Naked Singularities for the Einstein Vacuum Equations

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Part I: Weak Cosmic Censorship and Naked Singularities

The Weak Cosmic Censorship Conjecture (in $3+1$ Dimensions)

Conjecture (Weak Cosmic Censorship for Einstein Vacuum Equations)

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- A heuristic mechanism and also that fact that one should consider "generic" data was only understood much later. (Christodoulou 94,99)

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- A heuristic mechanism and also that fact that one should consider "generic" data was only understood much later. (Christodoulou 94,99)
- As with other fundamental questions in relativity (such as Strong Cosmic Censorship), the precise notion of regular initial data and singularity could affect the validity of the conjecture.

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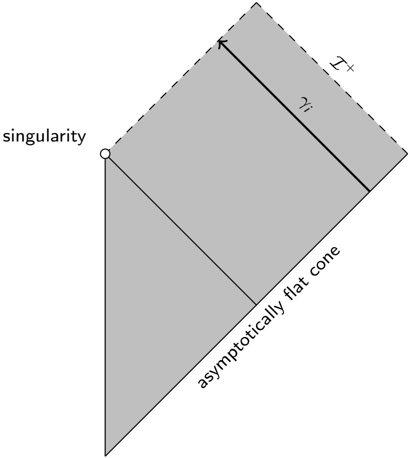
- Some necessary conditions for a function space to study weak cosmic censorship:
  1. Need well-posedness of the Cauchy problem in the space.
  2. Should reproduce familiar phenomenology (e.g. stability of Minkowski space).

Definition of a Naked Singularity

A singularity is naked if there exists $A > 0$ and a collection $\{\gamma_i\}$ of suitably normalized future oriented null geodesics which may start arbitrarily far out along an asymptotically flat cone and go extinct in affine time less than A.



Spherically Symmetric Matter Sourced Spacetimes

As a warm-up to studying the Einstein vacuum equations, Christodoulou explored this conjecture in the situation when the spacetime is assumed to be spherically symmetric and is sourced by a scalar field:

$$\mathrm{Ric}_{\mu\nu}\left(g\right) = \partial_\mu \phi \partial_\nu \phi, \qquad \Box_g \phi = 0.$$

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(See also large heuristic/numerical literature on naked singularities associated to critical phenomenon...)

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- Christodoulou's construction relied on a reduction, under a "twisted" self-similarity, of the spherically symmetric Einstein-scalar field system to a *two dimensional autonomous system* and then a corresponding phase plane analysis. For the Einstein vacuum equations, one cannot expect such a dramatic reduction, and thus a completely different approach is needed.

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- If one tries to construct such a spacetime dynamically, one needs to simultaneously solve a *low-regularity* (due to the presence of singularities) and a *global existence* problem (since one has to construct the maximal possible spacetime and show that no black hole region forms).

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In the rest of the talk we will

1. Review Christodoulou's solutions in more detail.
2. Present our naked singularities and compare with Christodoulou's solutions.

Part II: Christodoulou's Solutions

Symmetries of the Spherically Symmetric Einstein-scalar field system

A solution is described by a triple (h, r, ϕ) where h is a $1+1$ dimensional metric, r is the area radius function, and ϕ is the scalar field.

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Definition

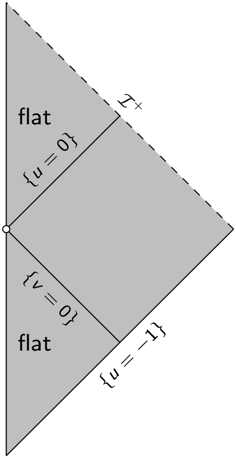
Let $k \in \mathbb{R}$. We say that a solution (h, r, ϕ) is "k-self-similar" if there exists a 1-parameter family of diffeomorphisms $\{f_s\}_{s>0}$ such that

$$f_s^* h = s^2 h, \qquad f_s^* r = sr, \qquad f_s^* \phi = \phi - k \log(s).$$

If $k = 0$, then we say that solutions are "scale-invariant."

Scale-invariant Solutions

The scale-invariant solutions may be written down explicitly and are parametrized by the value $a \doteq \partial_v (r\phi)|_{(u,v)=(-1,0)}$. For $a \ll 1$, the spacetime is as follows:



Though the scalar field is only continuous, these are perfectly physical solutions representing a spherical impulsive gravitational wave.

