The Teukolsky equation on Kerr black hole spacetimes

Rita Teixeira da Costa*

*Princeton University / University of Cambridge

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The Kerr family of black holes

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- $a = 0$: Schwarzschild,
- $|a| < M$: subextremal.

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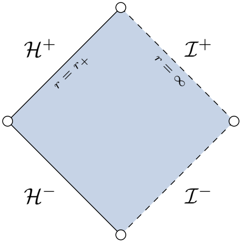

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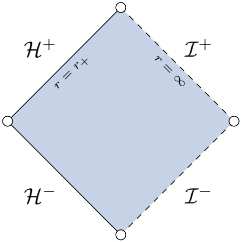

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- **unique** in the class of regular stationary, asymptotically flat black hole solutions to EVE;
- the **final state** of generic gravitational collapse.

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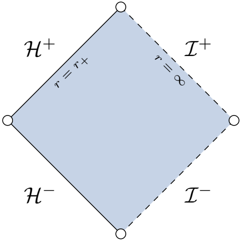

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- **unique** in the class of regular stationary, asymptotically flat black hole solutions to EVE;
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Understanding their **stability** properties is essential!

The Teukolsky master equation

To prove nonlinear stability, we must understand **linear** stability very, very well.

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$$\begin{aligned}\Big\{\Box_g + \frac{2s}{\rho^2}(r-M)\partial_r + \frac{2s}{\rho^2}\left(\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right)\partial_t \\ + \frac{2s}{\rho^2}\left[\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right]\partial_\phi + \frac{s}{\rho^2}\left(1 - s\cot^2\theta\right)\Big\}\alpha^{[s]} = 0\end{aligned} \tag{TE}$$

with $s = \pm 2$, which describes the dynamics of **gauge-invariant** curvature components.

Since $\alpha^{[s]} = 0$ for a Kerr metric, want to show stability of the zero solution to (TE). The rest is controllable by gauge.

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with $s = \pm 2$, which describes the dynamics of **gauge-invariant** curvature components. More generally, can take $s \in \frac{1}{2}\mathbb{Z}$:

- $s = 0$: scalar waves;
- $s = \pm 1$: some gauge-invariant electromagnetic quantities; ...

Since $\alpha^{[s]} = 0$ for a Kerr metric, want to show stability of the zero solution to (TE). The rest is controllable by gauge.

The Teukolsky radial ODE

(In)formally, a solution to (TE) is an infinite superposition of **modes**

$$\alpha^{[s],\,a\omega}_{m\Lambda} = e^{-i\omega t} \cdot e^{im\phi} \cdot S^{[s],\,a\omega}_{m\Lambda}(\theta) \cdot (r^2 + a^2)^{-1/2} \Delta^{-|s|/2} u^{[s]}(r)$$

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where $u^{[s]}$ solves the **Schrödinger-type ODE**

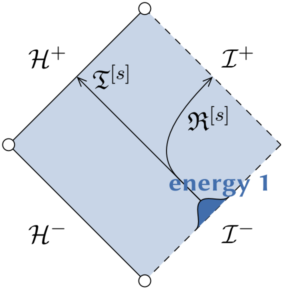
$$\left(u^{[s]}\right)'' + \left(\omega^2 - V^{[s]}\right) u^{[s]} = 0\,, \tag{TO}$$

$$V^{[s]} = \frac{\Delta\Lambda + 4Mram\omega - a^2m^2}{(r^2+a^2)^2} + s^2\frac{(r-M)^2}{(r^2+a^2)^2} + \frac{\Delta\left(a^2\Delta + 2Mr(r^2-a^2)\right)}{(r^2+a^2)^4}$$
$$- 2is\frac{\omega\left(r(r^2+a^2) + M(a^2-3r^2)\right) + am(r-M)}{(r^2+a^2)^2}\,,$$

with respect to a rescaled variable r^* ($r^* = \pm\infty$ when $r = \infty, r_+$).

A scattering interpretation of (in)stability

Consider the **scattering map** for each (ω, m, Λ).



