UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Illinois Center for Advanced Studies of the Universe

Manifestation of quantum effects in relativistic hydrodynamics

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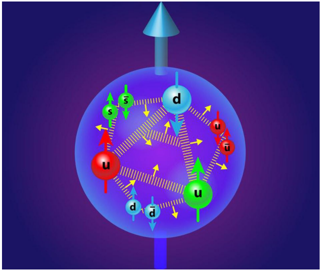
Gravity Seminars 2021-2022 - University of Southampton

November 4, 2021

Quantum $\Longleftrightarrow$ Classical



Assumptions of hydrodynamics

- **Conservation laws** in long-wavelength, low-energy limit
- **Separation of length scales**: Macroscopic L, Microscopic ℓ

Knudsen number: $\mathrm{Kn} = \dfrac{\ell}{L} \ll 1$ $\Longrightarrow$ **FLUID**

Navier-Stokes equations

- Conservation linear momentum

$$\rho(\partial_t + v_j \partial_j) v_i = \partial_j T_{ji}$$

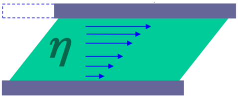
Assumption: Stress tensor is symmetric $T_{ij} = T_{ji}$

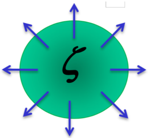
$$T_{ij} = \underbrace{-P\delta_{ij}}_{\text{ideal } \mathcal{O}(\mathrm{Kn}^0)} + \underbrace{\Pi_{ij}}_{\text{dissipation } \mathcal{O}(\mathrm{Kn}) + \mathcal{O}(\mathrm{Kn}^2) + \cdots}$$

- Truncation at first order $\mathcal{O}(\mathrm{Kn})$

$$\Pi_{ij} = \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \partial_k v_k \delta_{ij} \right) + \zeta \partial_k v_k \delta_{ij}$$

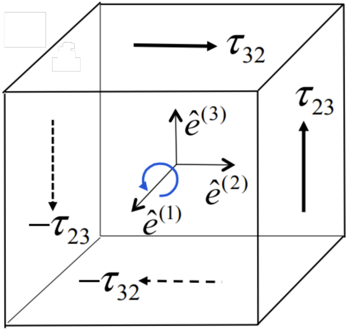
η - Shear viscosity , ζ - Bulk viscosity





What does it mean to have a symmetric stress tensor?

Symmetric Vs asymmetric stress tensor



- Antysimmetric stress tensor $T_{ij} \neq T_{ji} \Longrightarrow$ Nonzero net torque!
- Symmetric stress tensor $\Longrightarrow$ Angular momentum conservation is redundant

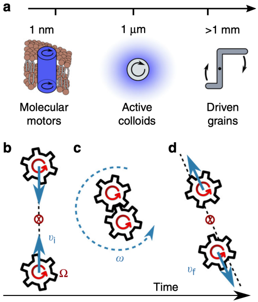
Hydrodynamics with internal angular momentum

- Conservation of total angular momentum
  Lukaszewicz, Micropolar Fluids, Theory and Applications (Birkhäuser Boston, 1999)

$$\rho(\partial_t + v_j\partial_j)S_i = \partial_j C_{ji} + \tilde{T}_i$$

S_i - Internal angular momentum (not necessarily of quantum nature)
C_{ij} - Couple stress tensor, $\tilde{T}_i = \epsilon_{ijk} T_{jk}$

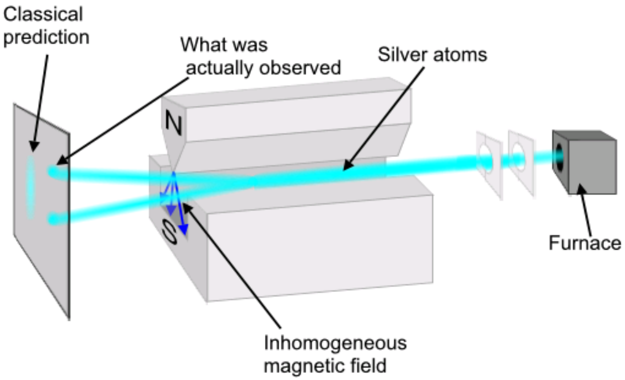
- Change of internal angular momentum due to C^{ji} and $\epsilon^{ijk} T^{jk}$



- Many applications: Micropolar fluids, Spintronics, chiral active fluids, ...
  R. Takahashi et al, Nature Physics 12, 52 (2016)
  D. Banerjee et al, Nature Commun 8, 1 (2017)

Quantum spin

- Stern-Gerlach Experiment (1922)

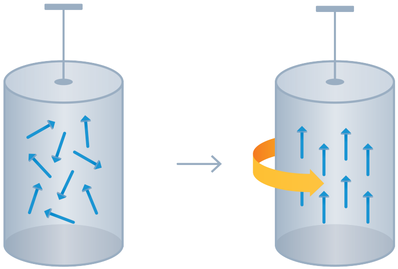


https://commons.wikimedia.org/w/index.php?curid=563880

- Two states: $|\uparrow\rangle, |\downarrow\rangle$
- Spin operator: $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

