{K}{3\sqrt{3}M}$$

Chaotic behaviour / instability
Lyapunov exponent λ

$$\frac{dr}{dt} \simeq -2\lambda(r - r_c), \quad \lambda = \frac{1}{6\sqrt{3}M}$$



Numerical computation of quasi-normal modes

Imposing in-going b.c. at horizon and out-going b.c. at infinity

$$\psi(r) = e^{i\omega(r-2M)}(r-2M)^{-iM\omega} r^{2iM\omega} \sum_{n=0}^{\infty} c_n \left(\frac{r-2M}{r}\right)^n,$$

(three-terms) recursion relation for $b_n = \frac{c_n}{c_{n-1}}$

$$\alpha_n b_n^{-1} + \beta_n + \gamma_n b_{n+1} = 0,$$

Leaver's method of continuous fractions

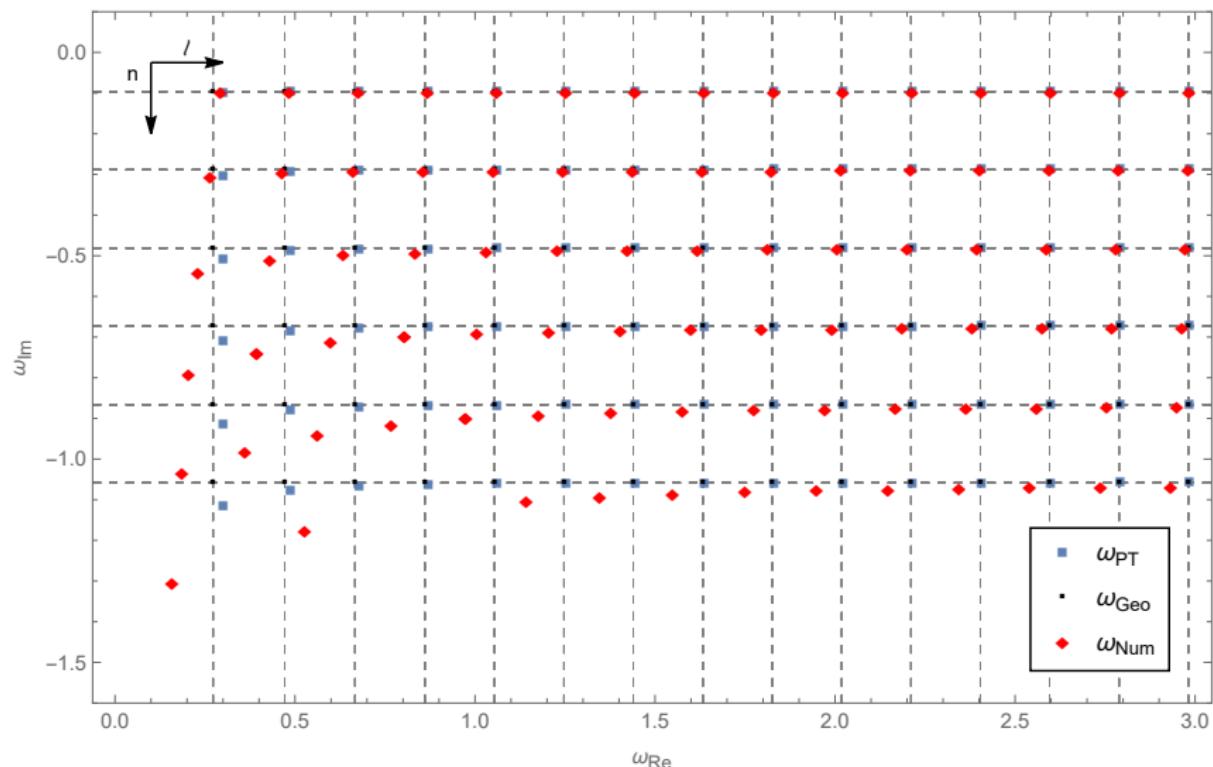
$$b_n = -\frac{\alpha_n}{\beta_n + \gamma_n b_{n+1}}, \quad b_n = -\frac{1}{\gamma_{n-1}} \left(\beta_{n-1} + \frac{\alpha_{n-1}}{b_{n-1}} \right)$$

ascending and descending relations for b_n , with overtone number n

Truncation at order N , eigenvalue equation for ω_{QNM}

Caveat: extremal BHs and branes

Pöschl-Teller, geodesic and numerical



The Seiberg-Witten/QNMs connection

State of the art

Bianchi, Consoli, Grillo, Morales [2105.04245] and [2109.09804]

see also

Isomonodromic approach (Carneiro da Cunha *et al.*)

- 1702.01016
- 1812.08921
- 2109.06929

Exact WKB quantization, Seiberg-Witten approach (Grassi *et al.*)

- 1908.07065
- 2006.06111

AGT correspondence (Bonelli *et al.*)

- 2105.04483

$\mathcal{N} = 2$ super Yang-Mills

$\mathcal{N} = 2$ SYM theories in 4-d with $G = SU(2)$ (N_f flavours, later on)

$$\mathcal{L} \propto \int d^4\theta \mathcal{F}(\Phi) , \quad \mathcal{F}_{\text{tree}}(\Phi) = \frac{1}{2} \text{tr} \tau \Phi^2$$

$$\Phi = \varphi + \lambda \theta + F_{\mu\nu} \theta \sigma^{\mu\nu} \tilde{\theta} + \dots , \quad \tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}$$

Coulomb branch $\langle \varphi \rangle = a$, $SU(2)$ breaks to $U(1)$, Matone relation

$$u(q=0) = \frac{1}{2} \text{tr} \langle \varphi^2 \rangle = a^2 , \quad u(q) = -q \frac{\partial \mathcal{F}(a, q)}{\partial q}$$

where $q = \exp(2\pi i \tau)$. Analytic prepotential

$$\mathcal{F} = \mathcal{F}_{\text{tree}} + \mathcal{F}_{\text{1-loop}} + \mathcal{F}_{\text{inst}}$$

Coulomb branch moduli space same monodromy as a torus ('elliptic curve')

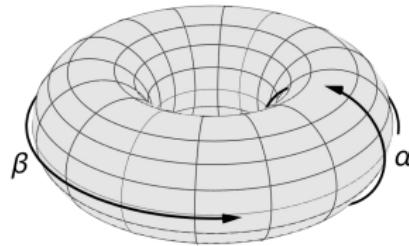
Classical Seiberg-Witten curve

Elliptic curve for $SU(2)$ theories

$$q y^2 P_L(x) + y P_0(x) + P_R(x) = 0$$

α - and β -cycles such that

$$a = \oint_{\alpha} \lambda \quad , \quad a_D = -\frac{1}{2\pi i} \frac{\partial \mathcal{F}(a, q)}{\partial a} = \oint_{\beta} \lambda \quad , \quad \lambda = \frac{x dy}{y}$$