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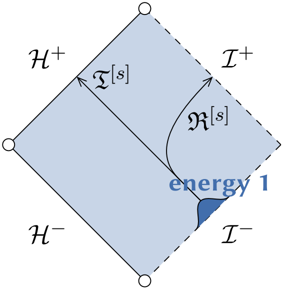

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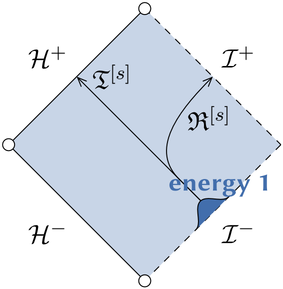

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Usually, this is forbidden by the **conserved energy**. E.g., if $a = 0$,

$$|\mathfrak{R}|^2 + |\mathfrak{T}|^2 = 1 \implies |\mathfrak{R}| \leq 1 < \infty.$$

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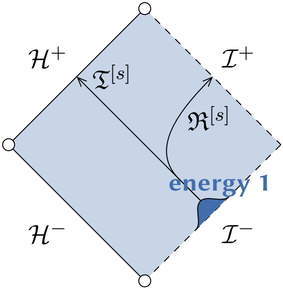

If $|\mathfrak{R}| = \infty$, have mode **instability**.

Usually, this is forbidden by the **conserved energy**. But if $a \neq 0$,

$$|\mathfrak{R}|^2 + \frac{\omega}{\omega - m\omega_+}|\mathfrak{T}|^2 = 1 \implies ??.$$

The **superradiance** condition

$$\frac{\omega}{\omega - m\omega_+} < 0, \qquad \omega_+ := \frac{a}{2M(M + \sqrt{M^2 - a^2})},$$

opens door to possible superradiant instabilities!

The ubiquity of instabilities

In fact, since Zeldovich/Teukolsky–Press '72 it is expected that **superradiant instabilities** are the **norm**...

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LETTERS TO NATURE

PHYSICAL SCIENCES

Floating Orbits, Superradiant Scattering and the Black-hole Bomb

The exception that confirms the rule

And yet...

PERTURBATIONS OF A ROTATING BLACK HOLE. II.
DYNAMICAL STABILITY OF THE KERR METRIC*

WILLIAM H. PRESS† AND SAUL A. TEUKOLSKY‡
California Institute of Technology, Pasadena

ABSTRACT

This paper tests the dynamical stability of rotating holes by numerical integration of the separable perturbation equations for the Kerr metric. No instabilities are found in any of the dozen or so lowest angular modes tested, for any value of specific angular momentum $0 \leq a < M$.
These stability results add credibility to the use of the Kerr metric in detailed astrophysical models.

no unstable modes were detected.

The exception that confirms the rule

And yet... even with good tools for the numerics,

Proc. R. Soc. Lond. A **402**, 285–298 (1985)
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An analytic representation for the quasi-normal modes of Kerr black holes

BY E. W. LEAVER

Department of Physics, University of Utah, Salt Lake City, Utah 84112, *U.S.A.*

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If superradiant instabilities are the **norm**, the (massless) Teukolsky equation appear to be the **exception**: why?

Mode stability

Mode stability à la Whiting I

Theorem (Whiting '89)

Fix $|a| < M$, and (ω, m, Λ) such that $\mathrm{Im}\,\omega > 0$. Then the **only solution** to (TO) with "good" bdry conditions is the **zero solution**.

