On the Relation Between Late-time Tails, Conserved Charges, and the Failure of Peeling

Based on the preprints The Case Against Smooth Null Infinity I-III
(arXiv: 2105.08079, 2105.08084, 2106.00035)

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Structure

1. Overview and Motivation

2. The Question of Late-Time Asymptotics/Tails

3. The Question of Early-Time Asymptotics/Peeling/Smooth Null Infinity

4. Bringing everything together

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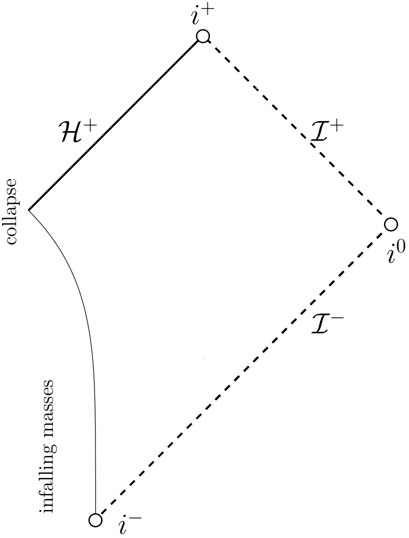
The aim of this talk is to discuss some **mathematical** work in the direction of understanding the ***physical*** *asymptotic behaviour of gravitational radiation* in gravitational collapse or similar astrophysical situations.

FOUR OVERARCHING QUESTIONS



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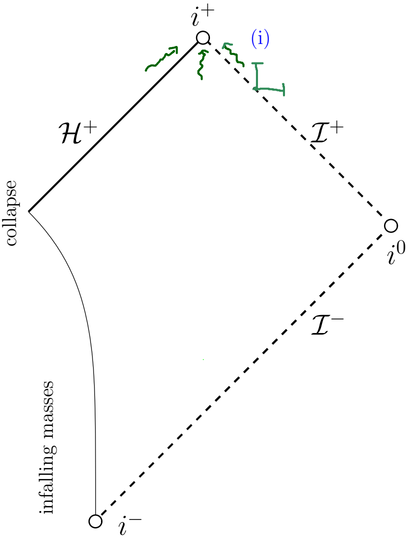

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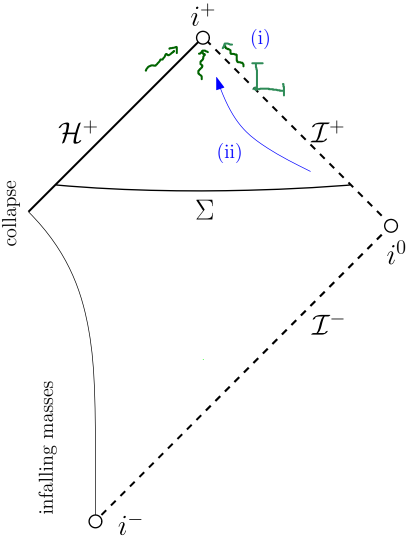

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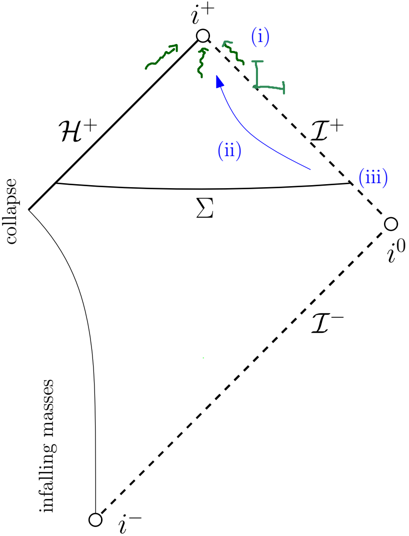

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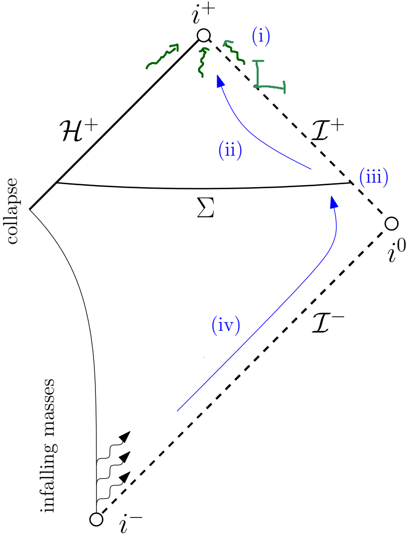