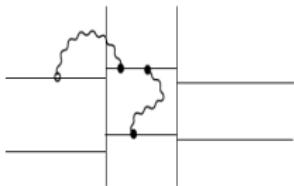
Brane description

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
NS5	—	—	—	—	—	—	·	·	·	·
D4	—	—	—	—	·	·	—	·	·	·

- D4-branes suspended between NS5-branes: gauge/colour ($SU(2)$)
- External D4-branes: matter hyper-multiplets / flavour ($N_f = (N_L, N_R)$)

Eg $SU(2)$ SYM theory with 3 fundamentals $N_f = (1, 2) \dots$ ubiquitous

$$P_L(x) = (x - m_3), \quad P_0(x) = x^2 - u + q \tilde{p}_0(x), \quad P_R(x) = (x - m_1)(x - m_2)$$



Brane diagram for $SU(2)$ SYM theory with four flavors $N_f = (2, 2)$

$$P_L(x) = (x - m_3)(x - m_4), \quad P_0(x) = x^2 - u + q p_0(x), \quad P_R(x) = (x - m_1)(x - m_2)$$

Quantum Seiberg-Witten curves

Nekrasov-Shatashvili Ω -background: non-trivial commutation relation

$$[\hat{x}, \ln \hat{y}] = \hbar \quad , \quad \hat{x} = \hbar y \frac{d}{dy} , \quad \epsilon_1 = \hbar , \quad \epsilon_2 = 0$$

“quantum” version of SW elliptic curve

$$[A(y)\hat{x}^2 + B(y)\hat{x} + C(y)] U(y) = 0$$

canonical form

$$\boxed{\psi''(y) + Q_{\text{SW}}(y) \psi(y) = 0} , \quad U(y) = \frac{1}{\sqrt{y}} \exp \left[-\frac{1}{2\hbar} \int^y \frac{B(y')}{y' A(y')} dy' \right] \psi(y)$$

with

$$Q_{\text{SW}}(y) = \boxed{-\frac{1}{\hbar^2} \frac{B^2 - 4AC}{4y^2 A^2} + \frac{2y(BA' - AB') + \hbar A^2}{4\hbar y^2 A^2}}$$

classical SW differential

$$\lambda(y) = \sqrt{\frac{B^2 - 4AC}{4y^2 A^2}} dy \underset{\hbar \rightarrow 0}{\simeq} i\hbar \sqrt{Q_{\text{SW}}(y)} dy$$

The gauge/gravity dictionary: AdS Kerr-Newman

Dictionary between gauge theory and gravity

$$Q_r(r) = Q_{\text{SW}} [y(r)] y'(r)^2 + \frac{y'''(r)}{2y'(r)} - \frac{3}{4} \left[\frac{y''(r)}{y'(r)} \right]^2$$

Eg ‘massive’ scalar perturbations of AdS Kerr-Newman BHs in $D = 4$

$$(\square - M_\phi^2) \Phi = 0, \quad M_\phi^2 = -\frac{2}{L^2} \quad [\Delta_{\mathcal{O}} = 1, 2, \text{eg } \phi^2, \psi^2]$$

Radial equation with four regular singularities (Heun Equation)

$$Q_r(r) = \frac{1}{\Delta_r^2} \left[\alpha_L^2 (a_{\mathcal{J}} m_\phi - \omega(a_{\mathcal{J}}^2 + r^2))^2 - \Delta_r (K^2 + r^2 M_\phi^2) - \frac{1}{2} \Delta_r \Delta_r'' + \frac{1}{4} \Delta_r'^2 \right]$$

with AdS scale L , mass M , charge Q , spin $Ma_{\mathcal{J}}$, separation constant K^2 and

$$\alpha_L = 1 - \frac{a_{\mathcal{J}}^2}{L^2}, \quad \Delta_r = (r^2 + a_{\mathcal{J}}^2) \left(1 + \frac{r^2}{L^2} \right) - 2Mr + Q^2 = L^{-2} \prod_{i=1}^4 (r - r_i)$$

The gauge/gravity dictionary

Same singularity structure as $N_f = (2, 2)$

$$Q_{\text{SW}}(y) = \sum_{i=1}^3 \left[\frac{\hat{\delta}_i + \frac{1}{4}}{(y - y_i)^2} + \frac{\nu_i}{y - y_i} \right], \quad y_i = \{0, -1, -1/q, (\infty)\}$$

with

$$\sum_{i=1}^3 \nu_i = 0, \quad \nu_2 = \frac{\nu_1 + q \left(\frac{1}{2} + \hat{\delta}_1 + \hat{\delta}_2 + \hat{\delta}_3 - \hat{\delta}_4 \right)}{q - 1}, \quad \nu_1 = -\frac{u}{\hbar^2} + f(q, m_i)$$

and

$$\hat{\delta}_1 = -\frac{(m_1 - m_2)^2}{4\hbar^2}, \quad \hat{\delta}_2 = -\frac{(m_1 + m_2)^2}{4\hbar^2}, \quad \hat{\delta}_3 = -\frac{(m_3 + m_4)^2}{4\hbar^2}, \quad \hat{\delta}_4 = -\frac{(m_3 - m_4)^2}{4\hbar^2}$$

The two equations map into one another using

$$y = \frac{r_{24}}{r_{12}} \frac{r - r_1}{r - r_4} \quad q = \frac{r_{12}r_{34}}{r_{24}r_{13}} \quad \hat{\delta}_i + \frac{1}{4} = \left. \frac{\Delta_r^2 Q_r}{\Delta'_r} \right|_{r_i} \quad \nu_1 = \frac{r_{12}r_{14}}{r_{24}} \text{Res}_{\{n\}} Q_r(r)$$

Cycles quantization and computation of QNMs

For $\hbar \rightarrow 0$, WKB expansion

$$\psi(y) \propto \exp \left[\int^y \sqrt{Q_{\text{SW}}(y')} dy' \right] \simeq \exp \left[-\frac{i}{\hbar} \int^y \lambda(y') dy' \right]$$

Bohr-Sommerfeld quantization condition

$$\oint_{\gamma} \lambda(y) = \hbar(n + \nu_{\gamma})$$

with $\gamma(y)$ enclosing y_{\pm} and y_0 such that

$$Q_{\text{SW}}(y_{\pm}) = 0, \quad Q'_{\text{SW}}(y_0) = 0, \quad y_{\pm} = y_0 + \hbar f_{\pm}(u, q, m_i) + \dots$$