Rotation and polarization

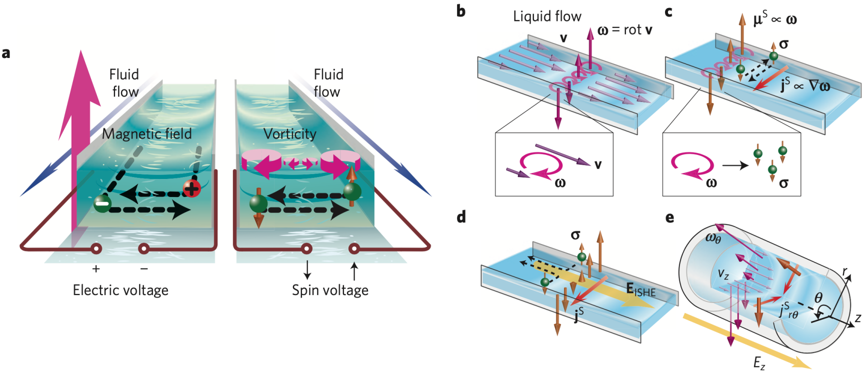
- Condensed matter: **Barnett effect**



Ferromagnet gets magnetized when it rotates

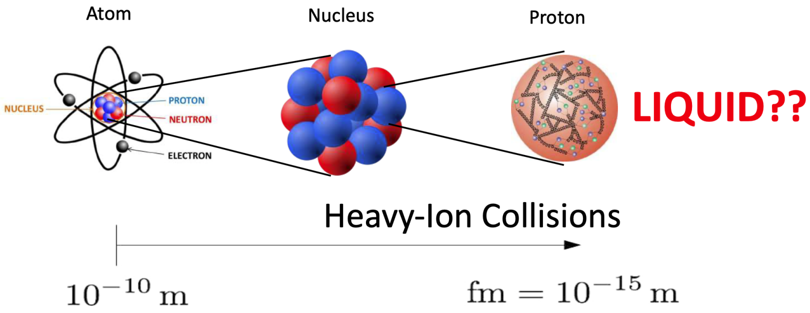
Polarization through rotation and hydrodynamics

R. Takahashi et al, Nature Physics 12, 52 (2016)



Frontiers of hydrodynamic behavior I

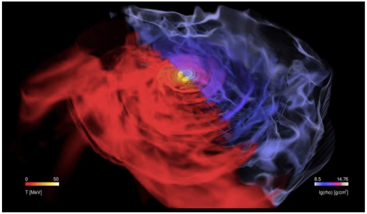
- Nuclear physics



- Redefinition of “**macroscopic**” scale

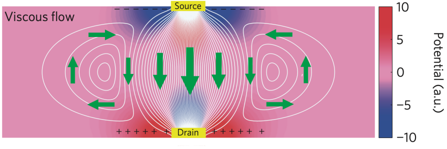
Frontiers of hydrodynamic behavior II

- Neutron star mergers



Picture by E. Most

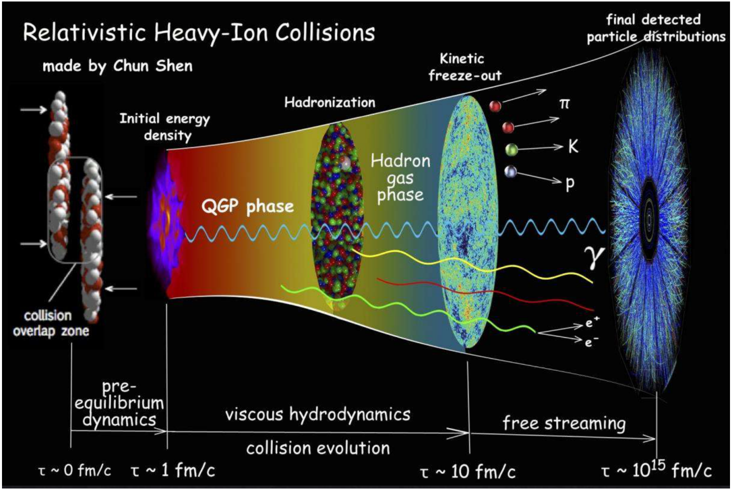
- Condensed matter: spintronics, graphene electron hydrodynamics, ...



Levitov, Falkovich, Nature Physics 12, 672 (2016)

Can one observe polarization through rotation in heavy-ion collisions? Yes!

Relativistic heavy-ion collisions

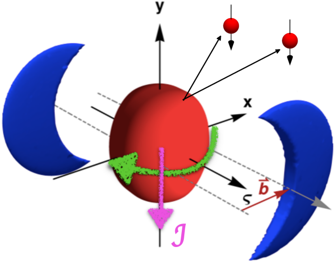


Picture by Chun Shen

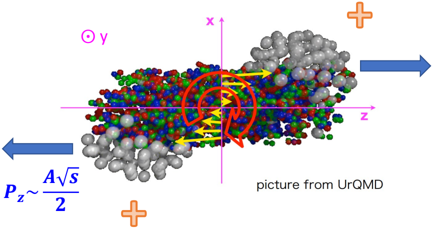
- Quark-gluon plasma $\Longrightarrow$ Early universe (up to 10 μs after Big Bang)
- Relativistic hydrodynamics is a powerful effective theory: $\partial_\mu T^{\mu\nu} = 0$

Heinz, Snellings, Ann. Rev. Nucl. Part. Sci. **63**, 123-151 (2013)

Noncentral heavy-ion collisions



picture from Florkowski, Ryblewski, Kumar, Prog. Part. Nucl. Phys. **108**, 103709 (2019)



Large global angular momentum

$$J \sim \frac{A\sqrt{s}}{2}\, b \sim 10^5 \hbar$$

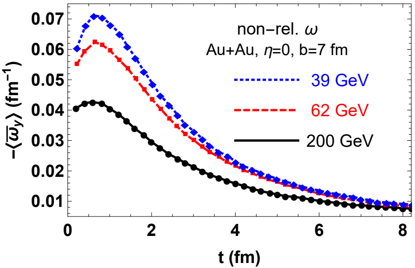
$\Rightarrow$ Vorticity of hot and dense matter $\Rightarrow$ particle polarization along vorticity

Vorticity

$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{v}$$

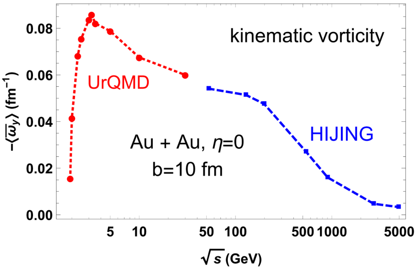
Vorticity in heavy-ion collisions

Time dependence



Jiang, Lin, Liao, PRC 94, 044910

Energy dependence



Deng, Huang, Ma, Zhang, PRC 101 064908
Deng, Huang, PRC 93 064907

see also, e.g., Huang, Liao, Wang, Xia, 2010.08937; Huang, 2002.07549; Becattini et al EPJC 75, 406; Csernai, Magas, Wang, PRC 87, 034906; Csernai, Wang, Bleicher, Stoecker, PRC 90, 021904; Ivanonv, Soldatov, PRC 95 054915