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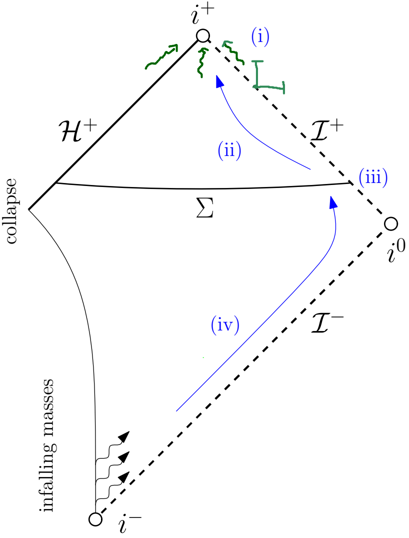

Aim of this talk is to show how all these questions are related and to provide answers to these questions within a simple model!

THE SETUP

- Consider linearised gravitational perturbations around the exterior of mass M-Schwarzschild (or Kerr): $g_M = -4(1-2M/r)dudv + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

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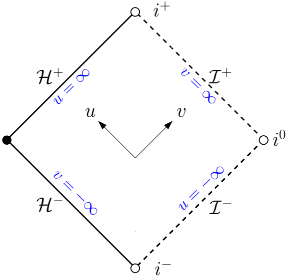

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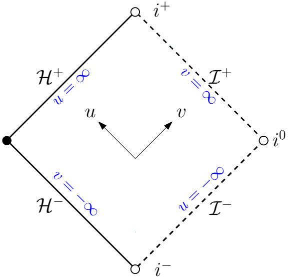

- It is well-known that the extremal components of the Weyl tensor, the Newman–Penrose scalars Ψ^0, Ψ^4, then satisfy decoupled equations:

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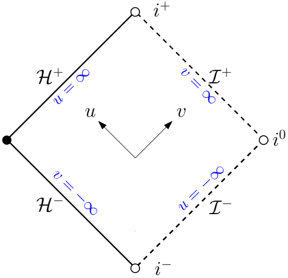

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- In this talk, we mostly focus on the simpler wave equation

$$\Box_g \phi (= \nabla^\mu \nabla_\mu \phi) = 0 \qquad \text{(Wave)}$$

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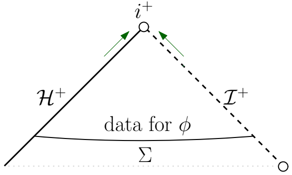

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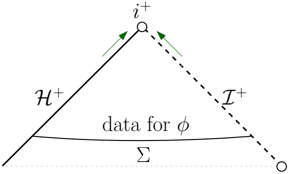

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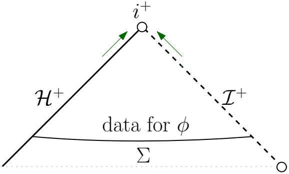

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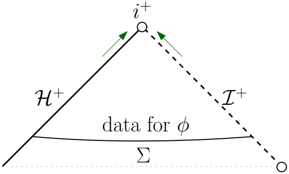

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Of course, the asymptotics one obtains will depend on the exact assumptions one makes on data. But what assumptions to make on data?

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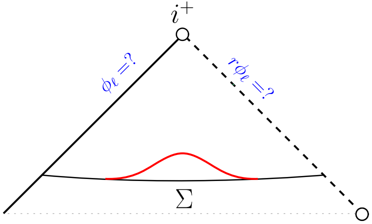

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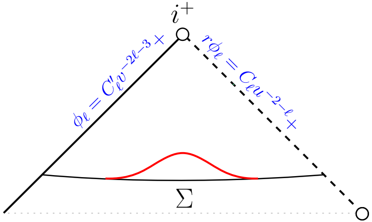