Solutions of Bounded Variation

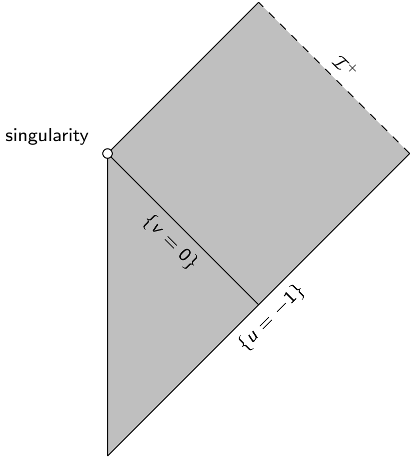
Theorem (Christodoulou, 1993)

The spherically symmetric Einstein-scalar field system is well-posed within the class of "solutions of bounded variation" (BV-solutions).

- Bounded variation spacetimes are locally modeled on scale-invariant solutions. That is, if you repeatedly rescale around a point on $\{r = 0\}$ you converge to a scale-invariant solution.

$0 < k \ll 1$ Self-Similar Solutions I

For $0 < k \ll 1$, the solutions cannot be written down explicitly. Nevertheless, Christodoulou showed that there exist solutions which correspond to naked singularities:



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2. The solution cannot be extended to the singular point and remain even a solution of bounded variation. For example:
   2.1 Hawking Mass Concentration at Singularity: $\lim_{r \to 0} \frac{m_H}{r} \sim k > 0$.
   2.2 Scalar-field blow-up: $\int_{-1}^{0} \left| \frac{\partial \phi}{\partial u} \right| |_{v=0} \, du = \infty$.

Final Remarks:

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Final Remarks:

- Christodoulou has to apply a suitable truncation to obtain an asymptotically flat solution.

- The limit as $k \to 0$ is *singular*. In fact, $\partial_v \phi \sim k^{-1/2}$!

Part III: Naked Singularities in Vacuum

Analogue of Self-Similar Solutions(?)

- We look for a solution $(\mathcal{M}, g)$ which possesses a conformally Killing vector field K:
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$$g = -2\Omega^2 \left(du \otimes dv + dv \otimes du\right) + \not{g}_{AB} \left(d\theta^A - b^A du\right) \otimes \left(d\theta^B - b^B du\right),$$
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- More explicitly, this implies that there exist $\mathring{\Omega}$, $\mathring{b}^A$, and $\mathring{\slashed{g}}_{AB}$ so that
$$\Omega\left(u, v, \theta^A\right) = \mathring{\Omega}\left(\frac{v}{u}, \theta^A\right), \qquad b^A\left(u, v, \theta^B\right) = u^{-1}\mathring{b}^A\left(\frac{v}{u}, \theta^B\right),$$
$$\slashed{g}_{AB}\left(u, v, \theta^C\right) = u^2 \mathring{\slashed{g}}_{AB}\left(\frac{v}{u}, \theta^C\right).$$

Null Constraint Equations Along $\{v = 0\}$

- Along $\{v = 0\}$ the null constraint equations implies that the following equation must hold along the sphere $\mathbb{S}^2$ at $(u, v) = (-1, 0)$:

$$\not{\mathrm{div}}\, b - \mathcal{L}_b \not{\mathrm{div}}\, b - \frac{1}{2}\left(\not{\mathrm{div}}\, b\right)^2 = \frac{1}{4}\left|\not{\nabla}\hat{\otimes} b\right|^2 + 4\mathcal{L}_b \log\Omega - 2\left(\mathcal{L}_b \log\Omega\right)\left(\not{\mathrm{div}}\, b\right).$$

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- After a further (scale-invariant) coordinate change one can take without loss of generality that

$$b = 0, \qquad \Omega = 1, \qquad g\!\!\!/_{AB}|_{v=0} = u^2 g\!\!\!/^{(\mathrm{round})}_{AB}.$$

This is a special rigidity associated to $3 + 1$ dimensional problems!

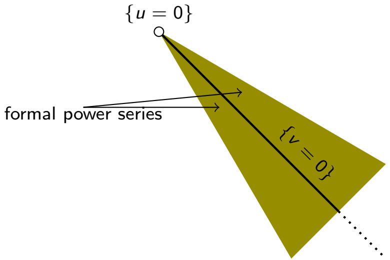
Fefferman–Graham Expansions

Theorem (Fefferman–Graham, 1985)

There exist formal power series in $\frac{v}{u}$ representing scale-invariant solutions. Furthermore, these formal ambient metrics are uniquely characterized by

$$\operatorname{tf}\left(\partial_v \not{g}\right)_{AB}\big|_{(u,v)=(-1,0)}.$$

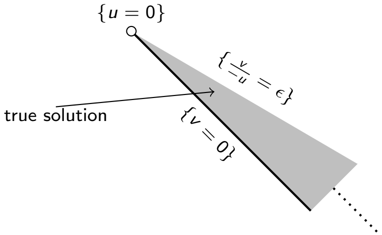
In the following diagram we have shaded in the region formally covered by FG's power series:



Existence of True Fefferman–Graham Spacetimes

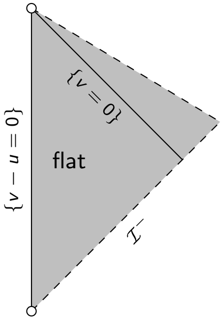
Theorem (Rodnianski–S., 2018)

All of the formal expansions of FG correspond to actual solutions in a region $\{u \in (-\infty, 0), \frac{v}{-u} \in [0, \epsilon)\}$ *for suitable* $0 < \epsilon \ll 1$.