- Vorticity decreases at high energies
- Extremely high vorticity: $\omega_y \sim 10^{-2}\,\mathrm{fm}^{-1} \sim 10^{21}\,s^{-1}$

Spin-vorticity coupling

$$\text{Effective interaction} \sim -\hbar\,\vec{\sigma}\cdot\vec{\omega}$$

$$\sim \text{Quantum} \cdot \text{Classical}$$

$\vec{\sigma}$ - Particle spin, $\vec{\omega}$ - Medium rotation

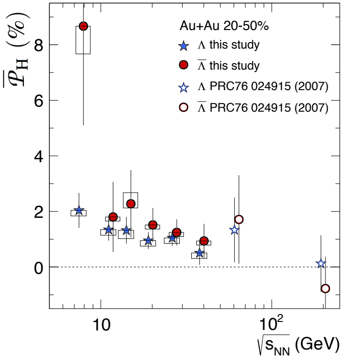
- Thermodynamic equilibrium statistical operator $\rho = \exp\left(-\frac{\hbar\vec{\sigma}\cdot\vec{\omega}}{k_B T}\right)$
  T - Temperature, k_B - Boltzmann constant

$$\vec{P} = \mathrm{Tr}(\rho\,\vec{\sigma}) \sim \frac{\hbar}{k_B T}\vec{\omega}$$

- $\Longrightarrow$ Λ-baryon polarization, ϕ and $K^{\star 0}$-mesons polarization
  Voloshin, 0410089; Liang, Wang, PRL **94**, 102301; Betz, Gyulassy, Torrieri PRC **76**, 044901; Becattini, Piccinini, Rizzo, PRC **77**, 024906

Experimental observation - Global Λ polarization

- Polarization along global angular momentum

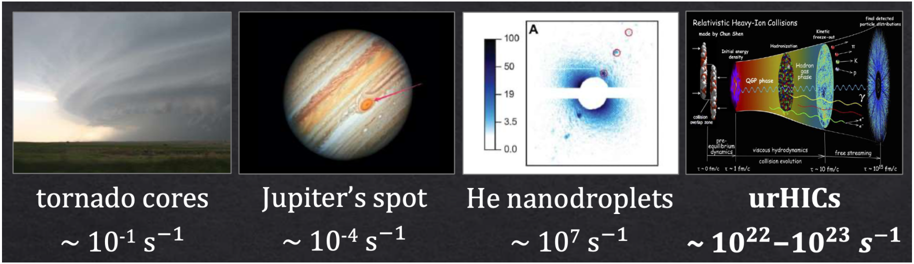


L. Adamczyk et al. (STAR), Nature 548 62-65 (2017)

- Weak decay: $\Lambda \to p + \pi^-$ angular distr.: $dN/d\cos\theta = \frac{1}{2}(1 + \alpha|\vec{\mathcal{P}}_H|\cos\theta)$
- Quark-gluon plasma is the "most vortical fluid ever observed"

$$\omega = (P_\Lambda + P_{\bar{\Lambda}})k_B T/\hbar \sim 10^{21}\,\mathrm{s}^{-1}$$

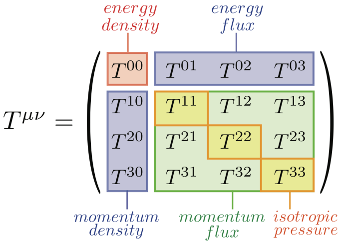
The most vortical fluid ever observed



Relativistic hydrodynamics

- Energy-momentum tensor $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$, Current $J^{\mu} = \langle \hat{J}^{\mu} \rangle$

$$\partial_\mu T^{\mu\nu} = 0$$
$$\partial_\mu J^{\mu} = 0$$



picture from Rezzolla, Zanotti, Relativistic Hydrodynamics

$$T^{\mu\nu} = \underbrace{\varepsilon u^\mu u^\nu + P\Delta^{\mu\nu}}_{\text{ideal part}} + \underbrace{\Pi^{\mu\nu}}_{\text{dissipation}}$$

$$J^{\mu} = \underbrace{n u^\mu}_{\text{ideal part}} + \underbrace{\mathcal{J}^{\mu}}_{\text{dissipation}}$$

- Unknowns ideal fluid: ε, u^μ $(u^\mu u_\mu = 1)$, n

What happens if spin is included as a dynamical variable in relativistic hydrodynamics?

Spin as a dynamical variable

Goal: Relativistic hydrodynamics (classical) with spin (quantum) as dynamical variable

Florkowski, Friman, Jaiswal, ES, PRC **97**, no. 4, 041901 (2018)
Florkowski, Friman, Jaiswal, Ryblewski, ES, PRD **97**, no. 11, 116017 (2018)
Florkowski, Becattini, ES, Acta Phys. Polon. B **49**, 1409 (2018)
Becattini, Florkowski, ES, PLB **789**, 419 (2019)
Bhadury, Forkowski, Jaiswal, Kumar, Ryblewski, 2002.03937, 2008.10976 (2020)
Singh, Sophys, Ryblewski, PRD **103**, no.7, 074024 (2021)
ES, Weickgenannt, EPJA, 57, 155 (2021)

Starting point: Kinetic theory from quantum field theory

Weickgenannt, Sheng, ES, Wang, Rischke, PRD **100**, no. 5, 056018 (2019)
Weickgenannt, ES, Sheng, Wang, Rischke, PRL, 2005.01506 (2020), 2103.04896 (2021)
Sheng, Weickgenannt, ES, Rischke, Wang, 2103.10636 (2021)