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To prove this, Whiting shows that (TO)

$$\left(\tilde{u}^{[s]}\right)'' + \left(\omega^2 - \tilde{V}^{[s]}\right)\tilde{u}^{[s]} = 0\,, \tag{TO}$$

$$\begin{aligned} V^{[s]} &= \frac{\Delta\Lambda + 4Mram\omega - a^2m^2}{(r^2+a^2)^2} + s^2\frac{(r-M)^2}{(r^2+a^2)^2} + \frac{\Delta\left(a^2\Delta + 2Mr(r^2-a^2)\right)}{(r^2+a^2)^4} \\ &\quad - 2is\frac{\omega\left(r(r^2+a^2) + M(a^2-3r^2)\right) + am(r-M)}{(r^2+a^2)^2}\,, \end{aligned}$$

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$$\left(u^{[s]}\right)'' + \left(\omega^2 - V^{[s]}\right) u^{[s]} = 0\,, \qquad (\widetilde{\text{TO}})$$

$$\tilde{V} = \frac{\Delta}{(r^2+a^2)^2}\left[\omega^2\left(\Delta + 4M^2\frac{2(r-M)}{r-r_+} + 4Mr_-\frac{2(r-M)}{r_+-r_-}\right)\right.$$
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Then, it is easy to conclude: $(\widetilde{\text{TO}})$ has **no superradiance**, so it can have no unstable modes; by isospectrality neither can (TO).

Mode stability à la Whiting II

The isospectrality is shown as follows: one checks (by hand) that the **injective** map $u^{[s]} \mapsto \tilde{u}^{[s]}$

$$\begin{aligned}\tilde{u}^{[s]} &= (x^2+a^2)^{1/2}(x-r_-)^{-s}(x-r_+)^{-2iM\omega}e^{-i\omega x} \\ &\quad\times\int_{r_+}^{\infty}\Big\{e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{i\frac{2Mr_-\omega-am}{r_+-r_-}}(r-r_+)^{-i\frac{2Mr_+\omega-am}{r_+-r_-}}e^{i\omega r} \\ &\qquad\times\Delta^{-s/2}(r^2+a^2)^{-1/2}u^{[s]}(r)\Big\}\,dr\end{aligned}$$

maps solutions of (TO) to solutions of $(\widetilde{\text{TO}})$.

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Same holds for $\omega \in \mathbb{R}\backslash\{0\}$, but need **regularization argument** (Shlapentokh-Rothman '15).

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Note: transformation breaks down as $|a| \to M$.

Mode stability à la Whiting III

But a similar strategy works! For $|a| = M$, the map (TdC '19)

$$\begin{aligned}\tilde{u}^{[s]} = &(x^2 + 2M^2)^{1/2}(x-M)^{-s}(x-2M)^{-2iM\omega} \\ &\times \int_M^\infty \Big\{ e^{\frac{2i\omega}{M}(x-M)(r-M)}(r-M)^{-2iM\omega} e^{2M^2 i(\omega - m\omega_+)(r-M)^{-1}} e^{i\omega r} \\ &\qquad \times \Delta^{-s/2}(r^2+a^2)^{-1/2} u^{[s]}(r) \Big\}\, dr\end{aligned}$$

defined whenever $\mathrm{Im}\,\omega \geq 0$ and $\omega \neq 0, m\omega_+$ also shows isospectrality of (TO) to a new, **superradiance-less** ODE.

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In summary,

Theorem (mode stability)

If $\operatorname{Im}\omega \geq 0$ with $\omega \neq 0$, then the **only solution** to (TO) with "good" boundary conditions, either for $|a| < M$, or for $|a| = M$ if additionally $\omega \neq m\omega_+$, is the **zero solution**.

A new approach (joint with M. Casals) I

But why should (TO) and $\widetilde{(\mathrm{TO})}$ be isospectral?

A new approach (joint with M. Casals) I

But why should (TO) and $(\widetilde{\mathrm{TO}})$ be isospectral? The answer might have been in the literature already...