Extensions to the Past

- It is natural to ask about "filling-in" the cone with a regular spacetime. There turns out to be a rigidity, and we must fill-in with a flat spacetime:



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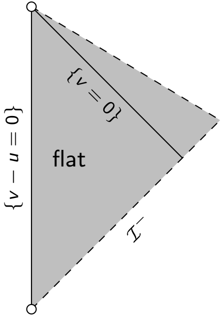
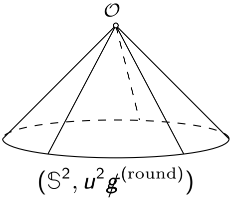


- This corresponds to a spherical impulsive wave; the metric will only be initially continuous across $\{v = 0\}$, and there is no real loss of regularity at the singularity. This is analogous to the case of $k = 0$ self-similar solutions in spherical symmetry.

Analogue of k-Self-Similar Solutions?

- All Fefferman–Graham solutions have the property that the scaling vector field K is *null* and tangent to the cone $\{v = 0\}$:



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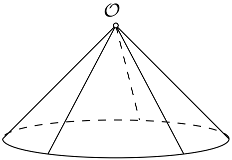
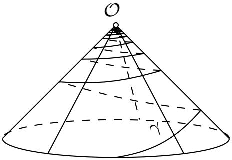


$(\mathbb{S}^2, u^2 g\!\!\!/^{(\mathrm{round})})$

- Natural to ask if we can construct a spacetime where the scaling symmetry is *spacelike* along the past cone of the singularity:



$(\mathbb{S}^2, g\!\!\!/)$

The flow of K yields a dilation of $g\!\!\!/$ only after a diffeomorphism! This can be considered analogous to the twisting of the k-self-similar solutions of Christodoulou.

How to Break the Rigidity

- Again we work in double-null coordinates

$$g = -2\Omega^2 \left(du \otimes dv + dv \otimes du\right) + \not{g}_{AB} \left(d\theta^A - b^A du\right) \otimes \left(d\theta^B - b^B du\right),$$

and require that $K = u\partial_u + v\partial_v$ is the generator of scaling symmetry.

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  for some $0 < \kappa \ll 1$.

- Of course, now the metric does not even extend continuously to $\{v = 0\}$ in these coordinates. But we can define a new coordinate $\hat{v} \doteq (1-2\kappa)^{-1}\, v^{1-2\kappa}$ and obtain

$$g = -2\Omega^2 v^{2\kappa}\left(du\otimes d\hat{v} + d\hat{v}\otimes du\right) + \slashed{g}_{AB}\left(d\theta^A - b^A du\right)\otimes\left(d\theta^B - b^B du\right).$$

  Now the metric extends to $\{v = 0\}$. (Though the form of the self-similar field is changed to $u\partial_u + (1-2\kappa)\,\hat{v}\partial_{\hat{v}}$!)

Null Constraints Revisited

- Now, along $\{\hat{v}=0\}$ the null constraint equations along the sphere $\mathbb{S}^2$ at $(u,v)=(-1,0)$ become:

$$
\begin{aligned}
\operatorname{d\!\!/iv} b - \mathcal{L}_b \operatorname{d\!\!/iv} b - \frac{1}{2}\left(\operatorname{d\!\!/iv} b\right)^2 = \\
\frac{1}{4}\left|\nabla\!\!\!/\hat{\otimes} b\right|^2 - \boxed{4\kappa} + 4\mathcal{L}_b \log\Omega - 2\left(\mathcal{L}_b \log\Omega\right)\left(\operatorname{d\!\!/iv} b\right).
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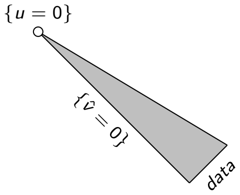
- The flexibility of κ allows us to break the rigidity, and we can find an infinite dimensional set of solutions. In the small data regime, we can essentially freely choose $g\!\!\!/$, $\operatorname{c\!\!/url}(b)$, and $v^{\kappa}\Omega$.