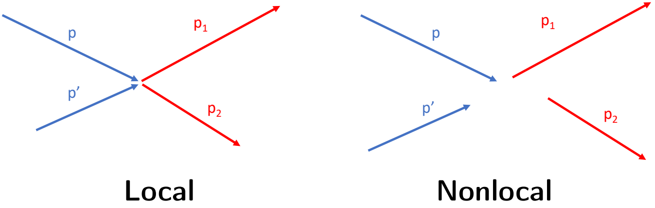
- Alternative approaches: Lagrangian formulation, entropy current, holography, ...
  Montenegro, Tinti, Torrieri, PRD 96, 056012 (2017)
  Montenegro, Torrieri, 2004.10195 (2020)
  Hattori, Hongo, Huang, Matsuo, Taya, PLB795, 100 (2019)
  Garbiso, Kaminski, JHEP **12**, 112 (2020)
  Fukushima, Pu, 2010.01608 (2020)
  Li, Stephanov, Yee, 2011.12318 (2020)
  Gallegos, Gürsoy, Yarom, arXiv:2101.04759 (2020)

Our results

Weickgenannt, ES, Sheng, Wang, Rischke, PRL, 2005.01506 (2020), 2103.04896 (2021)

- How do we describe the orbital-to-spin angular momentum conversion in kinetic theory?

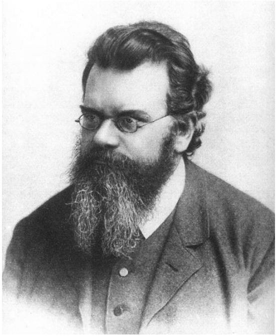
Nonlocal particle scatterings (finite impact parameter)



- And in hydrodynamics?

Antisymmetric part of energy-momentum tensor

Kinetic theory



Classical kinetic theory

- Distribution function

$$f(x,p)$$

- Boltzmann equation

$$\boxed{p_\mu \partial^\mu f(x,p) = C[f(x,p)]}$$

$C[f(x,p)]$ - Collision term

- Hydrodynamics from kinetic theory

$$T^{\mu\nu}(x) = \int d^4p\, p^\mu p^\nu f(x,p)$$

$$\partial_\mu T^{\mu\nu}(x) = \int d^4p\, p^\nu C[f(x,p)] = 0$$

How to formulate kinetic theory from quantum mechanics?

Wigner function - Quantum mechanics

"Quantum extension" of classical distribution function

$$W(x,p) = \int \frac{d\,y}{2\pi\hbar} e^{-\frac{i}{\hbar}p\cdot y} \psi^* \left(x+\frac{y}{2}\right) \psi \left(x-\frac{y}{2}\right)$$

Properties: $\displaystyle\int dp\, W(x,p) = |\psi(x)|^2, \qquad \int dx\, W(x,p) = |\psi(p)|^2$

Connected to probability!

- Expectation value of any operator $\hat{A}$

$$\langle \hat{A} \rangle = \int dx\, dp\, W(x,p) a(x,p)$$

- Schroedinger equation $\Longrightarrow$ Eqs. of motion $=$ Boltzmann eq.

$$\frac{\partial W(x,p)}{\partial t} + \frac{p}{m} \frac{\partial W(x,p)}{\partial x} = C$$

Wigner function - Quantum field theory

$$W_{\chi\sigma}(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle : \bar{\Psi}_\sigma \left(x+\frac{y}{2}\right) \Psi_\chi \left(x-\frac{y}{2}\right) : \right\rangle$$

- Dirac equation $\Longrightarrow$ Equations of motion for Wigner function
  Elze, Gyulassy, Vasak, Ann. Phys. 173 (1987) 462
  de Groot, van Leeuwen, van Weert, Relativistic Kinetic Theory. Principles and Applications

$$\left[\gamma \cdot \left(p + i\frac{\hbar}{2}\partial\right) - m\right] W(x,p) = \hbar \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle : \rho\left(x-\frac{y}{2}\right) \bar{\psi}\left(x+\frac{y}{2}\right) : \right\rangle$$

$\rho = -(1/\hbar)\partial\mathcal{L}_I/\partial\bar{\psi}$, $\mathcal{L}_I$ = interaction Lagrangian

$\Longrightarrow$ Boltzmann equation $\Longrightarrow$ Kinetic theory

$$p_\mu \partial^\mu W_{\chi\sigma}(x,p) = C_{\chi\sigma}$$

$\hbar$-expansion

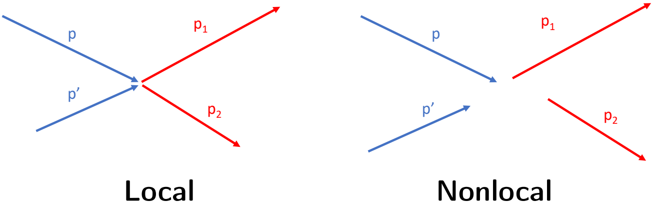
Weickgenannt, ES, Sheng, Wang, Rischke, PRL **127**, 052301 (2021); PRD **104**, 016022 (2021)

- Semiclassical expansion of Wigner function

$$W = W^{(0)} + \hbar W^{(1)} + \mathcal{O}(\hbar^2)$$

- Semiclassical and **nonlocal** expansion of collision kernel

$$C = C^{(0)}_{\text{local}} + \hbar C^{(1)}_{\text{local}} + \hbar C^{(1)}_{\text{nonlocal}} + \mathcal{O}(\hbar^2)$$



Spin in phase space

Weickgenannt, ES, Sheng, Wang, Rischke, PRL **127**, 052301 (2021); PRD **104**, 016022 (2021)

- Account for spin dynamics enlarge phase space
  J. Zamanian, M. Marklund, and G. Brodin, NJP 12, 043019 (2010)
  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)
- Introduce new phase-space variable $\mathfrak{s}^\mu$

$$W_{\chi\sigma}(x,p) \to \mathfrak{f}(x,p,\mathfrak{s})$$

- Boltzmann equation

$$p \cdot \partial\, \mathfrak{f}(x,p,\mathfrak{s}) = m\, \mathfrak{C}[\mathfrak{f}]$$

All dynamics in one scalar equation!