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An analytic representation for the quasi-normal modes of Kerr black holes

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A nontrivial mode exists if its frequency ω is a root of the continued fraction equation

$$0 = \beta_0^r - \frac{\alpha_0^r \gamma_1^r}{\beta_1^r -} \frac{\alpha_1^r \gamma_2^r}{\beta_2^r -} \frac{\alpha_2^r \gamma_3^r}{\beta_3^r -} \dots, \tag{27}$$

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The recursion coefficients are

$$\alpha_n^r=n^2+(c_0+1)\,n+c_0,$$
$$\beta_n^r=-2n^2+(c_1+2)\,n+c_3,$$
$$\gamma_n^r=n^2+(c_2-3)\,n+c_4-c_2+2,$$

and the intermediate constants c_n are defined by

$$c_0=1-s-\mathrm{i}\omega-\frac{2\mathrm{i}}{b}\left(\frac{\omega}{2}-am\right),\qquad c_1=-4+2\mathrm{i}\omega(2+b)+\frac{4\mathrm{i}}{b}\left(\frac{\omega}{2}-am\right),$$
$$c_2=s+3-3\mathrm{i}\omega-\frac{2\mathrm{i}}{b}\left(\frac{\omega}{2}-am\right),$$
$$c_3=\omega^2(4+2b-a^2)-2am\omega-s-1+(2+b)\mathrm{i}\omega-A_{lm}+\frac{4\omega+2\mathrm{i}}{b}\left(\frac{\omega}{2}-am\right),$$
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$$m_1 = s + 2iM\omega \,, \quad m_2 = i\frac{2M^2\omega - am}{\sqrt{M^2 - a^2}} \,, \quad m_3 = -s + 2iM\omega \,.$$

[Casals–TdC '21]

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The recursion coefficients satisfy

$$\begin{aligned} \alpha_{n-1}^r \gamma_n^r &= n \{ [n - \sigma_1(\boldsymbol{m})][n(n - \sigma_1(\boldsymbol{m})) + \sigma_2(\boldsymbol{m})] + \sigma_3(\boldsymbol{m}) \} , \\ \beta_n^r &= -\Lambda - s^2 + 8M^2\omega^2 + 2n(i\omega(r_+ - r_-) - n) \\ &\quad + [2n + 1 - i\omega(r_+ - r_-)]\sigma_1(\boldsymbol{m}) - \sigma_2(\boldsymbol{m}) , \end{aligned}$$

where $\sigma_i(\boldsymbol{m})$ are symmetric polynomials in $\boldsymbol{m} = (m_1, m_2, m_3)$
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[Casals–TdC '21]

A new approach (joint with M. Casals) IV

Theorem (Casals–TdC '21)

Take $|a| < M$. The point **spectrum** of

$$\left(u^{[s]}\right)'' + \left(\omega^2 - V^{[s]}\right) u^{[s]} = 0\,, \qquad \text{(TO)}$$

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for solutions with "good" boundary conditions

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- $m_2 \leftrightarrow m_3$: recovers Whiting's integral transformation;

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for solutions with "good" boundary conditions has **symmetries**

- $m_1 \leftrightarrow m_3$: recovers Teukolsky–Starobinsky identities;
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Hence, **mode stability** holds for (TE).

A new approach (joint with M. Casals) IV

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- compositions thereof.

(conjectured by Aminov, Grassi, Hatsuda '20)

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An aside: Kerr-dS

Hidden symmetries' method is more flexible:

Proposition (Casals–TdC '21)

Consider **subextremal Kerr-dS** (solves EVE with > 0 cosm. constant). Take $\eta_j = \frac{i\,(-1)^j(\omega - m\omega_j)}{2\kappa_j}$, and set

$$m_1 = s - \eta_1 - \eta_0\,, \quad m_2 = \eta_0 - \eta_1\,, \quad m_3 = -s - \eta_1 - \eta_0\,, \quad m_4 = \eta_0 + \eta_1 + 2\eta_2\,.$$

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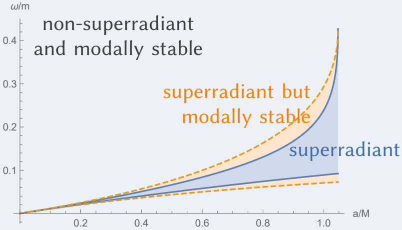