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- These late-time tails were originally predicted by Price and are called "Price's law" tails [Price, Gundlach, Pullin, Leaver...]
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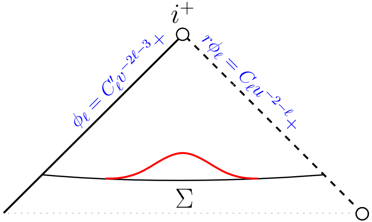

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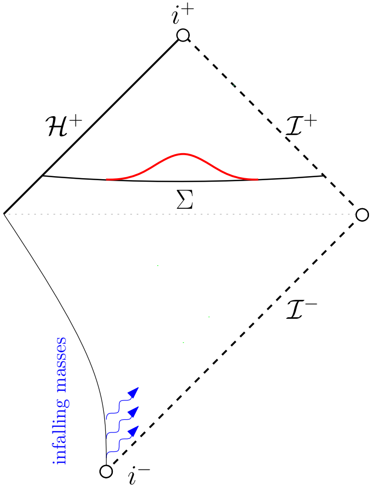
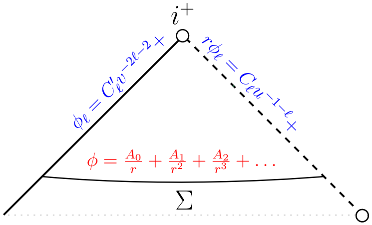


- But: Assumption of compact support not compatible with model of isolated system!

CASE (II): CONFORMALLY REGULAR/ PEELING INITIAL DATA

Capture asymptotic behaviour of data by requirement that its conformal structure be smoothly extendable to $\mathcal{I}^+$ [Penrose].

$\implies$ Sachs peeling: Data have asymptotic expansion in integer powers of $1/r$.



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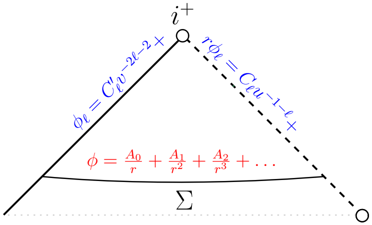

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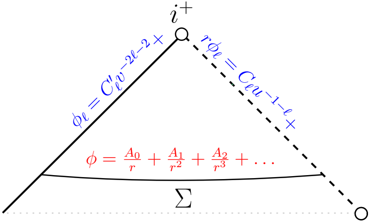

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- Decay rates one power slower than in case of localised data
- Constants C_ℓ, C'_ℓ are linear combinations of $A_1, \ldots, A_{\ell+1}$ (independent of mass M!)
- Faster decay for higher ℓ-modes related to existence of certain conserved charges. In Minkowski ($M = 0$):

$$\partial_u(r^{-2\ell}\partial_v(r^2\partial_v)^\ell(r\phi_\ell)) = 0 \tag{1}$$

Sketch of the proof I

- **Consider first** $\ell = 0 = M$**.** Then the conservation law $\partial_u(r^{-2\ell}\partial_v(r^2\partial_v)(r\phi_\ell)) = 0$ reads $\partial_u\partial_v(r\phi_0) = 0$.

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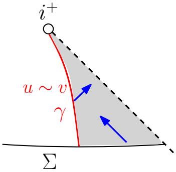

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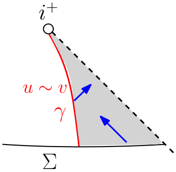

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- Finally, integrate this from γ:

$$r\phi_0 - r\phi_0|_\gamma \sim I_0^{\mathrm{NP}}[\phi]\left(\frac{1}{u} - \frac{1}{v}\right)$$

$$v \to \infty : \quad \Longrightarrow \quad r\phi_0|_{\mathcal{I}^+} \sim \frac{I_0^{\mathrm{NP}}[\phi]}{u}$$

Sketch of the proof II

- For higher ℓ-modes, can now perform a similar argument, but with $r\phi_0$ replaced by $(r^2\partial_v)^\ell(r\phi_\ell)$. (Recall $\partial_u(r^{-2\ell}\partial_v(r^2\partial_v)^\ell(r\phi_\ell)) = 0$ in Minkowski.)
- The main observation is that if the data are conformally regular ($\phi = \frac{A_0}{r} + \frac{A_1}{r^2} + \frac{A_2}{r^3} + \dots$), then

