Tidal deformation of black holes

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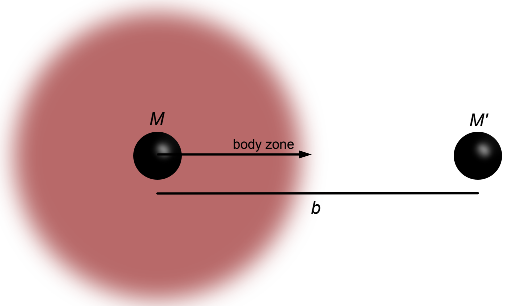
[arXiv:2012.10184; arXiv:2108:07328]

Body zone

We consider a binary system featuring a reference body of mass M and a companion body of mass M', with an orbital separation b.



We examine the gravitational field of the tidally deformed body in the body zone, a region of space limited by $r < r_{\max} \ll b$.

We work in the regime of **static tides**: the orbital period is much longer than the body's internal time scale.

Newtonian theory

The potential created by M' is expanded in powers of $r/b \ll 1$,

$$U^{\rm tidal} \sim -\frac{1}{2}\,\mathcal{E}\,r^2 P_2(\cos\theta), \qquad \mathcal{E} \propto M'/b^3 = \text{tidal moment}$$

Exterior potential of deformed body

$$U = \frac{M}{r} - \frac{1}{2}\left(r^2 + 2k_2\frac{R^5}{r^3}\right)\mathcal{E}\,P_2(\cos\theta)$$

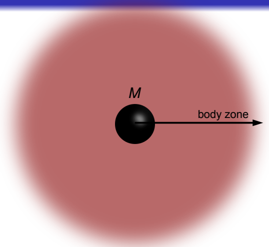
The **Love number** k_2 encapsulates details of internal constitution.

In Newtonian theory there is a one-to-one relation between Love numbers and tidally induced multipole moments,

$$\mathcal{Q} = -\frac{2}{3}k_2\,R^5\,\mathcal{E}$$

Relativistic theory

It is straightforward to promote U to the metric $g_{\alpha\beta}$ of a tidally deformed body.



$$g_{tt} = -f - f^2\left(r^2 A + 2k_2 \frac{R^5}{r^3} B\right)\mathcal{E}\, P_2(\cos\theta), \qquad f = 1 - \frac{2M}{r}$$

$$A = 1$$

$$B = -\frac{15}{16}\frac{r^5}{M^5}\ln f - \frac{5}{8}\frac{r(r-M)(3r^2 - 6Mr - 2M^2)}{M^4 f^2} = 1 + \cdots$$

The relativistic Love number k_2 is a (gauge invariant) property of the deformed metric. **Nothing more.**

Relativistic theory

Sam Gralla [arXiv:1710:11096] observed that a redefinition of the basis functions

$$r^2 A \to r^2 A + 2\lambda^5 r^{-3} B, \qquad r^{-3} B \to r^{-3} B$$

leads to a shift of the Love number: $k_2 \to k_2 - (\lambda/R)^5$.

Different definitions yield different values for the Love number.

With the "standard definition" of the basis functions, $k_2 = 0$ for a black hole.

From Love number to observable

How can k_2 — a property of the body-zone metric — be converted to an **observable** that would enter the binary's equations of motion and appear in the gravitational waves measured at infinity?

How can we link k_2 to a **useful and meaningful** notion of tidally induced quadrupole moment?

A priori, the relation $\mathcal{Q} = -\frac{2}{3} k_2 R^5 \mathcal{E}$ is **not defined** in GR.

To derive such a relation we require a notion of multipole moments for individual objects in a dynamical spacetime.