For finite \hbar , QNMs spectrum from quantisation of cycle a_{γ}

$$a_{\gamma} = c_1 a + c_2 a_D + \sum_{i=1}^4 d_i m_i = \oint_{\gamma} \lambda_{\hbar} = \hbar(n + \nu_{\gamma}), \quad c_i, d_i \in \mathbb{Z}$$

with ‘quantum’ SW differential $\lambda_{\hbar}(x)$ to be defined later on

Computation of a , a_D and $\mathcal{F}_{tree} + \mathcal{F}_{inst}$ (I)

Quantizing w.r.t x

$$\hat{y} = \exp \left[\hbar \frac{d}{dx} \right]$$

get difference equation

$$qM(x)W(x)W(x - \hbar) + P_0(x)W(x) + 1 = 0$$

for

$$W(x) = \frac{1}{P_R(x + \frac{\hbar}{2})} \frac{\tilde{U}(x)}{\tilde{U}(x + \hbar)}, \quad M(x) = P_L(x - \frac{\hbar}{2})P_R(x - \frac{\hbar}{2})$$

In continuous fraction form, small q expansion

$$W(x) = -\frac{1}{P_0(x)} \left(1 + \frac{qM(x)}{P_0(x)P_0(x - \hbar)} + O(q) \right)$$

Computation of a , a_D and $\mathcal{F}_{tree} + \mathcal{F}_{inst}$ (II)

Use $W(x)$ to define quantum SW differential

$$\lambda_{\hbar}(x) = -\frac{x}{2\pi i} d \log W(x)$$

with correct behavior at infinity

$$\lambda_{\hbar}(x) = \sum_{n=0}^{\infty} \frac{\langle \text{tr} \varphi^n \rangle}{x^n} = 2 + \frac{2u}{x^2} + \dots, \quad P_0(x) = x^2 + \dots$$

To compute tree-level and instanton contribution to \mathcal{F}_{NS} , first compute $a(u)$, then use Matone relation

$$a(u) = \oint_{\alpha} \lambda_{\hbar} = 2\pi i \sum_{n=0}^{\infty} \text{Res}_{\sqrt{u}+n\hbar} \lambda_{\hbar}(x), \quad u(a) = -q \frac{\partial \mathcal{F}_{NS}}{\partial q}$$

and, at leading order in q , find

$$\mathcal{F}_{tree} = -a^2 \log q, \quad \mathcal{F}_{inst} = q \left[\sum_{i=1}^4 \frac{\hbar m_i}{2} - \sum_{i < j} \frac{m_i m_j}{2} - \frac{2m_1 m_2 m_3 m_4}{4a^2 + \hbar^2} - \frac{4a^2 + 3\hbar^2}{8} \right]$$

$\mathcal{F}_{1\text{-loop}}$ and theories with $N_f < 4$

One-loop contribution from the Nekrasov-Shatashvili partition function

$$\frac{\partial \mathcal{F}_{\text{one-loop}}}{\partial a} = \hbar \log \frac{\Gamma\left(1 + \frac{a}{\hbar}\right)^2}{\Gamma\left(1 - \frac{a}{\hbar}\right)^2} \prod_{i=1}^4 \frac{\Gamma\left(\frac{1}{2} + \frac{m_i - a}{\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{m_i + a}{\hbar}\right)}$$

For $N_f < 4$ use successive decoupling limit(s)

$$q \rightarrow 0, \quad m_4 \rightarrow \infty, \quad q m_4 = \tilde{q}$$

get correction terms to tree-level and 1-loop prepotential

$$\mathcal{F}_{\text{tree}} = -a^2 \log\left(-\frac{\tilde{q}}{\hbar}\right), \quad \frac{\partial \mathcal{F}_{\text{1-loop}}}{\partial a} = \hbar \log \frac{\Gamma\left(1 + \frac{a}{\hbar}\right)^2}{\Gamma\left(1 - \frac{a}{\hbar}\right)^2} \prod_{i=1}^{[[3]]} \frac{\Gamma\left(\frac{1}{2} + \frac{m_i - a}{\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{m_i + a}{\hbar}\right)}$$

Let's have a go with 'exact' solutions for $q = 0$

Exact solutions: 3-dimensional Spherical Harmonics (I)

Setting $\chi = \cos \theta$

$$\left[\frac{\partial}{\partial \chi} \left((1 - \chi^2) \frac{\partial}{\partial \chi} \right) + A - \frac{m_\phi^2}{1 - \chi^2} \right] U(\chi) = 0$$

put in canonical form with

$$Q_\chi(\chi) = \frac{1 - m_\phi^2 + (1 - \chi^2)A}{(1 - \chi^2)^2}, \quad U(\chi) = \frac{1}{\sqrt{1 - \chi^2}} \phi(\chi)$$

and map to SW curve for $SU(2)$ with $N_f = (1, 2)$ and $y = \frac{1}{2}(\chi - 1)$

$$q = 0, \quad \frac{u}{\hbar^2} = A + \frac{1}{4}, \quad m_1 = m_3 = 0, \quad \frac{m_2}{\hbar} = |m_\phi|$$

For $\hbar = 0$, WKB approximation

$$Q_\chi(\chi_c, A_c) = Q'_\chi(\chi_c, A_c) = 0, \quad \chi_c = 0, \quad A_c = m_\phi^2 - 1$$

Translate condition on A_c back into gauge language

$$\sqrt{u} - m_2 = a - m_2 = 0$$

Exact solutions: 3-dimensional Spherical Harmonics (II)

Identify correct (classically degenerating) cycle γ

$$a_\gamma = a - m_2 = \hbar \left(n_\theta + \frac{1}{2} \right)$$

exact SW quantization yields

$$A = (n_\theta + |m_\phi|)(n_\theta + |m_\phi| + 1) (= K^2), \quad n_\theta + |m_\phi| = \ell$$

NB: Very useful when dealing with rotating BHs

$$Q_\chi(\chi) = \frac{(1 - \chi^2) ([[\mathbf{a}_\mathcal{J}^2 \omega^2 \chi^2]] + A) - m_\phi^2 + 1}{(1 - \chi^2)^2}$$

with $A = \ell(\ell + 1) + \mathcal{O}(a_\mathcal{J}\omega)$ and

$$\frac{q}{\hbar} = a_\mathcal{J}\omega, \quad \frac{u}{\hbar^2} = A + \frac{1}{4} + a_\mathcal{J}\omega(a_\mathcal{J}\omega + 2(1 - m_\phi))$$