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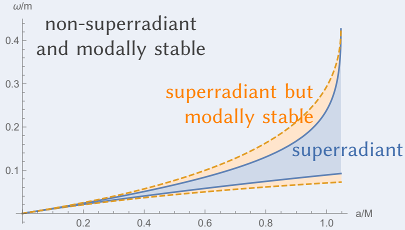
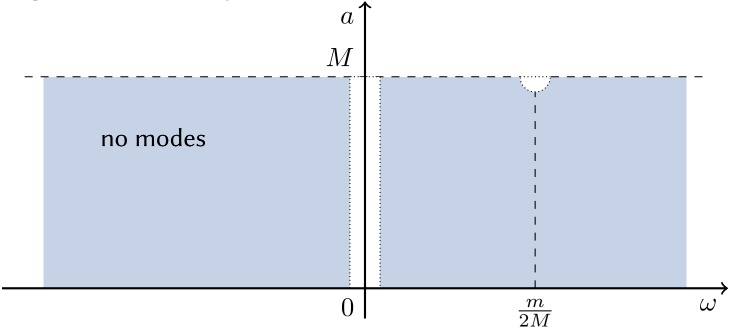


Open: (dis)prove mode stability for the blue region.

Beyond mode stability

Shortcomings of mode stability

Through mode stability:



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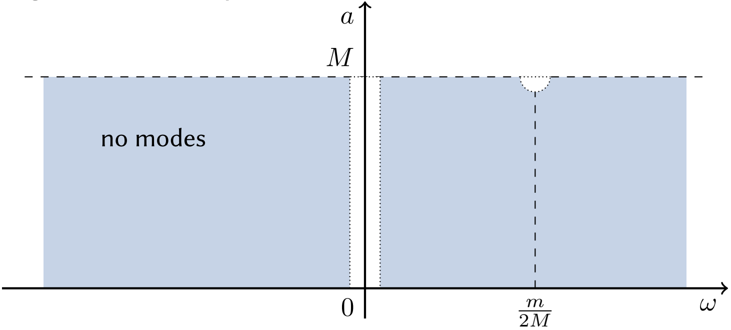

But one could have modes at

- $|\omega| + |m| + |\Lambda| \to \infty$, i.e. high frequencies;
- $\omega \to 0$, i.e. low frequencies;
- $\omega \to m\omega_+$, $|a| \to M$, both high and low!

A complete picture for $|a| < M$

Theorem (Shlapentokh-Rothman–TdC ’20)

For $s = 0, \pm 1, \pm 2$, there are also no modes at either of

- high frequencies: $|\omega| + |m| + |\Lambda| \to \infty$;
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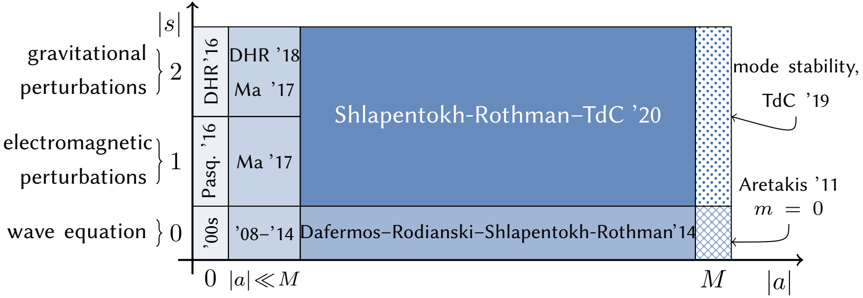
Thus, solutions to (TE) are bounded and, in fact, decay in time.

The shoulders of giants

Solutions of (TE) remain bounded in time and, in fact, decay if...



Beyond the Teukolsky equations:

- **Nonlinear** stability in $a = 0$ subfamily: Dafermos–Holzegel–Rodnianski–Taylor '21. ($|a| \ll M$ underway, Klainerman–Szeftel, Giorgi...)
- Decay of linearized EVE for $|a| \ll M$: Andersson–Bäckdahl–Blue–Ma and Häfner–Hintz–Vasy '19.
- **Coupled** electromagnetic+gravitational perturb: Giorgi '19...