Subtraction prescription

The following prescription was proposed:

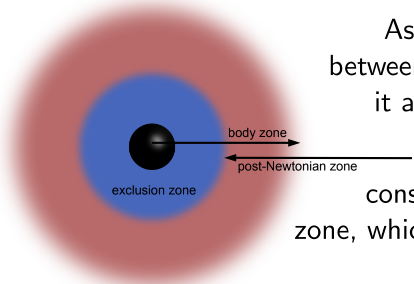
- Subtract "the tidal field" from the perturbed metric
- Calculate the Geroch-Hansen multipole moments for the now asymptotically flat spacetime

This prescription is **ambiguous**: What is "the tidal field"? Is it $r^2 A$? Or is it $r^2 A + 2\lambda^5 r^{-3} B$ for some λ? It is impossible to say.

This fundamental ambiguity leads to an ambiguous definition of tidally induced multipole moments.

The prescription is also limited to a linearized description of the tidal deformation; the subtraction procedure will no longer work in a nonlinear treatment.

Post-Newtonian prescription



Assume that the **mutual gravity** between bodies is **weak**, and describe it as a post-Newtonian expansion.

The post-Newtonian metric is constructed in the post-Newtonian zone, which excludes the exclusion zone.

Einstein field equations (harmonic coordinates: $\partial_\beta \mathfrak{g}^{\alpha\beta} = 0$)

$$\nabla^2 \mathfrak{g}^{\alpha\beta} = \partial_{tt}\mathfrak{g}^{\alpha\beta} + 16\pi\tau^{\alpha\beta}[\cancel{\text{matter}}, \text{field}], \qquad \mathfrak{g}^{\alpha\beta} = \sqrt{-g}\,g^{\alpha\beta}$$

Information about the compact body is fed through decaying solutions to Laplace's equation: $\nabla^2 \mathfrak{g}^{\alpha\beta} = 0$.

When viewed in the PN zone, the body's gravitational field looks like the field of a point mass with a **multipole structure**.

Post-Newtonian prescription

There is an overlap between the post-Newtonian and body zones.

The post-Newtonian metric is matched to the body's deformed metric in the overlap.

The matching determines the multipole structure.

Tidally induced quadrupole moment (1PN order)

$$\mathcal{Q} = -\frac{2}{3} k_2\, R^5\, \mathcal{E} + O(\ddot{\mathcal{E}}) + O(\mathcal{E}^2) + O(1\text{PN})$$

The tidally induced multipole moments will appear in the binary's equations of motion and gravitational waves.

The relativistic Love number k_2 is finally linked to an observable.

Black hole

Does $\mathcal{Q} = -\frac{2}{3} k_2 R^5 \mathcal{E} + \cdots$ apply to a black hole?

No! Because $\mathcal{Q}$ was calculated through 1PN order only, while the right-hand side (with $R = 2M$) is of 5PN order.

Way out: Contrive the black hole to be in a static binary by giving it a charge Q, letting the companion have a charge q, and balancing gravitational attraction against electrostatic repulsion.

$$qQ \simeq mM \qquad \Longrightarrow \qquad Q/M \simeq m/q \ll 1$$

Black hole

Construct the spacetime in full GR as a perturbation of the Reissner-Nordström solution, convert to harmonic coordinates, expand in PN orders, and read off the multipole moments.

Final result

All tidally induced multipole moments vanish to all PN orders.

Does the result apply to real binaries involving black holes? **Yes!**

- The relation between $\mathcal{Q}$ and $\mathcal{E}$ is generic; it applies to a realistic tidal environment as well as a contrived one.
- The static limit captures the leading term in an expansion in powers of (internal time scale)/(orbital period).
- The black hole's charge-to-mass ratio can be as small as desired (though not identially zero).

Conclusion

The Love numbers of a compact body are defined in full general relativity as a property of the deformed metric in the body zone.

There is no direct link to tidally induced multipole moments.

These moments are defined in post-Newtonian theory when the mutual gravity between body and companion is weak. (The body's own gravity can be strong.)

The link between Love numbers and tidally induced multipole moments is obtained by matching the body's deformed metric to the post-Newtonian metric.

For a black hole, the moments vanish to all PN orders.