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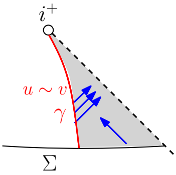

- Can again extend this a bit away from $\mathcal{I}^+$: $\partial_v(r^2\partial_v)^\ell(r\phi_\ell)\sim v^{-2}$ in depicted region.
- Finally, integrate this $\ell+1$ times from γ, each time picking up a $1/u$-factor:

$$r\phi_\ell|_{\mathcal{I}^+}\sim\frac{I_\ell^{\mathrm{NP}}[\phi]}{u^{\ell+1}}$$

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- Thus, roughly speaking, the ℓ-th mode of $r\Psi^4$ behaves like the $\ell + 2$-nd mode of $r\phi$.
- Similarly, the ℓ-th mode of $r^5\Psi^0$ behaves like the $\ell - 2$-nd mode of $r\phi$. (Recall that the lowest angular mode for $\Psi^{|s|-s}$ is $\ell = 2 = |s|$.)

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- For instance, for compactly supported data, one would get

$$r\Psi^4_{\ell=2}|_{\mathcal{I}^+} \sim u^{-6} (\sim r\phi_{\ell=4}|_{\mathcal{I}^+}).$$

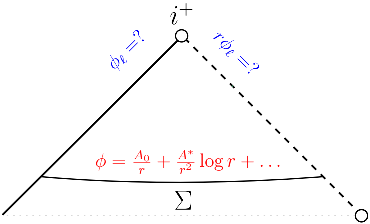
  For conformally smooth data, one would get

$$r\Psi^4_{\ell=2}|_{\mathcal{I}^+} \sim u^{-5} (\sim r\phi_{\ell=4}|_{\mathcal{I}^+}).$$

  This has recently been proved by [Ma–Zhang].

Case (iii): Conformally irregular initial data

What happens if we assume data that are not conformally regular?



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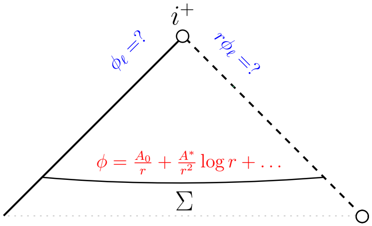
What happens if we assume data that are not conformally regular?



Let's revisit the previous proof!

Sketch of the proof I

- **Consider first** $\ell = 0 = M$**.** Then the conservation law $\partial_u(r^{-2\ell}\partial_v(r^2\partial_v)(r\phi_\ell)) = 0$ reads $\partial_u\partial_v(r\phi_0) = 0$.
- Since we have on data that $\partial_v(r\phi_0) \sim -\frac{A_1}{r^2} \sim -\frac{A_1}{v^2}$, we thus get that $\partial_v(r\phi_0) \sim -\frac{A_1}{v^2}$ everywhere.
- If $M \neq 0$, no longer have global conservation law. Instead:

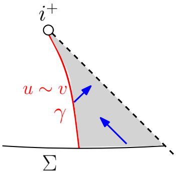
$$\partial_u\partial_v(r\phi_0) = -\left(1 - \frac{2M}{r}\right)\frac{2M \cdot r\phi_0}{r^3} \tag{6}$$

- **If** $M \neq 0$**,** no longer have global conservation law. Instead:

$$v^2 \cdot \partial_u\partial_v(r\phi_0) = -\left(1 - \frac{2M}{r}\right)\frac{2M \cdot r\phi_0}{r^3} \cdot v^2 \to 0 \tag{7}$$

- This implies the conservation of the $\ell = 0$-Newman–Penrose charge:

$$\lim_{v\to\infty} v^2\partial_v(r\phi_0) =: I_0^{\mathrm{NP}}[\phi] \equiv -A_1 \tag{8}$$