Nonlocal collisions

Weickgenannt, ES, Sheng, Wang, Rischke, PRL **127**, 052301 (2021); PRD **104**, 016022 (2021)

$$\mathfrak{C}[\mathfrak{f}] = \mathfrak{C}_{\text{local}}[\mathfrak{f}] + \hbar\, \mathfrak{C}_{\text{nonlocal}}[\mathfrak{f}]$$

- Long calculation $\Longrightarrow$ Intuitive result in low-density approximation:

$$\mathfrak{C}[\mathfrak{f}] = \int d\Gamma_1 d\Gamma_2 d\Gamma'\, \mathcal{W}\left[\mathfrak{f}(x+\Delta_1, p_1, \mathfrak{s}_1)\mathfrak{f}(x+\Delta_2, p_2, \mathfrak{s}_2) - \mathfrak{f}(x+\Delta, p, \mathfrak{s})\mathfrak{f}(x+\Delta', p', \mathfrak{s}')\right]$$
$$+ \int d\Gamma_2\, dS_1(p)\, \mathfrak{W}\, \mathfrak{f}(x+\Delta_1, p, \mathfrak{s}_1)\mathfrak{f}(x+\Delta_2, p_2, \mathfrak{s}_2)$$

$d\Gamma \equiv d^4p\, dS(p)$

- Structure: Momentum and spin exchange + Spin exchange only
- Nonlocal Collisions $\Longrightarrow$ Displacement $\Delta \sim \mathcal{O}(\hbar) \sim \mathcal{O}(\partial)$
- $\mathcal{W}$, $\mathfrak{W}$ vacuum transition probabilities, depend on phase-space spins

Equilibrium distribution function

Weickgenannt, ES, Sheng, Wang, Rischke, PRL **127**, 052301 (2021); PRD **104**, 016022 (2021)

- **Equilibrium condition:** $\mathfrak{C}[\mathfrak{f}] = 0$
- Ansatz for distribution function
  F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, AP. 338, 32 (2013)
  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)

$$\mathfrak{f}_{eq}(x,p,\mathfrak{s}) \propto \exp\left[\underbrace{-\beta(x)\cdot p}_{\text{Energy-momentum}} \underbrace{+\frac{\hbar}{4}\Omega_{\mu\nu}(x)\Sigma_{\mathfrak{s}}^{\mu\nu}}_{\text{Total angular momentum}} \right]\delta(p^2 - M^2)$$

  - M - mass (possibly modified by interactions)
  - $\beta^\mu = u^\mu/T$, Spin potential $\Omega^{\mu\nu}$
  - Spin-dipole-moment tensor $\Sigma_{\mathfrak{s}}^{\mu\nu} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha \mathfrak{s}_\beta$

- Insert into $\mathfrak{C}[\mathfrak{f}]$ and expand up to $\mathcal{O}(\hbar)$

Condition for $\mathfrak{C}[\mathfrak{f}] = 0 \Longrightarrow$ **Global equilibrium**

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0 \qquad \Omega_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu) = \text{const.}$$

System gets polarized through rotations!

Equilibrium distribution function

Weickgenannt, ES, Sheng, Wang, Rischke, PRL **127**, 052301 (2021); PRD **104**, 016022 (2021)

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  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)

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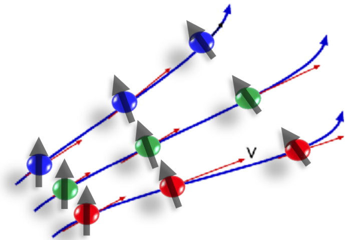
System gets polarized through rotations!

Relativistic spin hydrodynamics



Canonical energy-momentum and spin tensor

Action $\Rightarrow$ Poincaré symmetry $\Rightarrow$ Noether's th. $\Rightarrow$ Conservation laws

- **Conservation of energy and momentum:**
  Canonical energy-momentum tensor $\hat{T}_C^{\mu\nu}(x)$

$$\partial_\mu \hat{T}_C^{\mu\nu}(x) = 0$$

- **Conservation of total angular momentum:**
  Canonical total angular momentum tensor ("orbital"+"spin")

$$\hat{J}_C^{\lambda,\mu\nu}(x) = x^\mu \hat{T}_C^{\lambda\nu}(x) - x^\nu \hat{T}_C^{\lambda\mu}(x) + \hat{S}_C^{\lambda,\mu\nu}(x)$$

$$\partial_\lambda \hat{J}_C^{\lambda,\mu\nu}(x) = 0 \Longrightarrow \partial_\lambda \hat{S}_C^{\lambda,\mu\nu}(x) = \hat{T}_C^{\nu\mu}(x) - \hat{T}_C^{\mu\nu}(x)$$

Pseudo-gauge transformations

ES, Weickgenannt, EPJA, 57, 155 (2021) (Review)

Densities are not uniquely defined $\Longrightarrow$ Relocalization

F. W. Hehl, Rep. Mat. Phys. 9, 55 (1976)

$$\hat{T}'^{\mu\nu}(x) = \hat{T}_C^{\mu\nu}(x) + \frac{1}{2}\partial_\lambda \left[\hat{\Phi}^{\lambda,\mu\nu}(x) + \hat{\Phi}^{\mu,\nu\lambda}(x) + \hat{\Phi}^{\nu,\mu\lambda}(x)\right]$$

$$\hat{S}'^{\lambda,\mu\nu} = \hat{S}_C^{\lambda,\mu\nu}(x) - \hat{\Phi}^{\lambda,\mu\nu}(x) + \partial_\rho \hat{Z}^{\mu\nu,\lambda\rho}(x)$$