Exact solutions: (Near) super-radiant modes (I)

Scalar perturbations of asymptotically flat Kerr-Newman BH ($L \rightarrow \infty$)

$$Q_r(r) = \frac{1}{\Delta_r^2} \left[(a_{\mathcal{J}} m_\phi - \omega(a_{\mathcal{J}}^2 + r^2))^2 - \Delta_r K^2 - \frac{1}{2} \Delta_r \Delta_r'' + \frac{1}{4} \Delta_r'^2 \right]$$

with

$$\Delta_r = r^2 + a_{\mathcal{J}}^2 - 2Mr + Q^2 = (r - r_+)(r - r_-)$$

once again map to $N_f = (1, 2)$ with

$$\begin{aligned} \frac{q}{\hbar} &= 2i\omega(r_+ - r_-), & \frac{m_2}{\hbar} &= -\frac{i[(r_+^2 + r_-^2 + 2a_{\mathcal{J}}^2)\omega - 2a_{\mathcal{J}} m_\phi]}{r_+ - r_-} \\ \frac{u}{\hbar^2} &= A + f(\omega, a_{\mathcal{J}}, m_\phi, r_\pm), & \frac{m_1}{\hbar} &= \frac{m_3}{\hbar} = -i(r_+ + r_-) \end{aligned}$$

At extremality $r_+ = r_-$... decoupling limit

$$q \rightarrow 0, \quad m_2 \rightarrow \infty, \quad q m_2 = \tilde{q}$$

Keeping m_2 finite, get 'super-radiant' frequencies

$$\boxed{\omega_{SR} = \frac{m_\phi a_{\mathcal{J}}}{r_+^2 + a_{\mathcal{J}}^2} = m_\phi \Omega_\phi}$$

Exact solutions: Near Super-Radiant modes (II)

Perturbations of near-extremal KN BHs characterized by NSR frequencies

$$\omega = \omega_{\text{SR}} + \nu \delta_r, \quad \delta_r = r_+ - r_- \ll r_H \approx M$$

Solve differential equation exactly for $r \gg r_+ \gg \delta_r$ (Confluent hypergeometric) and for $r = r_+ + \delta_r \tau$ (Hypergeometric), match in intermediate region

$$[-2i\omega_{\text{SR}}\delta_r]^{-2\alpha} \frac{\Gamma(2\alpha)^2 \Gamma(\bar{A}) \Gamma(\bar{C} - \bar{B})^2}{\Gamma(-2\alpha)^2 \Gamma(\bar{B}) \Gamma(\bar{C} - \bar{A})^2} = 1, \quad \alpha = \sqrt{A + \frac{1}{4} - (a_{\mathcal{J}}^2 + 6r_+^2)\omega_{\text{SR}}^2}$$

Since $\delta_r^{-\alpha}$ diverges, require $\bar{B} = n + \delta_r \eta$ and get quantization condition

$$\omega_{\text{NSR}} = \omega_{\text{SR}} (1 + 4\pi r_+ T_{\text{BH}}) - 2\pi i T_{\text{BH}} \left(\alpha + n + \frac{1}{2} \right), \quad T_{\text{BH}} = \frac{\delta_r}{4\pi(r_+^2 + a_{\mathcal{J}}^2)}$$

Re-derive more easily with quantum SW, keep leading terms in δ_r

Since $q = 0$ only tree-level and 1-loop terms of \mathcal{F} contribute ... $a_D = \hbar n$

$$\exp \left[\frac{1}{\hbar} \frac{\partial \mathcal{F}}{\partial a} \right] = \left(-\frac{q}{\hbar} \right)^{\frac{2\sqrt{u}}{\hbar}} \frac{\Gamma \left(1 + \frac{\sqrt{u}}{\hbar} \right)^2}{\Gamma \left(1 - \frac{\sqrt{u}}{\hbar} \right)^2} \prod_{i=1}^3 \frac{\Gamma \left(\frac{1}{2} + \frac{m_i - \sqrt{u}}{\hbar} \right)}{\Gamma \left(\frac{1}{2} + \frac{m_i + \sqrt{u}}{\hbar} \right)} = 1 = \exp \left[-\frac{2\pi i a_D}{\hbar} \right]$$

Numerical solutions: Kerr-Newman black hole

For ‘neutral, massless’ scalar perturbations of Kerr-Newman BHs

- Both radial and angular equations described by $N_f = (1, 2)$ and $G = SU(2)$
- Identify $a_\chi = a - m_2 = \hbar(n_\chi + \frac{1}{2})$ as correct cycle for angular equation
- Identify $a_r = a_D = \hbar n_r$ as correct cycle for radial equation

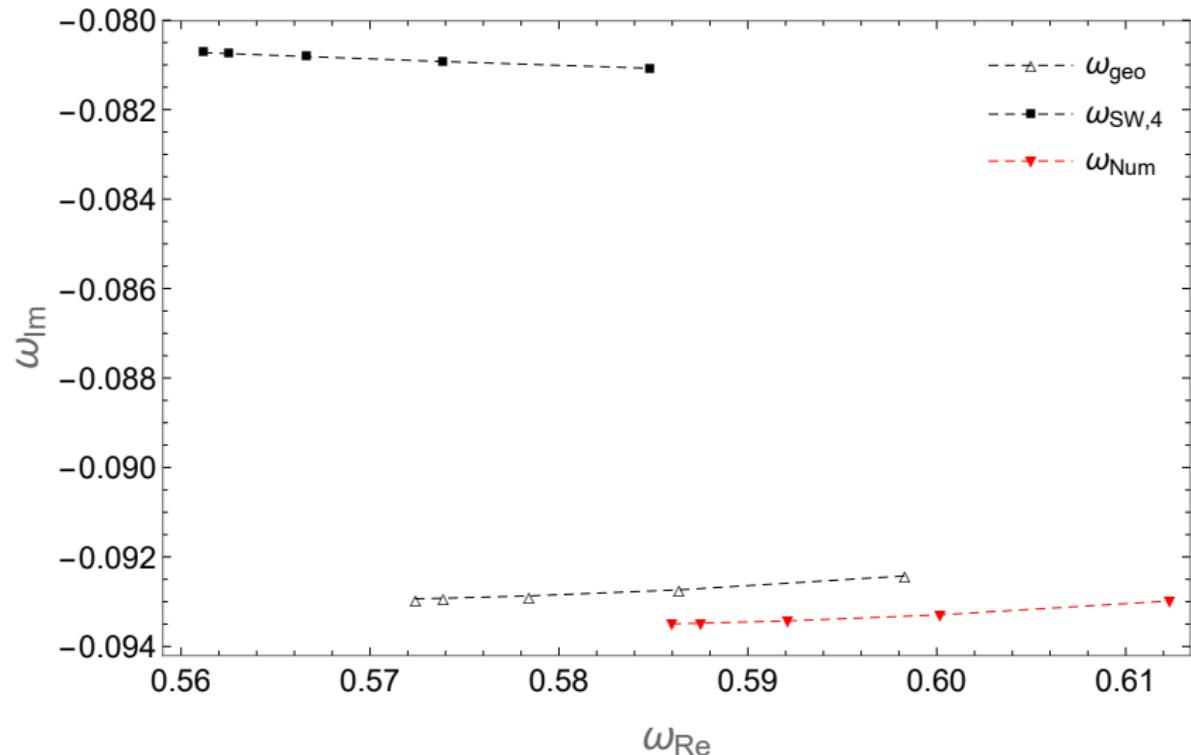
Numerical analysis using $\mathcal{F}_{\text{inst}}$ up to q^4 , for ‘equatorial’ perturbations