- Can moreover extend this conservation law a bit away from $\mathcal{I}^+$: $\partial_v(r\phi_0) \sim I_0^{\mathrm{NP}}[\phi]v^{-2}$ in depicted region.
- Finally, integrate this from γ:

$$r\phi_0 - r\phi_0|_\gamma \sim I_0^{\mathrm{NP}}[\phi]\left(\frac{1}{u} - \frac{1}{v}\right)$$

$$v \to \infty : \quad \Longrightarrow \; r\phi_0|_{\mathcal{I}^+} \sim \frac{I_0^{\mathrm{NP}}[\phi]}{u}$$

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- **If** $M \neq 0$**,** no longer have global conservation law. Instead:

$$v^2\log^{-1}v\cdot\partial_u\partial_v(r\phi_0) = -\left(1-\frac{2M}{r}\right)\frac{2M\cdot r\phi_0}{r^3}\cdot v^2\log^{-1}v \to 0 \tag{10}$$

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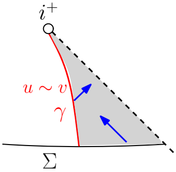

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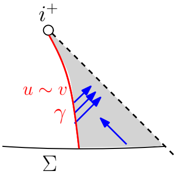
Sketch of the proof II

- For higher ℓ-modes, can now perform a similar argument, but with $r\phi_0$ replaced by $(r^2\partial_v)^\ell(r\phi_\ell)$. (Recall $\partial_u(r^{-2\ell}\partial_v(r^2\partial_v)^\ell(r\phi_\ell)) = 0$ in Minkowski.)
- The main observation is that if the data are conformally regular ($\phi = \frac{A_0}{r} + \frac{A_1}{r^2} + \frac{A_2}{r^3} + \dots$), then

$$\partial_v(r^2\partial_v)^\ell(r\phi_\ell)|_\Sigma \sim r^{-2} \sim v^{-2} \tag{12}$$

for any ℓ, even though extra r-weights are introduced!



- Can again extend this a bit away from $\mathcal{I}^+$: $\partial_v(r^2\partial_v)^\ell(r\phi_\ell) \sim v^{-2}$ in depicted region.
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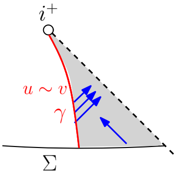
for any $\ell > 1$, so extra r-weights are introduced!



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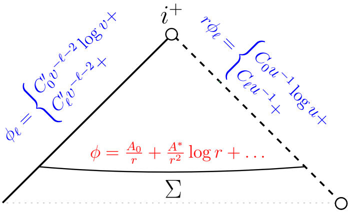

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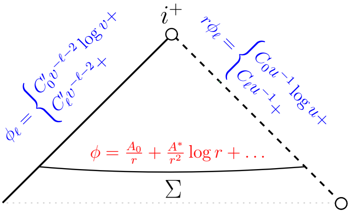

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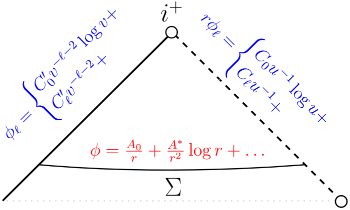

- Constants C_ℓ, C_ℓ' are nonvanishing multiples of A^*.
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$\implies$ If your solution is conformally irregular, then the cause of this irregularity is precisely what you would measure in the late-time tails!

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  - For instance, it is expected that in the non-linear setting, the non-stationary terms will dominate for higher ℓ-modes if the data are compactly supported. [Bizoń–Chmaj–Rostworowski, upcoming work by Luk–Oh]
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  - One might expect that if the data are instead sufficiently conformally irregular, then the linear effects (which are moreover completely Minkowskian) continue to dominate!
- We will now try and understand *dynamically* what the behaviour towards $\mathcal{I}^+$ should be!

