Hyperbolic mass and Maskit gluings

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University of Vienna

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based on joint work with Erwann Delay and Rafaela Wutte

Energy for locally asymptotically hyperbolic manifolds

space-dimension n

Theorem (with E. Delay, arXiv:1901.05263)

The energy-momentum vector of conformally compact n-dimensional asymptotically locally hyperbolic manifolds (M, g) with spherical infinity *and with scalar curvature $R(g)$ satisfying $R(g) \geq -n(n-1)$, $n \geq 3$, is timelike future-pointing or vanishes.*

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There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) with

$$R(g) = -6 \,,$$

with arbitrarily high genus and with negative total mass

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1. Known since 2001 for spin manifolds by Witten-type methods (Wang, PTC-Herzlich).
2. Different story if topology at infinity is not spherical.
3. *Huang, Jang, Martin (2019): lightlike cannot occur*
4. Generalises to many ends and boundaries with $H < n-1$ (PTC, Galloway, 2107.05603 [gr-qc])
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5. if $n \geq 7$, needs the higher-dimensional asymptotically flat positive energy theorem (Lohkamp, Schoen & Yau)

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5. key idea: the "Maskit gluing" by Isenberg, Lee & Stavrov (2010)

Other topologies at infinity?

In dimension $3 + 1$:

- positivity of energy in the spherical case is taken care of by the last theorem
- toroidal case: the Horowitz-Myers metrics provide examples with negative mass without black hole boundaries
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in dim 3 $\equiv$ "asymptotically Birmingham-Kottler"

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time-symmetric vacuum general relativistic initial data with suitably normalised negative cosmological constant

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not clear how to generalise this to higher dims

Energy for asymptotically hyperbolic manifolds

Why should we care?

- *Asymptotically hyperbolic manifolds* are ubiquitous in nowadays theoretical physics (supergravities, string theory, holography, CFT/AdS).
- They appear naturally as spacelike hypersurfaces in solutions of Einstein equations, with or without a cosmological constant Λ:
  hyperbolic space itself occurs as a *"static slice"* of the Anti-de Sitter spacetime ($\Lambda < 0$), or as a *hyperboloid* in Minkowski spacetime $\Lambda = 0$.
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Model metrics: Kottler-Birmingham metrics

Static vacuum solutions of Einstein equations with a negative cosmological constant

$$\mathbf{g}_m = -V_m^2 dt^2 + V_m^{-2} dr^2 + r^2 h_\kappa \,, \qquad V_m^2 = r^2 + \kappa - \frac{2m}{r^{n-2}} \,.$$

where h_κ is a t- and r-independent Einstein metric on a $(n-1)$-dim compact manifold, with scalar curvature $R(h) = (n-1)(n-2)\kappa$.

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- *asymptotically BK* is the same as locally asymptotically hyperbolic in space-dimension 3
- and is a special case of locally asymptotically hyperbolic in higher dimensions

Model metrics: Horowitz-Myers Instantons

$$\mathbf{g}_m = -V_m^2 dt^2 + V_m^{-2} dr^2 + r^2(d\theta^2 + h_0') \,, \quad V_m^2 = r^2 \not{+} \not{k} - \frac{2m}{r^{n-2}} \,.$$

where h_0' is a t-, θ-, and r-independent Ricci flat metric on a $(n-3)$-dim compact manifold.

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Woolgar's version of the Horowitz-Myers conjecture

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- the metric at conformal infinity is

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- The mass relative to $\mathbf{g}_0$ can be arbitrarily negative, proportional to the negative of m.
- Horowitz-Myers conjecture: these are minima of energy at prescribed conformal structure at infinity.

Idea: use "gluing at infinity"

"Maskit gluing"

Theorem (Isenberg, Lee & Stavrov 2010, PTC, Delay, arXiv:1511.07858)

Given two asymptotically hyperbolic manifolds with constant scalar curvature (or general relativistic vacuum initial data sets) one can construct a new one by making a connected sum at the conformal boundary at infinity. The construction can be localised by a Carlotto-Schoen type hyperbolic gluing.