$$\chi = 0, \quad \ell = m_\phi = 2$$

(Setting $M = 1$ in the plots) find

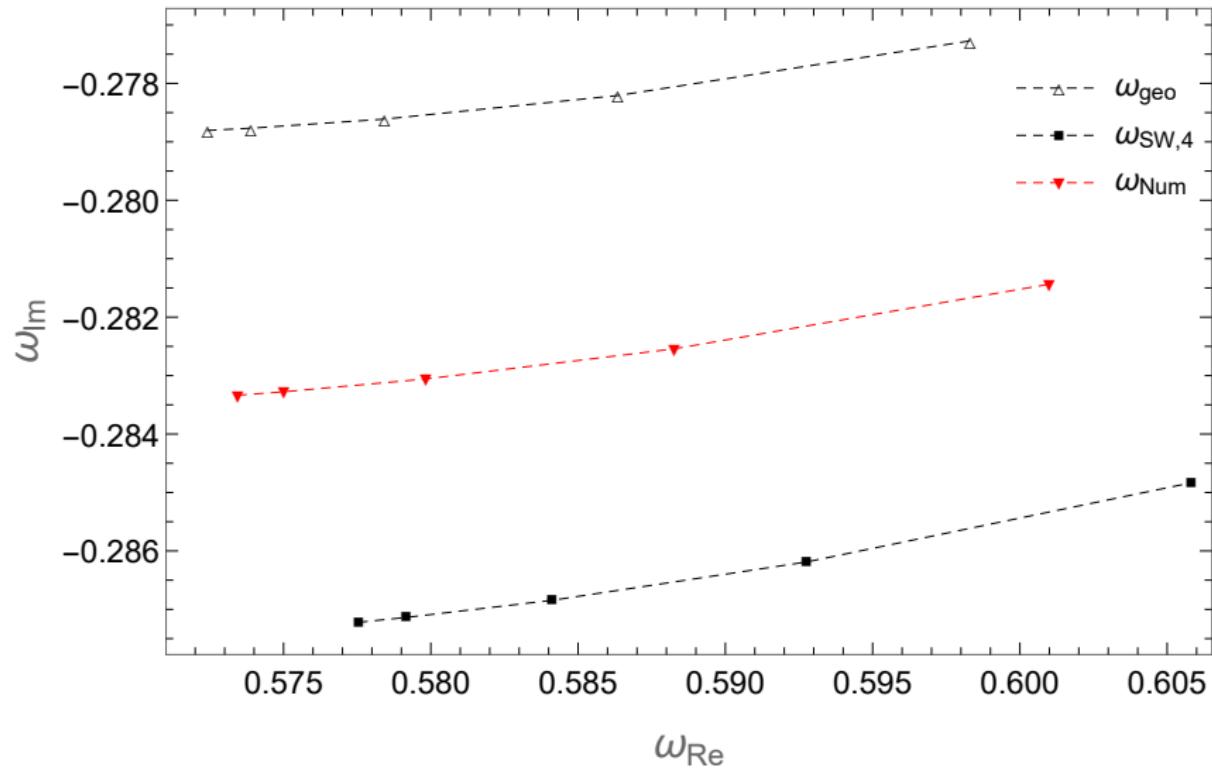
- Good agreement for first overtone number $n_r = 0$
- Very good agreement for second overtone number $n_r = 1$

Plot: Kerr-Newman black hole - $a_J = 0.5$, $n_r = 0$



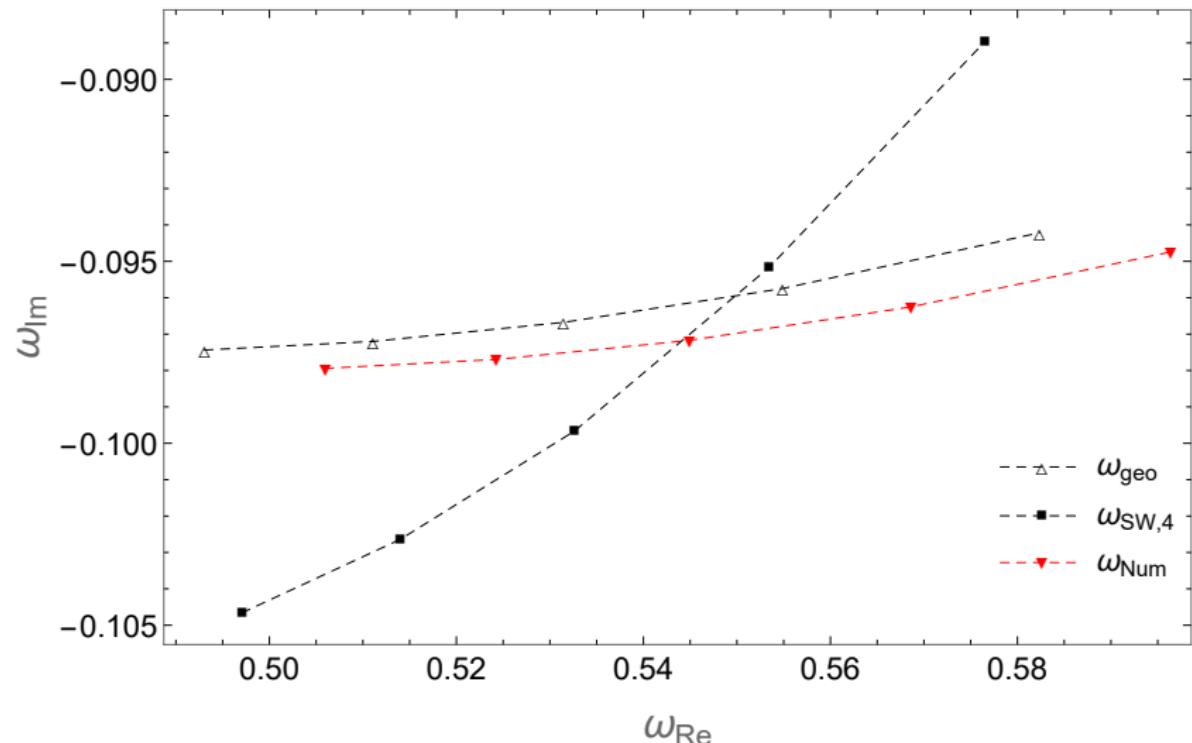
$a_J = 0.5$, Q runs from 0.1 to 0.4 with $\Delta Q = 0.1$