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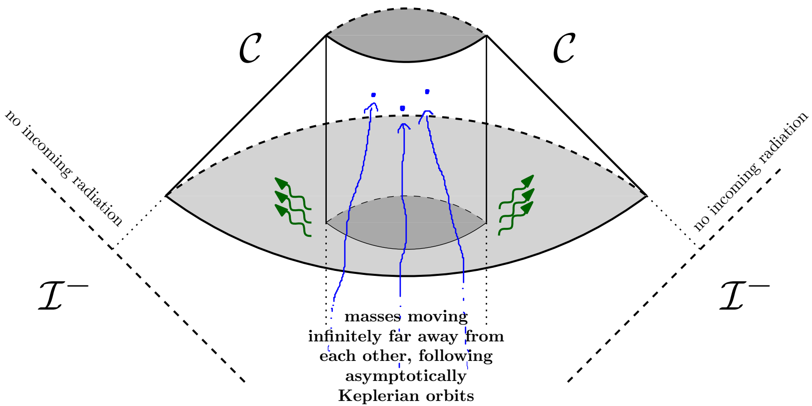
The schematic picture



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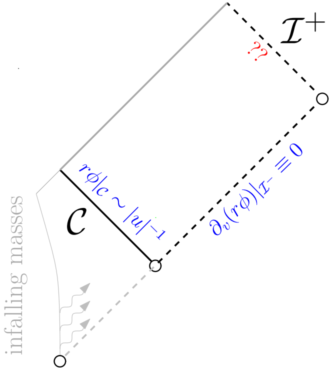

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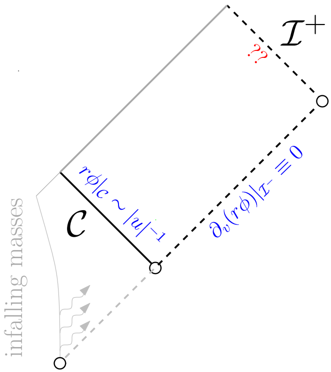

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This is a scattering problem that gives rise to a unique solution by existing scattering theory [Dafermos–Rodnianski–Shlapentokh-Rothman]!

- **Sketch of $r\phi|_{\mathcal{C}} = Q \cdot |u|^{-1}$:** If the masses follow asymptotically hyperbolic Keplerian orbits, i.e. if their relative velocities tend to constants, then along $\mathcal{I}^+$ the quadrupole approximation predicts

$$\frac{dE}{dt} \sim -\dddot{Q}_{ij}^{\mathrm{TT}} \dddot{Q}_{ij}^{\mathrm{TT}} = -C|u|^{-4} + \dots \quad \text{as } u \to -\infty \tag{14}$$

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- In the case of the scalar field, energy decay along $\mathcal{I}^+$ measured by flux of Noether current associated to time translations:

$$\frac{dE_{\text{scalar}}}{dt} = -\int_{\mathbb{S}^2} (\partial_u(r\phi))^2|_{\mathcal{I}^+} \tag{15}$$

- It follows that $\partial_u(r\phi)|_{\mathcal{I}^+} \sim |u|^{-2}$, and so $r\phi|_{\mathcal{I}^+} \sim |u|^{-1}$. One can make a more elaborate argument at the level of gravitational perturbations to show this rate on $\mathcal{C}$ instead of $\mathcal{I}^+$.

Analysis of the corresponding solution

- For simplicity, focus on spherically symmetric part ϕ_0, and recall that

$$\partial_u \partial_v (r\phi_0) = -2M \left(1 - \frac{2M}{r}\right) \frac{r\phi_0}{r^3}. \tag{16}$$

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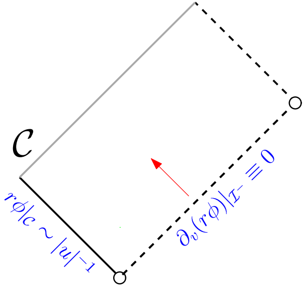

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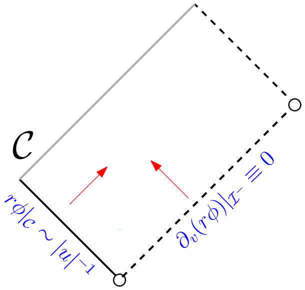

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ANALYSIS OF THE CORRESPONDING SOLUTION

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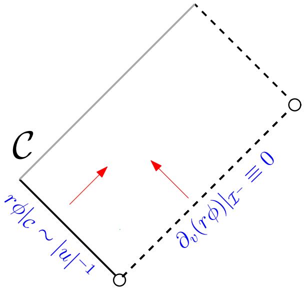