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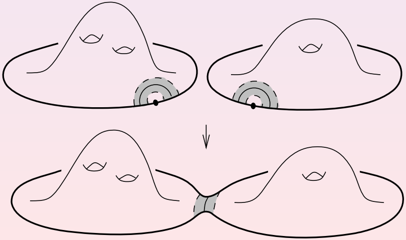

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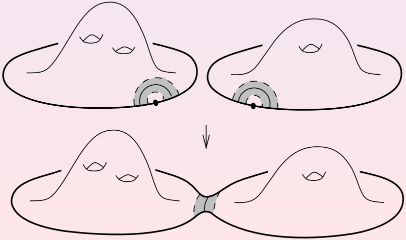

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Question: What is the energy-momentum of the new initial data set?

1. Spacetime variational methods: “Noether charge” *à la Wald (∼ 1990)* $\equiv$ geometric Hamiltonian methods *à la Kijowski-Tulczyjew (1979)*

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2. A convenient geometric formula for total energy E:
   if g approaches a *Kottler-Birmingham* metric with $m = 0$

$$E \quad = \quad -\frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{r=R} D^j V \, (\mathrm{R}^i{}_j - \frac{\mathrm{R}}{n}\delta^i_j) dS_i \, .$$

where $\mathrm{R}^i{}_j$ is the Ricci tensor of g and

$$V = \sqrt{r^2+\kappa}\,, \qquad \kappa \in \{0, \pm 1\}\,. \qquad (**)$$

Energy-momentum, spherical conformal infinity

Total energy-momentum $p_{(\mu)}$:

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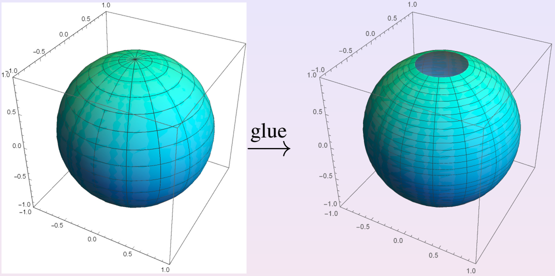
- $p_{(\mu)}$ transforms as a Lorentz vector under isometries of the hyperbolic metric.

Positive energy for asymptotically hyperbolic manifolds

Energy-momentum vector and localised Maskit gluing



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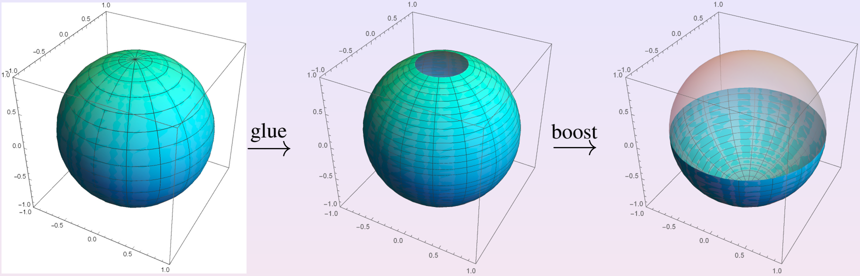

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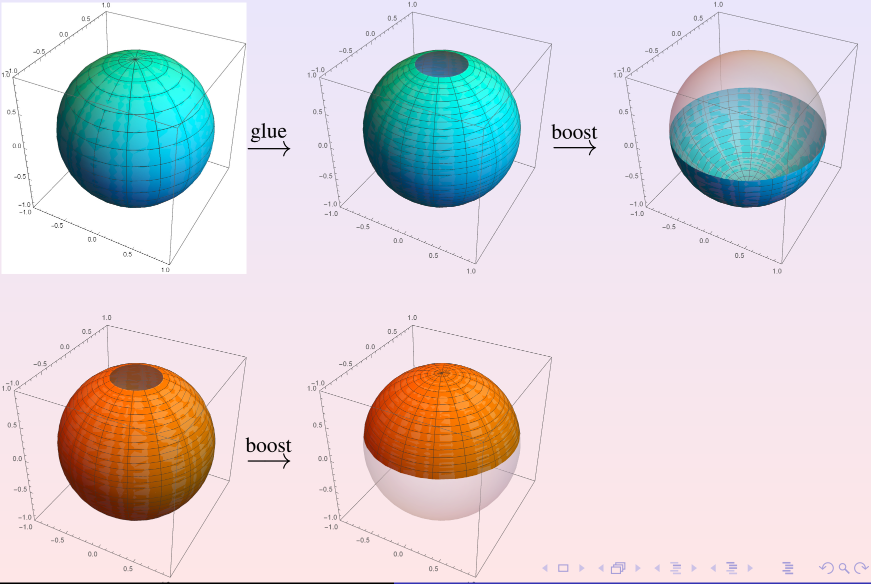