Plot: Kerr-Newman black hole - $a_J = 0.5$, $n_r = 1$



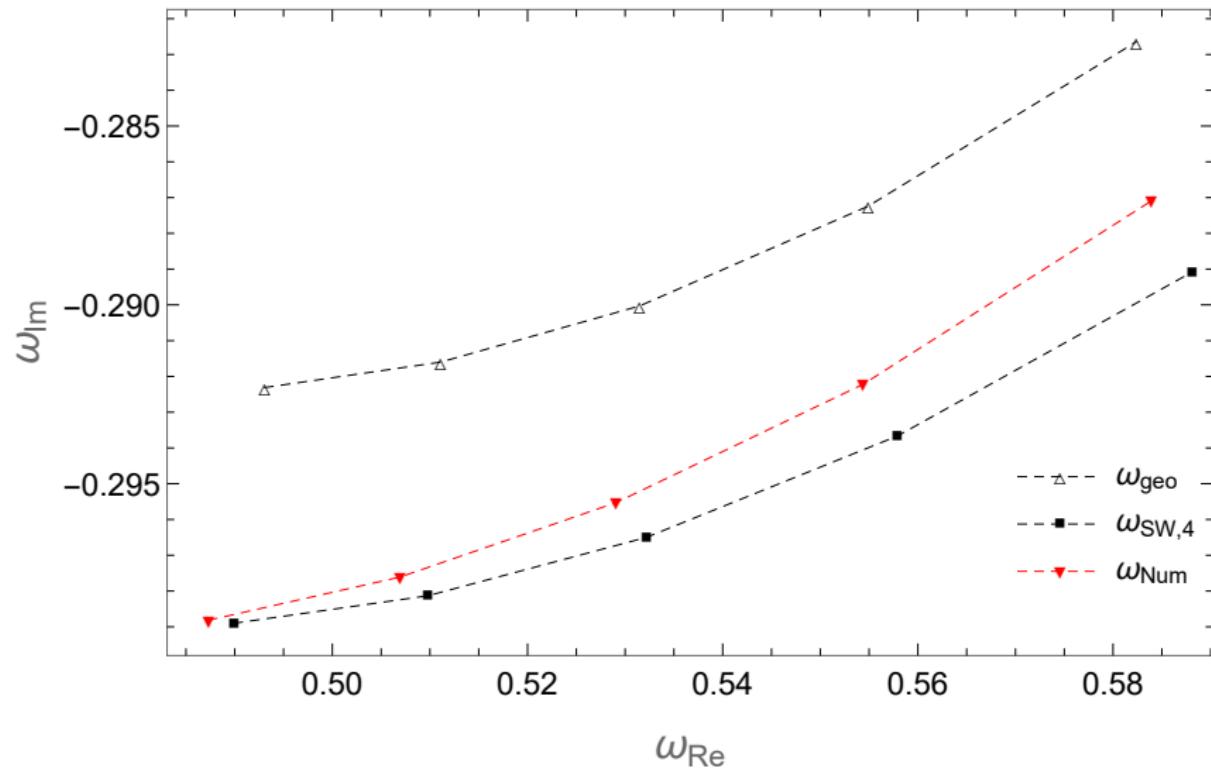
$a_J = 0.5$, Q runs from 0.1 to 0.4 with $\Delta Q = 0.1$

Plot: Kerr-Newman black hole - $Q = 0.5$, $n_r = 0$



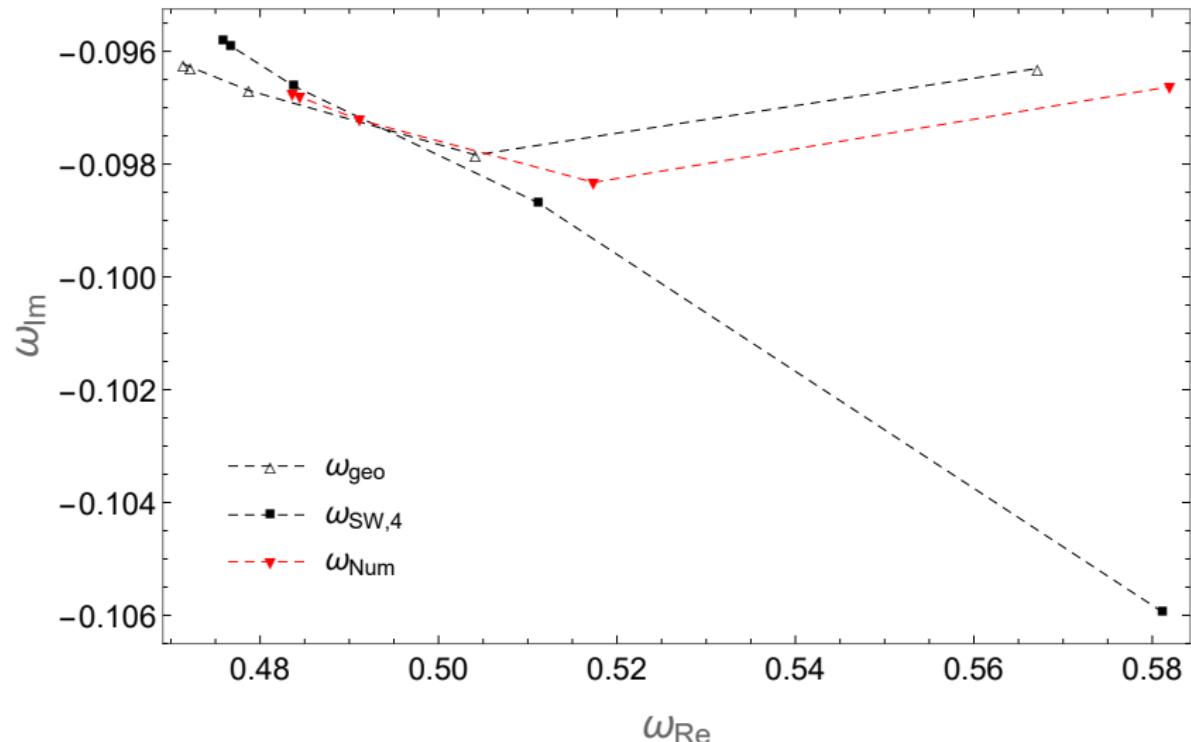
$Q = 0.5$, $a_{\mathcal{J}}$ runs from 0.1 to 0.4 with $\Delta a_{\mathcal{J}} = 0.1$