$\hat{\Phi}^{\lambda,\mu\nu} = -\hat{\Phi}^{\lambda,\nu\mu}$, $\hat{Z}^{\mu\nu,\lambda\rho} = -\hat{Z}^{\nu\mu,\lambda\rho} = -\hat{Z}^{\mu\nu,\rho\lambda}$

- Global charges are left invariant
- Conservation laws $\quad \partial_\mu \hat{T}'^{\mu\nu} = 0$, $\partial_\lambda \hat{S}'^{\lambda,\mu\nu} = \hat{T}'^{\nu\mu} - \hat{T}'^{\mu\nu}$

Relativistic spin hydrodynamics

Florkowski, Friman, Jaiswal, ES, PRC 97, 041901 (2018)
ES, Weickgenannt, EPJA, 57, 155 (2021) (Review)

- Hydrodynamic densities from quantum field theory

$$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle \qquad S^{\lambda,\mu\nu} = \langle \hat{S}^{\lambda,\mu\nu} \rangle$$

- **10 hydro eqs.**: 4 Energy-momentum + 6 Total angular momentum

$$\partial_\mu T^{\mu\nu} = 0 \qquad \hbar\, \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- **10 unknowns**: $\beta^\mu = u^\mu / T$ and $\Omega^{\mu\nu}$

- $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$ are NOT uniquely defined
  $\Longrightarrow$ Pseudo-gauge transformations
  - Canonical - **Problem:** Total spin is not a tensor for free fields
  - **Solution**: Hilgevoord, Wouthuysen, NP 40 (1963) 1

Relativistic spin hydrodynamics

Florkowski, Friman, Jaiswal, ES, PRC 97, 041901 (2018)
ES, Weickgenannt, EPJA, 57, 155 (2021) (Review)

- Hydrodynamic densities from quantum field theory

$$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle \qquad S^{\lambda,\mu\nu} = \langle \hat{S}^{\lambda,\mu\nu} \rangle$$

- **10 hydro eqs.**: 4 Energy-momentum + 6 Total angular momentum

$$\partial_\mu T^{\mu\nu} = 0 \qquad \hbar\, \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- **10 unknowns**: $\beta^\mu = u^\mu / T$ and $\Omega^{\mu\nu}$

- $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$ are NOT uniquely defined
  $\Longrightarrow$ Pseudo-gauge transformations
  - Canonical - **Problem:** Total spin is not a tensor for free fields
  - **Solution**: Hilgevoord, Wouthuysen, NP 40 (1963) 1

Spin hydro with Hilgevoord-Wouthuysen currents

Weickgenannt, ES, Sheng, Wang, Rischke, PRL **127**, 052301 (2021);
ES, Weickgenannt, EPJA, 57, 155 (2021) (Review)

- Equations of motion from kinetic theory

$$\partial_\mu T^{\mu\nu}_{\mathrm{HW}} = \int d\Gamma \, p^\nu \, \mathfrak{C}[\mathfrak{f}] = 0$$

$$\hbar \, \partial_\lambda S^{\lambda,\mu\nu}_{\mathrm{HW}} = \int d\Gamma \, \frac{\hbar}{2} \Sigma^{\mu\nu}_{\mathfrak{s}} \, \mathfrak{C}[\mathfrak{f}] = T^{[\nu\mu]}_{\mathrm{HW}}$$

$\Sigma^{\mu\nu}_{\mathfrak{s}} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha \mathfrak{s}_\beta$

- For free fields or global equilibrium $\mathfrak{C}[\mathfrak{f}] = 0 \Longrightarrow S^{\lambda,\mu\nu}_{\mathrm{HW}} = 0$ (unlike for the canonical currents)
- Nonzero nonlocal collision term $\Longleftrightarrow T^{[\nu\mu]}_{\mathrm{HW}} \neq 0 \Longleftrightarrow$ Spin not conserved
- Nonlocal collisions away from global equilibrium $\mathfrak{C}[\mathfrak{f}] \neq 0 \Longrightarrow$ Dissipative dynamics

Fluid gets polarized through rotation!

Scales

- Knudsen number

$$\mathrm{Kn} \equiv \frac{\ell_{\mathrm{mfp}}}{L_{\mathrm{hydro}}} \ll 1$$

- Deviations from equilibrium $\Longrightarrow$ Dissipative effects

$$\frac{\delta f}{f_{\mathrm{eq}}} \sim \mathcal{O}(\mathrm{Kn})$$

- Δ - New scale associated to spin effects and nonlocality

$$\kappa \equiv \frac{\Delta}{L_{\mathrm{hydro}}} \sim \frac{\hbar f^{(1)}}{f^{(0)}}$$

- If $\Delta \lesssim \ell_{\mathrm{mfp}}$

$$\boxed{\kappa \lesssim \mathrm{Kn}}$$

Spin effects $\Longrightarrow$ No standard notion of local equilibrium!

Relativistic dissipative spin hydrodynamics

- First-order (Navier-Stokes-like) theory
  Hattori, Hongo, Huang, Matsuo, Taya, PLB795, 100
  Hongo, Huang, Kaminski, Stephanov, Yee, 2107.14231