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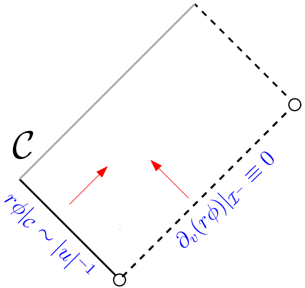

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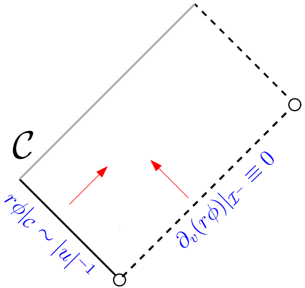

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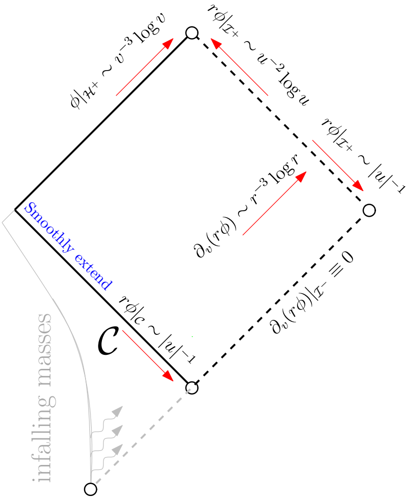

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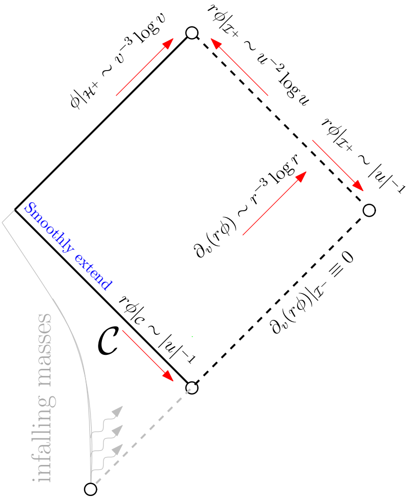

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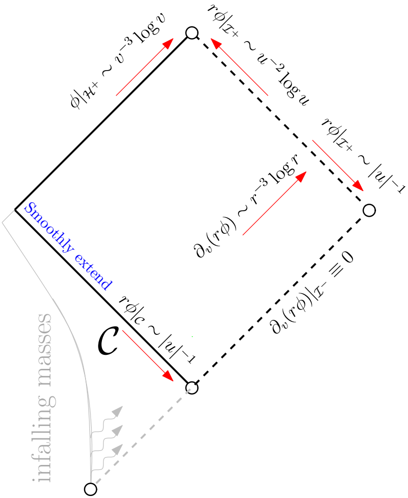

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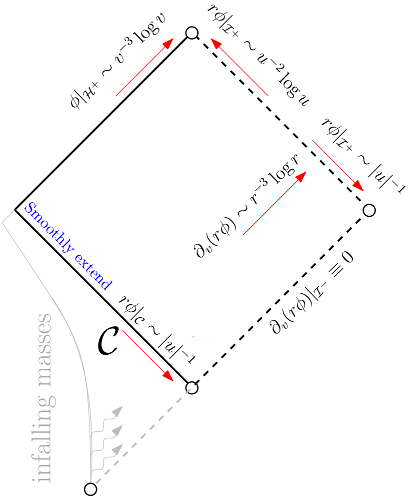

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- To be contrasted with Price's law for compactly supported Cauchy data: $r\phi|_{\mathcal{I}^+} = Cu^{-2} + \dots$

The Situation for gravitational perturbations

- Recall first the **peeling rates** for the Weyl tensor components Ψ_i:

$$\Psi^i = \mathcal{O}\left(\frac{1}{r^{5-i}}\right) \quad \text{near } \mathcal{I}^+$$
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- **Quadrupole Approximation:** The Bondi mass loss formula on $\mathcal{I}^+$ states that

$$\frac{dM}{du} = -\int_{\mathbb{S}^2} |N|^2 \, d\Omega \ (\sim |u|^{-4} \text{ by quadrupole approx.}), \tag{19}$$

where the News function N satisfies $\left.\frac{dN}{du}\right|_{\mathcal{I}^+} = r\Psi^4\big|_{\mathcal{I}^+}$