Plot: Kerr-Newman black hole - $Q = 0.5$, $n_r = 1$



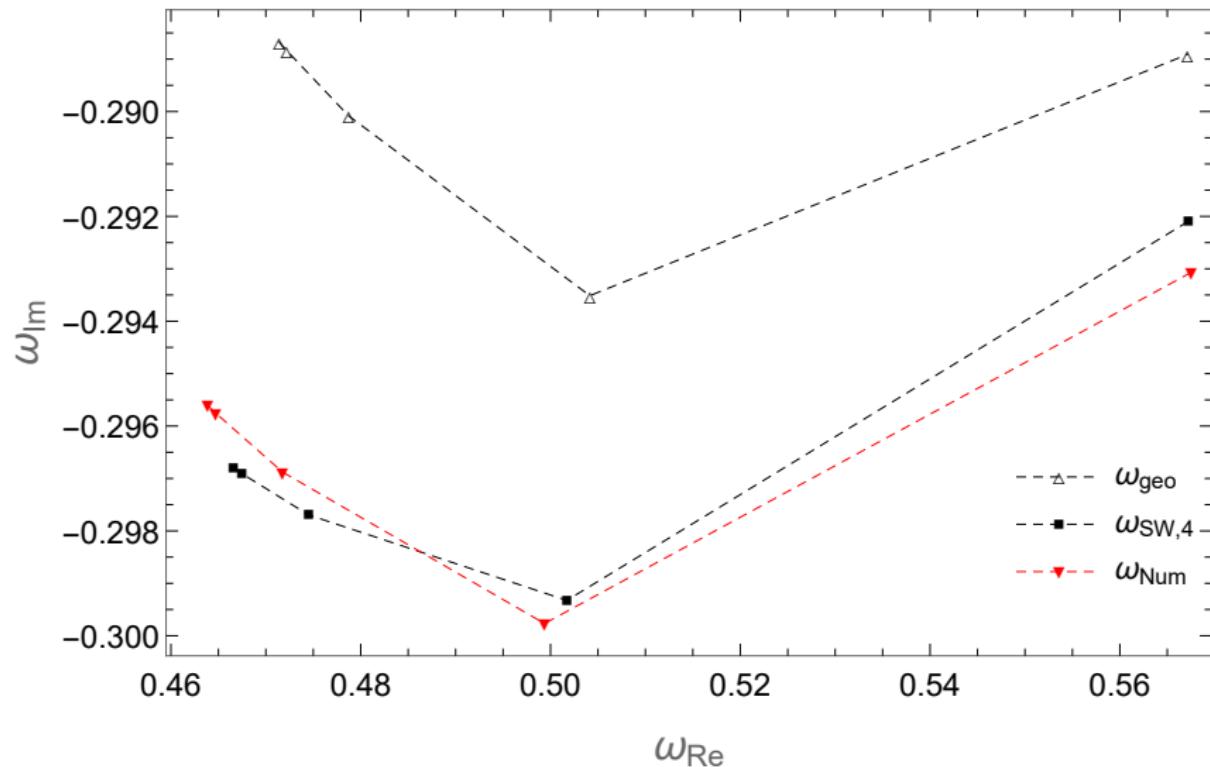
$Q = 0.5$, $a_{\mathcal{J}}$ runs from 0.1 to 0.4 with $\Delta a_{\mathcal{J}} = 0.1$

Plot: Reissner-Nördtrom black hole - $n_r = 0$



Q runs from 0 to 0.9 with $\Delta Q = 0.3$

Plot: Reissner-Nördtrom black hole - $n_r = 1$



Q runs from 0 to 0.9 with $\Delta Q = 0.3$

Other dictionaries: QNMs of branes and fuzz balls vs quantum SW

Extend gauge/gravity dictionary for scalar perturbations, using $G = SU(2)$ only

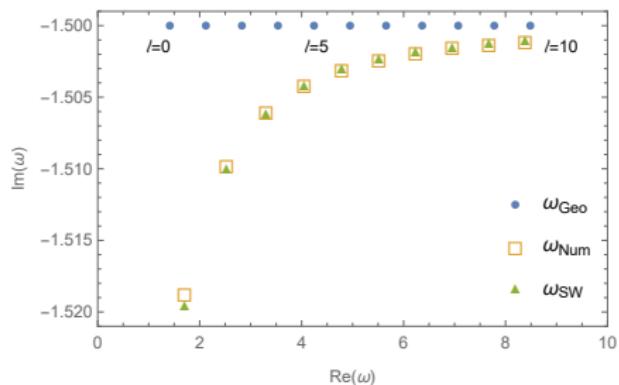
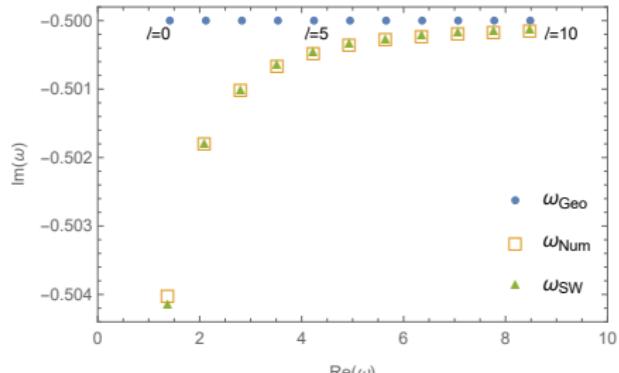
(!) Radial equation

- D3-branes $N_f = (0, 0)$ (DCHE, Mathieu equation, ... Couch-Torrence)
- Intersecting D3-branes (4-charge BH in 4-dimensions) $N_f = (1, 1)$
- CCLP (general 5-dimensional charged and rotating BH) $N_f = (0, 2)$
- BMPV (supersymmetric, extremal 5-dimensional BH) $N_f = (0, 1)$
- D1-D5 fuzzball (smooth, horizonless 6-dimensional geometry) $N_f = (0, 2)$
- D1-D5 (D3-D3') BH $N_f = (0, 0)$
- JMaRT (smooth, horizonless 6-dimensional geometry) $N_f = (0, 2)$

Angular equation 'deformed' version of spherical harmonics equation

- All 4-dimensional geometries (S^2): $N_f = (1, 2)$
- All 5-dimensional and (5+circle)-dimensional geometries (S^3): $N_f = (0, 2)$

QNMs of D3-branes: full agreement!