- Second-order (transient) theory
  Weickgenannt, Wagner, ES, Rischke, to appear

$$
\begin{aligned}
&\tau_{\mathfrak{q}} \Delta^{\mu}_{\rho} \Delta^{\nu\lambda}_{\alpha\beta} \frac{d}{d\tau} \mathfrak{q}^{\langle\rho\rangle\alpha\beta} + \mathfrak{q}^{\langle\mu\rangle\nu\lambda} \\
&= - \mathfrak{d}^{(2)} \beta_0 \sigma_{\rho}{}^{\langle\nu} \epsilon^{\lambda\rangle\mu\alpha\rho} u_\alpha + \mathfrak{K}^{(2)}_{\Omega I} \tilde{\Omega}^{\langle\mu\rangle\langle\nu} I^{\lambda\rangle} + \mathfrak{K}^{(2)}_{\nabla\Omega} \Delta^{\nu\lambda}_{\alpha\beta} \nabla^{\alpha} \tilde{\Omega}^{\mu\beta} - \mathfrak{K}^{(2)}_{\omega\sigma} \sigma^{\nu\lambda} \omega^{\mu}_{0} \\
&\quad - \mathfrak{K}^{(2)}_{\Omega\Pi} \tilde{\Omega}^{\langle\mu\rangle\langle\nu} \left(-\Pi \dot{u}^{\lambda\rangle} + \nabla^{\lambda\rangle} \Pi - \Delta^{\lambda\rangle}_{\alpha} \partial_\beta \pi^{\alpha\beta} \right) + \mathfrak{g}^{(2)}_1 \mathfrak{z}^{\mu\langle\nu} F^{\lambda\rangle} + \mathfrak{g}^{(2)}_2 \mathfrak{z}^{\mu\langle\nu} I^{\lambda\rangle} \\
&\quad + \mathfrak{g}^{(2)}_3 \Delta^{\mu}_{\rho} \Delta^{\nu\lambda}_{\alpha\beta} \nabla^{\beta} \mathfrak{z}^{\rho\alpha} + \mathfrak{g}^{(2)}_4 \mathfrak{q}^{\langle\mu\rangle\nu\lambda} \theta + \mathfrak{g}^{(2)}_5 \mathfrak{q}^{\langle\mu\rangle\rho\langle\nu} \sigma^{\lambda\rangle}_{\rho} + 2\tau_{\mathfrak{q}} \mathfrak{q}^{\langle\mu\rangle\rho\langle\nu} \omega^{\lambda\rangle}_{\rho} \\
&\quad + \mathfrak{g}^{(2)}_6 \mathfrak{p}^{\langle\mu\rangle} \sigma^{\nu\lambda} - 6\mathfrak{g}^{(2)}_7 \mathfrak{q}^{\rho\mu}{}_{\rho} \sigma^{\nu\lambda} + \mathfrak{g}^{(2)}_8 F^{\mu} \mathfrak{z}^{\langle\nu\lambda\rangle} + \mathfrak{g}^{(2)}_9 \mathfrak{p}^{\langle\nu} \nabla^{\lambda\rangle} u^{\mu} + \mathfrak{g}^{(2)}_{10} \mathfrak{q}^{\rho\langle\nu}{}_{\rho} \nabla^{\lambda\rangle} u^{\mu}
\end{aligned}
$$

Nonrelativistic limit

- $p^\mu \to m(1, \boldsymbol{v})$, $\Sigma_{\mathfrak{s}}^{\mu\nu} \to \epsilon^{ijk}\mathfrak{s}^k$

$$T_{\mathrm{HW}}^{[ji]} = m\epsilon^{ijk}\partial^0\left\langle \frac{\hbar}{2}\mathfrak{s}^k \right\rangle + m\epsilon^{ijk}\partial^l\left\langle v^l\frac{\hbar}{2}\mathfrak{s}^k \right\rangle$$

with $\langle ... \rangle \equiv (m^2/2\pi\sqrt{3}) \int d^3v\, d^3\mathfrak{s}\, \delta(\mathfrak{s}^2 - 3)\,(...)f$

- Agreement with phenomenological result of nonrelativistic kinetic theory.

  S. Hess and L. Waldmann, Zeitschrift für Naturforschung A 26, 1057 (1971)

- Comparison with micropolar fluids

  G. Lukaszewicz, Micropolar Fluids, Theory and Applications (Birkhäuser Boston, 1999)

$$\varrho\left(\partial^0 + u^j\partial^j\right) S^i = \partial^j C^{ji} + \epsilon^{ijk} T^{jk}$$

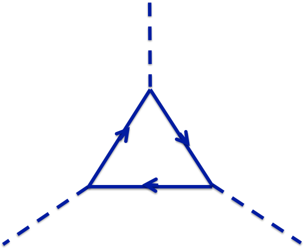
$\Longrightarrow$ Internal angular momentum $\varrho\,\ell^i = m\left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle$,

$\Longrightarrow$ Couple stress tensor $C^{ji} = -\left\langle \frac{\hbar}{2}\mathfrak{s}^i p^j \right\rangle + m\left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle u^j$.

Are the relativistic spin hydrodynamic theories present in the literature causal and stable?

We don't know yet...

Chiral (Anomalous) Hydrodynamics



Quantum Anomaly

- Symmetry of Lagrangian $\Longrightarrow$ Noether's theorem $\Longrightarrow$ Conservation law

$$\partial_\mu J^\mu = 0$$

  Associated charge Q constant in time
- Is this ensured when we consider quantum corrections?
  Not necessarily!

$$\partial_\mu J^\mu \neq 0$$

Quantum anomaly: Classical symmetry destroyed by quantum corrections

Chiral anomaly

- **Massless** QED/QCD Lagrangian is invariant under axial $U_A(1)$

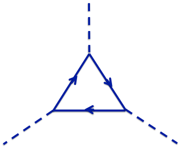
$$\psi = \begin{pmatrix} up \\ down \\ strange \end{pmatrix} \quad \rightarrow \psi' = e^{-i\theta\gamma_5}\psi$$

- Noether's theorem $\Longrightarrow$ Conservation axial vector $J^\mu_A = \bar{\psi}\gamma^\mu\gamma_5\psi$

$$\partial_\mu J^\mu_A = 0$$

- Quantum corrections

$$\partial_\mu J^\mu_A = C E_\mu B^\mu$$

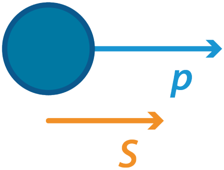


E^μ - electric field, B^μ - magnetic field, C - Anomaly coefficient

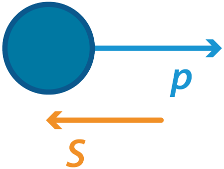
$$\frac{d(\text{Right-handed - Left-handed})}{dt} \neq 0$$