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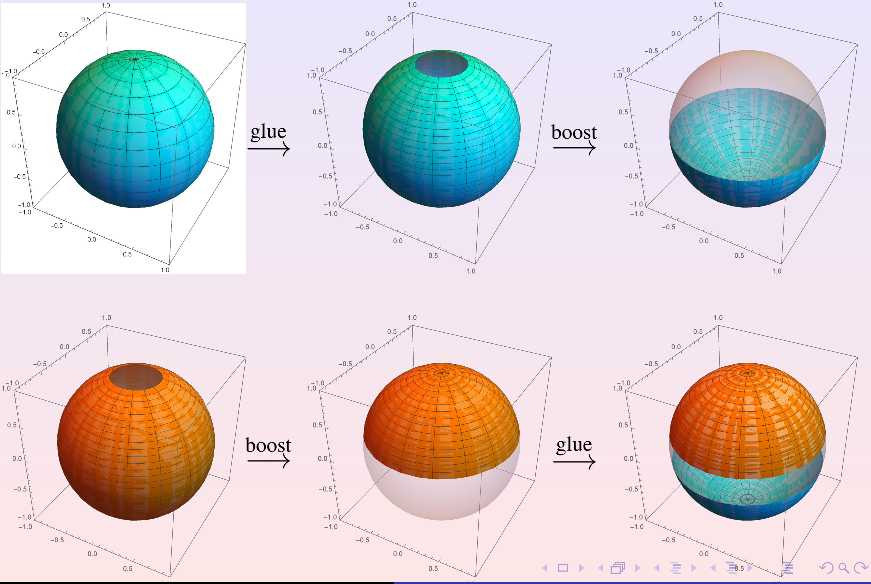

Now energy-mometum is obviously additive

$$p_{(\mu)} = -\frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{r=R} D^j V_{(\mu)} \left(\mathrm{R}^i{}_j - \frac{\mathrm{R}}{n}\delta^i_j\right) dS_i \,.$$

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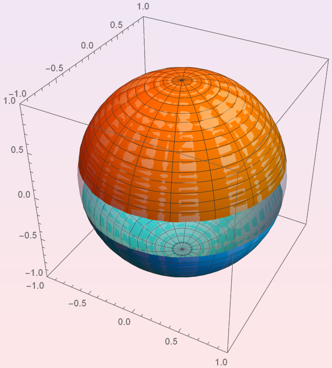
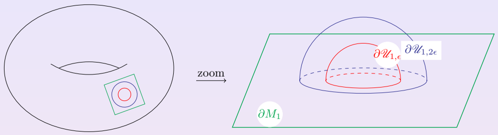


Carlotto-Schoen type gluing, toroidal infinity



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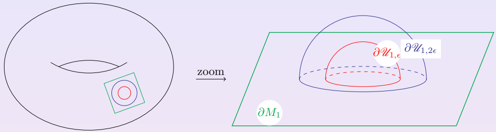

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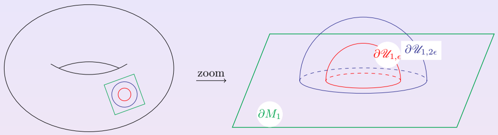

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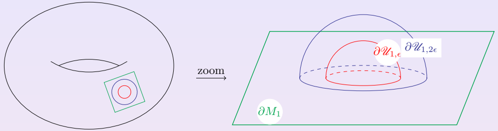