The AGT picture

AGT duality between $\mathcal{N} = 2$ quiver theories and 2-dimensional Liouville theory

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}$$

Chiral vertex operators

$$V_{\alpha_i} = e^{2\alpha_i \phi}, \quad h_i = \alpha_i(Q - \alpha_i)$$

Write $n+3$ -points function in terms of Conformal Blocks

$$\left\langle \prod_{i=0}^{n+2} V_{\alpha_i}(z_i) \right\rangle = \sum_{p_1 \cdots p_n} \langle p_0 | V_{\alpha_0}(z_0) | p_1 \rangle \cdots \langle p_n | V_{\alpha_{n+2}}(z_{n+2}) | p_{n+1} \rangle = \sum_{p_1 \cdots p_n} |\mathcal{C}_{p_0 \cdots p_{n+1}}^{\alpha_1 \cdots \alpha_{n+1}}|^2$$

AGT: conformal blocks \sim (ratio of) quiver partition functions

$$\mathcal{C}_{p_0 \cdots p_{n+1}}^{\alpha_1 \cdots \alpha_{n+1}}(\{z_i\}) \prod_{j=1}^n z_j^{-\Delta_{p_j} + \Delta_j + \Delta_{p_j+1}} = \frac{Z_{\text{inst}}(\{\vec{a}_i\}, \{q_i\})}{Z_{U(1)}(\{q_i\})}$$

The AGT picture: quantum SW curves

Consider $SU(2) \times SU(2)$ quiver, wave function as conformal block involving degenerate field with $\alpha_3 = -\frac{b}{2}$

$$\psi(y) = \mathcal{C}_{p_0 \dots p_3}^{\alpha_1 \dots \alpha_3} (\{z_i\}) , \quad p_1 = p_2 \mp \frac{b}{2}$$

with

$$z_0 = \infty , \quad z_1 = 1 , \quad z_2 = q , \quad z_3 = y , \quad z_4 = 0$$

Since $(L_{-1}^2 + b^2 L_2) V_{\alpha_3} \sim 0$ (null), $\psi(y)$ satisfies BPZ equation

$$\psi''(y; \{z_i\}) + b^2 \sum_{i \neq 1}^{n+1} \left[\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \partial_{z_i} \right] \psi(y; \{z_i\}) = 0$$

map to quantum SW curve / differential equation

$$b^2 \Delta_i = \delta_i , \quad \nu_i = b^2 c_i , \quad \partial_{z_i} \psi(y; \{z_i\}) = c_i \psi(y; \{z_i\})$$

Non-rotating BHs and branes and photon-spheres

Extremal Reissner-Nordström BHs ($Q = M$), with $u = r - Q$

$$ds^2 = -\frac{u^2 dt^2}{(u+Q)^2} + \frac{(u+Q)^2}{u^2} [du^2 + u^2 d\Omega_2^2]$$

enjoy symmetry under conformal inversions $ds^2 \rightarrow W(u)ds^2$ [Couch, Torrence]

$$u \rightarrow Q^2/u$$

exchanging horizon $u = 0$ ($r = Q$) with infinity, keeping photon-sphere $u = Q$ ($r = 2Q$) fixed. Quite remarkably for massless geodesics

$$\Delta\phi_{fall}(J, E) = \int_0^{u_i} \frac{J du}{u^2 P_u(u; J, E)} = \int_{Q^2/u_i}^{\infty} \frac{J du}{u^2 P_u(u; J, E)} = \Delta\phi_{scatt}(J, E)$$

Same identity valid for other extremal geometries in $D \geq 4$

- D3-branes $u_c = L$... AdS/CFT
- D3-D3' ‘small’ BHs $u_c^2 = L_3 L_{3'}$
- 4-charge STU BHs (intersecting D3-branes) with $Q_1 Q_2 = Q_3 Q_4$ or permutations, $u_c^4 = Q_1 Q_2 Q_3 Q_4$

Fixed locus $u = u_c$: photon-sphere ... ‘light-ring’, massive BPS probes OK [MB, Di

Rotating BHs and branes and photon-halos

Extremal Kerr-Newmann ($M^2 = a^2 + Q^2$) BHs NO conformal inversion symmetry of the metric BUT radial wave equation invariant [Couch, Torrence]

Fixed locus depends on impact parameters ($b = J/E$, $b_z = J_z/E$)

Critical 'radius' not fixed: $r_c \in [r_{min}, r_{Max}]$ depending also on θ_{obs} ... photon-halo!!

Equality of radial actions, same E, J, J_z (no analytic continuation)

$$S_R^{in}(u_i, u_f; E, J, J_z) = \int_{u_i}^{u_f} P_u du = \int_{u_c^2(b, b_z)/u_i}^{u_c^2(b, b_z)/u_f} P_u du = S_R^{out}(u'_i, u'_f; E, J, J_z)$$

Yet $\Delta\phi_{fall} \neq \Delta\phi_{scatt}$ due to (divergent) bdry terms in $\partial S/\partial J$ at $u = 0$... classical renormalisation

Extremal (nonBPS) rotating STU BHs enjoy generalised CT inversion symmetry

[Cvetic, Pope, Saha] when $Q_1 Q_2 = Q_3 Q_4$ [MB, Di Russo] ... stay tuned

Yet, NO GCT inversion symmetry for (non)rotating 5-d and 6-d BHs (different behaviour at horizon and infinity ... Freudenthal duality?) and fuzz-balls (no horizon, after all)

Conclusions and outlook

SW/QNMs connection

- New approach to gravity perturbations of NON-extremal BHs and branes (HE, CHE, DCHE/CT)
- Pretty good numerical results with (relatively) small numbers of instantons
- Deeper connection between gauge theories and gravity ... Electric-magnetic duality between M2-branes and M5-branes, Kerr/CFT correspondence, Couch-Torrence symmetry and AdS/CFT
- Not-only QNMs but also tidal Love numbers and grey-body factors, AGT correspondence

Future directions

- Quivers and higher-rank gauge groups, using AGT
- Non-separable systems (e.g. multi-center geometries)
- More robust physical interpretation

Thanks!