Chirality

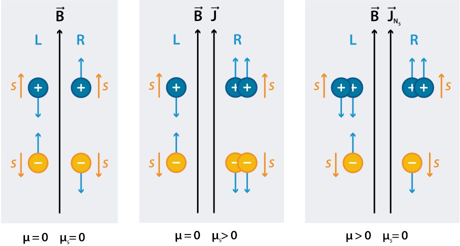
Right-handed:



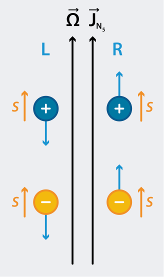
Left-handed:



- Chiral magnetic effect: $\vec{J}_V = \frac{1}{2\pi^2}\mu_5 \vec{B}$ (μ_5 - Axial chemical potential)



- Chiral vortical effect



$$\vec{J}_V = \frac{1}{\pi^2}\mu_5 \mu \vec{\omega}$$

$$\vec{J}_A = \left(\frac{\mu_V^2 + \mu_A^2}{2\pi^2} + \frac{T^2}{6}\right)\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{v}$$

Chiral hydrodynamics from kinetic theory

e.g., Chen, Son, Stephanov, PRL 115, no.2, 021601 (2015); Yang, PRD 98, no.7, 076019 (2018)

- Consider ensemble of particles (and antiparticles) with chirality $\pm$
- Distribution function

$$f_{\mathrm{eq},\pm}(x,p) = [\exp(g_\pm) + 1]^{-1}$$

$$g_\pm(x,p) = -\beta \cdot p - \frac{\mu_\pm}{T} - \underbrace{\frac{1}{2} S^{\mu\nu} \varpi_{\mu\nu}}_{\text{Spin-vorticity coupling}}$$

$S^{\mu\nu}$ - Rank-2 spin tensor,
Thermal vorticity - $\varpi^{\mu\nu} = -\frac{1}{2}(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu)$

- Hydrodynamic currents $T^{\mu\nu}$, J^μ from distribution function

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \xi_T(\omega^\mu u^\nu + \omega^\nu u^\mu)$$
$$J_V^\mu = n_V u^\mu + \xi_V \omega^\mu$$
$$J_A^\mu = n_A u^\mu + \xi_A \omega^\mu$$

This is great. But can one solve it?

Initial-value problem

- **Nonlinear** system of partial differential equations

$$\mathcal{A}(\Psi, \partial)\Psi = \mathcal{B}$$

  - Ψ - vector of unknowns
  - $\mathcal{A}(\Psi, \partial)$ - Principal part contains higher-order derivatives
  - $\mathcal{B}$ - Coefficients depending on unknowns and lower-order derivatives

- Arbitrary initial data along hypersurface $\Sigma = \{\phi(x) = 0\}$

- **To find solution:** Need to express higher order derivatives in terms of lower-order ones

  $\Longrightarrow \mathcal{A}$ must be invertible

$$\text{Characteristic determinant} = \det[\mathcal{A}(\Psi_0, \varphi)] \neq 0$$

  Ψ_0 initial data, $\varphi_\mu = \partial_\mu \phi$

Well-posedness

(see e.g. R. Wald, General Relativity)

Initial-value problem is locally well-posed in some function space (e.g., analytic functions) along a hypersurface Σ if

1) **Arbitrary** initial data on Σ $\Rightarrow$ Exists unique solution in a neighborhood of Σ
2) "Small changes" of initial data $\Rightarrow$ "Small changes" of solution
3) For relativistic theories causality must hold

- Examples of well-posed theories: Ideal hydrodynamics, Klein Gordon, GR

- Given **arbitrary** initial data, if

$$\det[\mathcal{A}(\Psi_0, \varphi)] = 0 \quad \Longrightarrow \quad \text{System is ill-posed and acausal}$$

$\Longrightarrow$ No unique solution for **arbitrary** initial data

Ideal chiral hydrodynamics from kinetic theory

Consider the full nonlinear system

ES, Bemfica, Disconzi, Noronha, 2104.02110 (2021)

$$
\begin{aligned}
T^{\mu\nu} &= \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \xi_T(\omega^\mu u^\nu + \omega^\nu u^\mu) \\
J_V^\mu &= n_V u^\mu + \xi_V \omega^\mu \\
J_A^\mu &= n_A u^\mu + \xi_A \omega^\mu
\end{aligned}
$$

e.g., Chen, Son, Stephanov, PRL **115**, no.2, 021601 (2015); Yang, PRD **98**, no.7, 076019 (2018)

V, A - Vector- and axial-vector current

- One can prove

Characteristic determinant $= 0$

System is ill-posed and acausal!

- **Conventional ideal case**: $\xi_T, \xi_V, \xi_A = 0 \implies$ Well-posed, causal and stable

Ideal chiral hydrodynamics in Landau frame

ES, Bemfica, Disconzi, Noronha, 2104.02110 (2021)

- Shift of velocity

$$u^\mu = u_L^\mu - \frac{\xi_T \omega_L^\mu}{\varepsilon + P}$$

- Constitutive relations in Landau frame

$$\begin{aligned} T^{\mu\nu} &= \varepsilon u_L^\mu u_L^\nu + P\Delta^{\mu\nu} \\ J_V^\mu &= n_V u_L^\mu + \xi_V \omega_L^\mu \\ J_A^\mu &= n_A u_L^\mu + \xi_A \omega_L^\mu \end{aligned}$$

The theory is well posed, causal and stable!

- Definition of hydrodynamic variables (hydrodynamic frames) matter even in the ideal case

Conclusions

- Vorticity and spin polarization are inherently connected
- New era where spin degrees of freedom must be taken into account in kinetic theory/hydrodynamics - New experimental observables in heavy-ion collisions
- Quantum field theory calculations suggest that spin hydrodynamics is always dissipative in the presence of nonlocal particle collisions
- Spin brings new conceptual problems for the emergence of hydrodynamics
- Causality and stability of relativistic spin and chiral hydrodynamics?
- Possible applications in condensed matter: electron hydrodynamics in graphene