Low frequencies ($|m| + |\Lambda| \lesssim 1, \omega \to 0$)

If $|\omega| \neq 0$, (TO) has nontrivial **long-range** $\operatorname{Im} V^{[s]}$,

$$V^{[s]} = \frac{\Delta\Lambda + 4Mram\omega - a^2m^2}{(r^2+a^2)^2} + s^2\frac{(r-M)^2}{(r^2+a^2)^2} + \frac{\Delta\left(a^2\Delta + 2Mr(r^2-a^2)\right)}{(r^2+a^2)^4} - 2is\frac{\omega\left(r(r^2+a^2) + M(a^2-3r^2)\right) + am(r-M)}{(r^2+a^2)^2}\,.$$

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To deal with this, we use an idea of **Chandrasekhar**: set

$$\psi^{[s]}_{(k)} \equiv \left(k \text{ adequetely } r\text{-weighted null derivatives of } u^{[s]}\right) ,$$

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for $k = 0, \ldots, |s|$. Then $\Psi^{[s]} \equiv \psi^{[s]}_{(|s|)}$ satisfies

$$\left(\Psi^{[s]}\right)'' + \left(\omega^2 - \mathcal{V}^{[s]}\right) \Psi^{[s]} = a \sum_{k=0}^{|s|-1} \mathfrak{J}^{[s]}_k \left(\psi_{(k)}, im\psi_{(k)}\right) , \tag{RWO}$$

with $\mathcal{V}^{[s]}$ **real and short-range**.

High frequencies ($|\omega| + |m| + |\Lambda| \to \infty$) I

New potential $\mathcal{V}^{[s]}$ is like the scalar wave's: e.g. have **trapping** at high frequencies.

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Luckily for Kerr stability,

- trapping is unstable
- if (ω, m, Λ) is superradiant, it cannot be trapped*.

[Dafermos–Rodnianski '09, DRSR '14]

*This **disjointness** breaks down as $|a| \to M$.

High frequencies ($|\omega| + |m| + |\Lambda| \to \infty$) II

Coupling terms in

$$\left(\Psi^{[s]}\right)'' + \left(\omega^2 - \mathcal{V}^{[s]}\right)\Psi^{[s]} = a \sum_{k=0}^{|s|-1} \mathfrak{J}_k^{[s]}\left(\psi_{(k)}, im\psi_{(k)}\right)\,, \quad \text{(RWO)}$$

means the correspondence $\Psi^{[s]} \leftrightarrow$ scalar wave is not perfect

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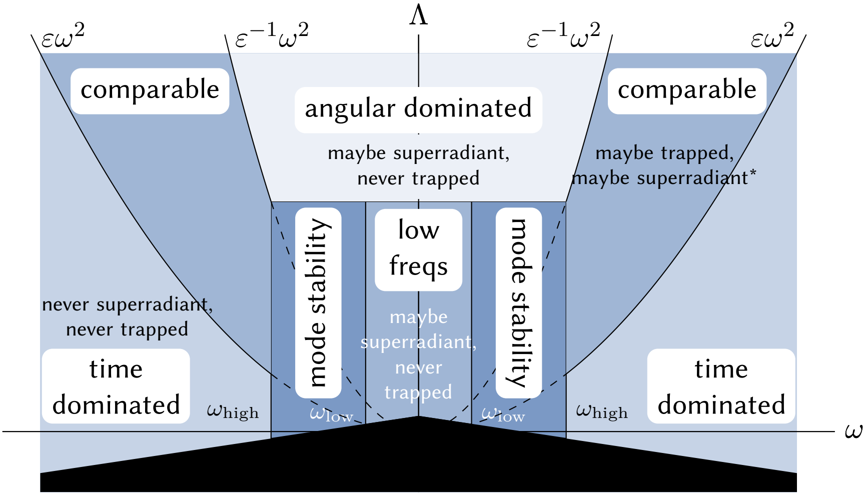
Energies are bad on their own, but turn out to go well together.

In practice...



* Quantitatively disjoint if $|a| < M$, but lose control as $\omega \to m\omega_+$, $|a| \to M$.

Thank you!