We have the following:

Theorem (Rodnianski–S., 2019)

Along $\{\hat{v} = 0\}$ we prescribe exactly self-similar data and give transversal data satisfying a suitable matching condition. Then we always have existence of the solution in a scale-invariant region:

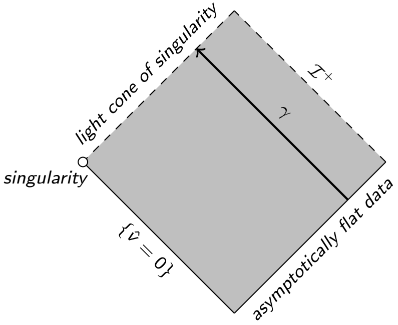


Extending to a Global Spacetime: Exterior of a Naked Singularity

We may globalize our construction to the future.

Theorem (Rodnianski–S., 2019)

There exists a naked singularity exterior: If we suitably extend the transversal data to an asymptotically flat cone, the corresponding maximal spacetime is contained in the region below:



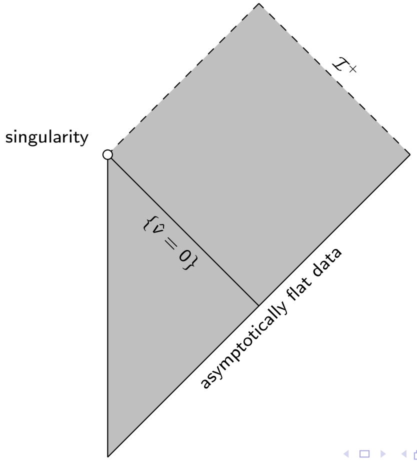
The singularity is naked in that arbitrarily far out ingoing null curves originating from the initial data (such as γ) intersect the future light cone of the singularity in time 1.

Filling in the Light Cone

It is also of great interest to extend our solutions to the interior of the cone $\{\hat{v} = 0\}$:

Theorem (S., 2022)

There exists a naked singularity interior: The solution in the previous theorem may be extended to the interior of the cone $\{\hat{v} = 0\}$*, and in this extension the initial data forms a complete asymptotically flat cone:*



Comparison with Christodoulou's Solutions I: Regularity of Initial Data

- Across the cone $\{v = 0\}$ the initial data of our solutions are $C^{1,\gamma}$ for $\gamma \sim \epsilon^2$. The limited regularity is only in the v-direction; one can take arbitrarily many derivatives tangent to $\mathbb{S}^2$. This is qualitatively similar to Christodoulou's solutions.

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- The initial data for Christodoulou's solutions are large in that $\partial_v \phi|_{(u,v)=(-1,0)} \sim k^{-1/2}$. Similarly, our initial data is large in that $\mathrm{tf}\left(\partial_v \not{g}\right)|_{(u,v)=(-1,0)} \sim \epsilon^{-1}$.

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- Of course, for the Einstein vacuum equations, we cannot appeal to a well-posedness result for BV-solutions. However, away from the "axis," we can appeal to work of Luk–Rodnianski which allows for the v-derivative of the metric to only be in L^2, as long as one has sufficient angular regularity. Furthermore, we expect that if we require Hölder-continuity in v (as well as sufficient additional angular regularity) then one may establish a well-posedness statement which includes the initial data we have.

Comparison with Christodoulou's Solution II: Nature of the Singularity

- In drawing analogies between the Einstein vacuum equations and the spherically symmetric Einstein-scalar field equations, the standard rules for comparison are

$$\partial_v \phi \leftrightarrow \mathrm{tf}\left(\partial_v \not{g}\right) = \hat{\chi}, \qquad \partial_u \phi \leftrightarrow \mathrm{tf}\left(\partial_u \not{g}\right) = \underline{\hat{\chi}}.$$

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- Christodoulou's solutions have $\int_u |\partial_u \phi| \, du = \infty$ at the singularity. Our solutions have that $\int_\gamma \left|\Omega \underline{\hat{\chi}}\right| \, ds = \infty$ for suitable null geodesics γ which converge to the singularity. (Also can find Jacobi fields which blow-up along suitable null geodesics.)

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- Another aspect of the singularity for Christodoulou's solution is that along the past cone of the singularity, we have $\frac{m_H}{r} \sim \epsilon^2$, where m_H is the Hawking mass. We similarly have that $\frac{m_H\left(\mathbb{S}^2_{u,0}\right)}{\sqrt{\mathrm{Area}\left(\mathbb{S}^2_{u,0}\right)}} \sim \epsilon^2$.

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- The above considerations are formally consistent with a C^0-singularity of the metric.

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3. Is it possible to construct naked singularities with smooth initial data? One expects so, but this remains an open problem.