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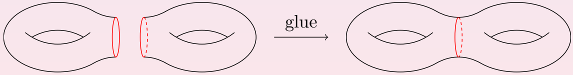
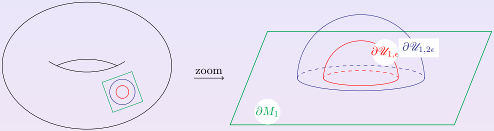
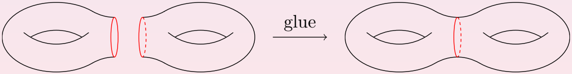
- the boundary of the red half-ball is totally geodesic
- the hyperbolic metric extends smoothly when any two such boundaries touch



Carlotto-Schoen type gluing, toroidal infinity



- the metric is exactly hyperbolic inside the red half-ball
- the boundary of the red half-ball is totally geodesic
- the hyperbolic metric extends smoothly when any two such boundaries touch



- the *initial mass* is defined with respect to a toroidal BK metric; the *final one* with respect to a genus-two BK metric!

Mass formula

- initial toroidal background: $b = \frac{dr^2}{r^2} + r^2 \underbrace{\left(d\theta^2 + d\varphi^2\right)}_{=:h_0}$

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- the *initial mass* is defined with respect to b; the *final one* with respect to $\bar b$

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Mass formula, space dimensions 3, somewhat more generally:

Theorem

Let g be asymptotic to two backgrounds,

$$b = \frac{dr^2}{r^2+\kappa} + r^2 h_\kappa \text{ and } \bar b = \frac{d\bar r^2}{\bar r^2+\bar\kappa} + \bar r^2 h_{\bar\kappa}\,, \text{ with } h_{\bar\kappa} = e^\omega h_\kappa\,.$$

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$$E = -\frac{1}{16\pi}\lim_{R\to\infty}\int_{r=R} D^j r\,(R^i{}_j - \frac{R}{3}\delta^i_j)dS_i\,.$$

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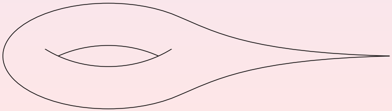
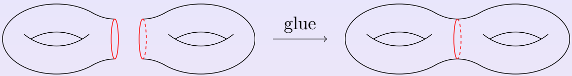


Gluing torii



- mass of each half of the glued manifold

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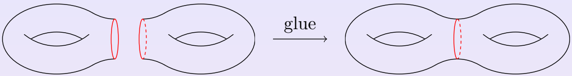

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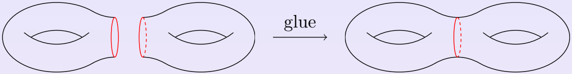
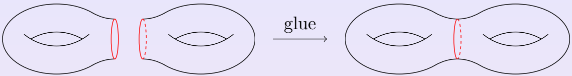


- mass of each half of the glued manifold, ω depends on ϵ ⚠

$$E = -\frac{1}{16\pi}\lim_{R\to\infty}\int_{\{r=R\}\times\left(\mathbb{T}^2\setminus D(p,\epsilon)\right)} D^j(e^{-\omega_\epsilon/2}r)\,(\mathrm{R}^i{}_j - \frac{\mathrm{R}}{3}\delta^i_j)dS_i\,.$$

Gluing torii, ϵ small needed



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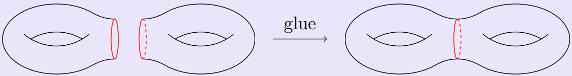
Gluing torii, limit $\epsilon \to 0$ needed



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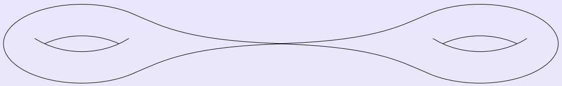
Taking the limit $\epsilon \to 0$



Theorem (PTC, E. Delay, R. Wutte)

When Maskit-gluing two Horowitz-Myers metrics *with mass parameter m,* e^{ω_ϵ} *tends to the conformal factor* e^{ω_0} *of a* punctured torus *as* ϵ *tends to zero, with*

$$\bar{E} \to -\frac{m}{4\pi}\int_{\mathbb{T}^2} e^{-\omega_0/2} d\mu_{h_0} < 0 \quad (1)$$

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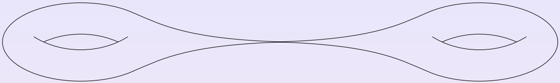