THE SITUATION FOR GRAVITATIONAL PERTURBATIONS

- Recall first the **peeling rates** for the Weyl tensor components Ψ_i:

$$\Psi^i = \mathcal{O}\left(\frac{1}{r^{5-i}}\right) \quad \text{near } \mathcal{I}^+$$
$$\Psi^i = \mathcal{O}\left(\frac{1}{r^{1+i}}\right) \quad \text{near } \mathcal{I}^-$$

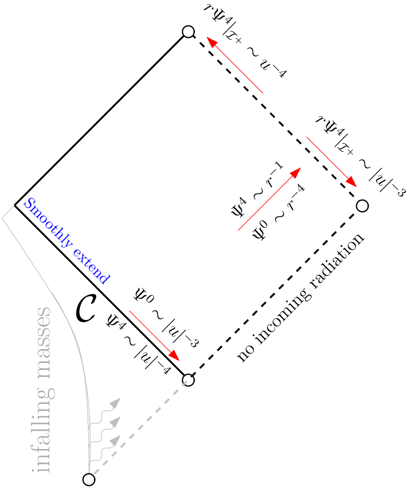
- Recall also that **Price's law** predicts that $r\Psi^4|_{\mathcal{I}^+} \sim u^{-6}$
- **Quadrupole Approximation:** The Bondi mass loss formula on $\mathcal{I}^+$ states that

$$\frac{dM}{du} = -\int_{\mathbb{S}^2} |N|^2 \, d\Omega \ (\sim |u|^{-4} \text{ by quadrupole approx.}), \tag{19}$$

  where the News function N satisfies $\left.\frac{dN}{du}\right|_{\mathcal{I}^+} = r\Psi^4\big|_{\mathcal{I}^+}$
- This already suggests that $r\Psi^4 \sim |u|^{-3}$ near $\mathcal{I}^-$, violating peeling. Indeed, this is what is suggested by perturbative arguments as well [Walker–Will, Damour, Christodoulou...]

Situation for gravitational perturbations

Conjecture (K. '21).



- Under physical setup (infalling masses coming from infinitely far away at i^-), Ψ^4 fails to peel near $\mathcal{I}^-$, and Ψ^0 fails to peel near $\mathcal{I}^+$. In particular, the radiation field $r^5\Psi^0|_{\mathcal{I}^+}$ is not defined.

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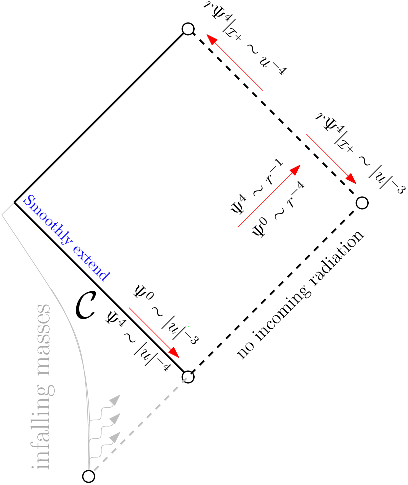

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- This failure of smoothness/peeling translates into something measurable at late times: $r\Psi^4|_{\mathcal{I}^+} \sim MQu^{-4} + \dots$
- To be contrasted with Price's law for compactly supported Cauchy data: $r\Psi^4|_{\mathcal{I}^+} = Cu^{-6} + \dots$

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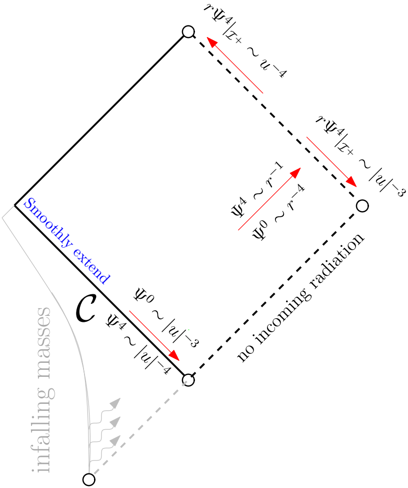

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Thank you for your attention!

Space for questions and comments