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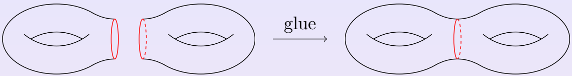
It thus follows that the final mass is negative for ϵ small enough

Taking the limit $\epsilon \to 0$



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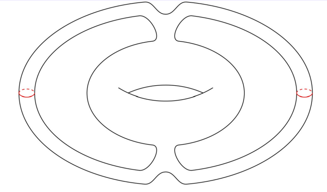
The construction can be iterated

Gluing with several punctures



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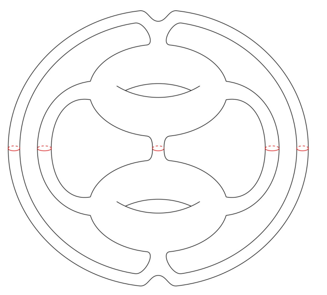
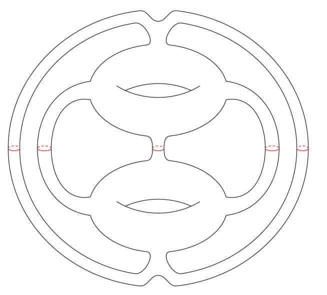
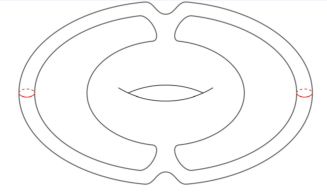


"Topological instability at the conformal boundary"?





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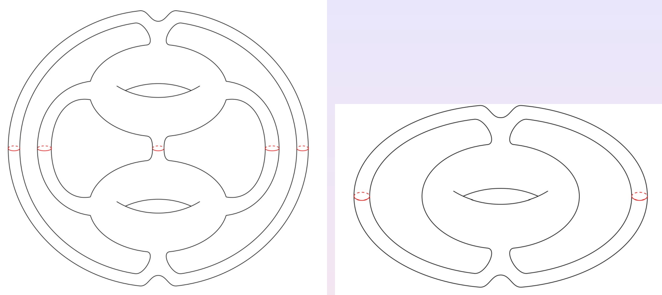
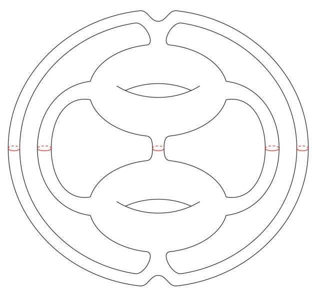
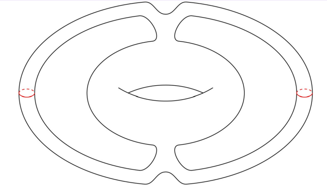


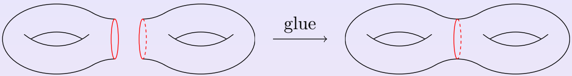
The above construction can be used to lower the total mass of an ALH manifold by a localised deformation near the conformal boundary at infinity, *for geometries with very thin necks*

“Topological instability at the conformal boundary”?

The existing higher-genus-mass-inequalities, which include conditions such as *existence of a strictly negative mass aspect function* (Lee & Neves; Gibbons), or *product topology* (Galloway et al.), cannot be improved without further conditions

Conjectures:



- For any genus of the conformal boundary at infinity there exists $m_c \leq 0$, depending only upon the conformal class of conformal infinity, so that for conformally compact vacuum asymptotically locally hyperbolic initial data sets we have

$$E \geq m_c \,,$$

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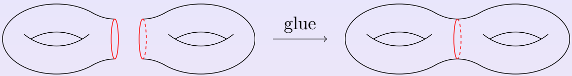

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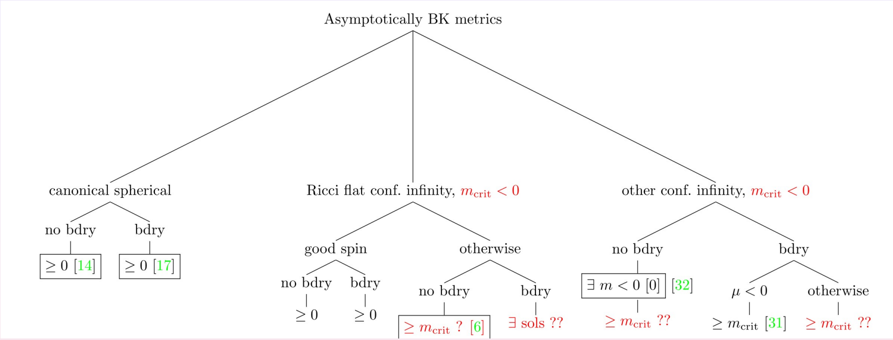
- m_c is *attained on a static metric*.

Hyperbolic mass, asymptotically Birmingham-Kottler metrics

Conformally compact, with or without black-hole boundary

Hyperbolic mass, asymptotically Birmingham-Kottler metrics

Conformally compact, with or without black-hole boundary



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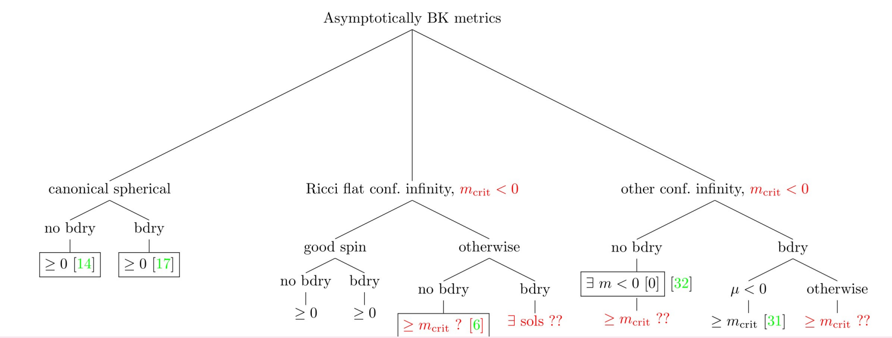

Negative mass solutions:

- toroidal: Horowitz-Myers (1998)

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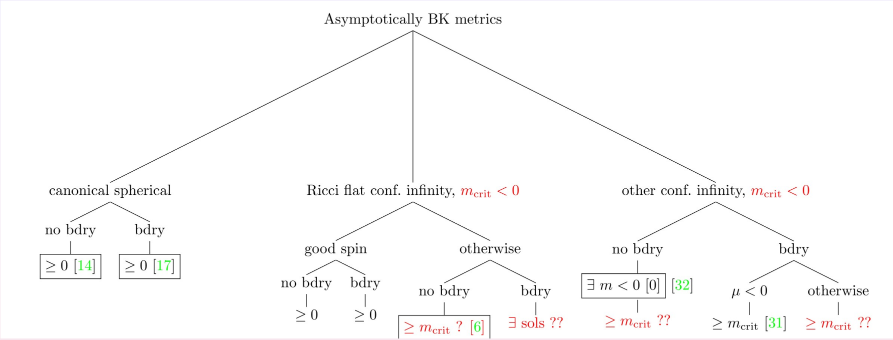

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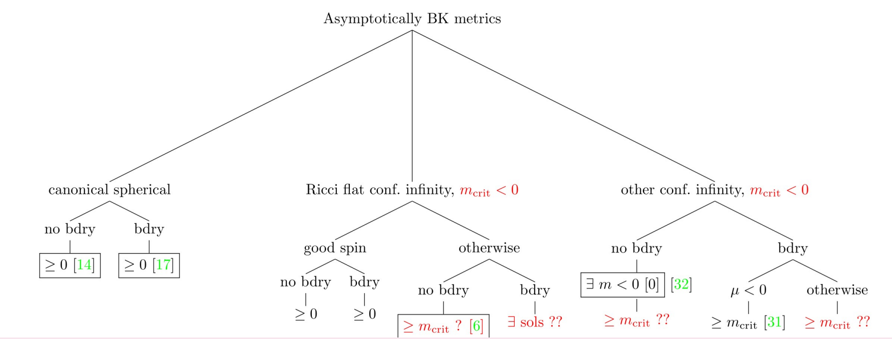

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- higher genus: PTC, Delay, Wutte (XII 2021), dim 3+1