School of
Mathematical
Sciences

UNIVERSITY OF
Southampton

Senior Challenge '20
Year 10 or below

Illustrations by Theo Chaddock

Rules

1) Senior Challenge '20 should be attempted at home in your own time.
2) Your entry must be your own work, though of course you may ask for help on how to get started or for the meanings of unfamiliar words.
3) Entries without any working out at all or written on this sheet **will not be marked**.
4) It is possible to win a prize or certificate even if you have not completed all of the questions, so hand in your entry even if it is not quite finished.
5) Please make sure that you staple your pages together and you must write **your name and school neatly on every page.**

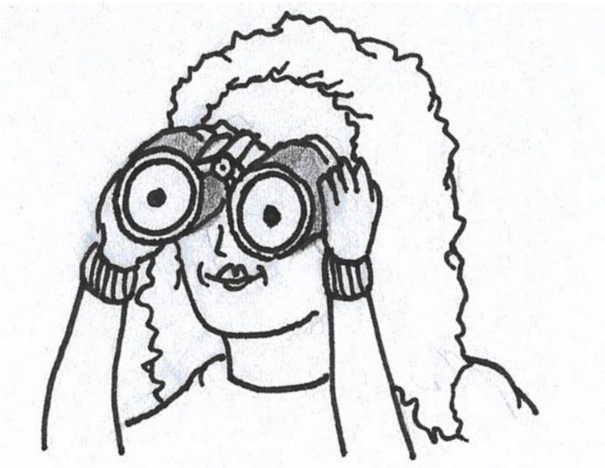
Either you or your maths teacher needs to return your entry by 18th March to this address:

Challenge '20 Entries
Dr Adam Pound
School of Mathematical Sciences
University of Southampton
University Road
Southampton
SO17 1BJ

A Prize-Giving Evening will be held at the University of Southampton on 3rd June.
We hope that you enjoy the questions.

1. **Flicks for Four Families**

   Four families go to the cinema. The Reid family take 2 adults and 2 children for £18; the McGhee family take 1 senior, 2 adults and 1 child for £18.50; and the Griffiths family take 1 senior and 3 adults for £19.50. How much will it cost for the Linton family to take 1 senior, 2 adults and 3 children?



2. **I Can See Here From My House**

   Sarah claimed she could see Russia from her house in Wasilla (Alaska) although the nearest point in Alaska to Russia (called Wales, oddly enough) is 635 miles away from Wasilla.
   If you square 635, and subtract 1 from the result, you get a number divisible by four (try it).
   Show that the same is true of 7.
   Can you prove that the same is true of ANY odd number?

3. **Spectacular!**

   Six students, Oleg, Bonnie, Pete, Georgia, Rachel and Simon, all need new glasses.
   The opticians have 6 pairs of glasses, each with a different coloured frame: red, blue, green, silver, pink and orange.
   Each of the boys has 2 colours that he'd prefer and each girl has 1 colour she'd prefer.
   Oleg prefers silver and pink,
   Simon prefers green and red,
   Pete prefers orange and blue,
   Bonnie prefers orange,
   Rachel prefers pink,
   and Georgia prefers blue.
   One of the girls (who has an r in her name) realises that not everyone can have their preference. How do they know?
   She then offers to choose a different colour so that everyone else can have their preference.
   Who was it and what colour did they choose?
   Which colour does each student ultimately select?

4. Whatever the Weather

In a network of weather stations, each station is required to know how many hours of daylight there were at each of the others. Unfortunately, their email system has been infected by a virus and they can only communicate via phone calls.
For three stations, show that three separate calls are required so that all three sets of results are known at all three stations.
For five stations, show that at least six separate calls are required.
What is the minimum number of calls if there are seven stations?
Extend your reasoning for n stations.

5. Staying Out of the Spotlight

Andrew is standing in the centre of a circle. Above the edge of the circle there are 3 spotlights, pointed straight down. These are equally spaced around the circle and can be moved around it in either direction by Nicole whilst remaining the same distance from each other. The spotlights can move twice as fast as Andrew. Andrew gets caught if a spotlight hits him. Determine whether Andrew can get out of the circle without being caught.

6. Say What You See

Lilly, Shaun and Ellen are stood in a line, one behind the other, looking in the same direction along the line.
Lilly can see both Shaun and Ellen; Shaun can only see Ellen, who cannot see either of the other two.
They are blindfolded and a hat is placed on each of their heads. They are told that the hats are selected at random from a box containing two red hats and three green hats.
The blindfolds are removed, and they are asked to state, without turning around or conferring, what colour hat they themselves are wearing.
After a pause, during which no-one has spoken, Ellen announces that she knows the colour of her hat. What is the colour of her hat and how does she know this?

7. **Clwydian Conundrum**

Lesley is standing at the hill fort on the top of Foel Fenlli, looking out along the Clwydian range of hills. The information sign next to her tells her that the two hilltops she can see are in fact both 2613m away from her (in a straight line). She notes that Moel Famau is on a bearing of 352° and Moel Gyw is on a bearing of 166°. What is the straight-line distance between the two hill tops?
It is a bright sunny day. What is the furthest away visible object?

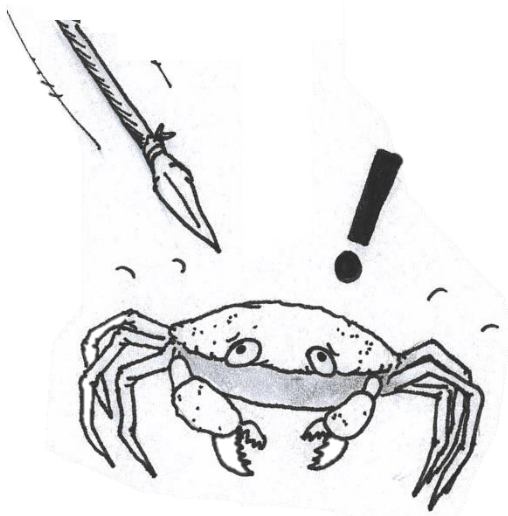
8. **Catching Crabs**

Michael (an amateur physicist) has decided to go spearfishing. His friend Rob loves crab so he decides to try to spear one.
Michael sees a crab on the seabed, which he knows is 3m below the water's surface; however, his knowledge of physics tells him that things may not be as they appear.
Michael knows that when light leaves the water, it 'refracts' (changes direction) at the surface. Thinking back to his physics lessons, Michael remembers that

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

(for water n=4/3 and for air n=1)

Michael is looking at the water at an angle of 45° from the vertical. At what angle from the vertical is the light travelling through the water?
Michael knows he will need to aim his spear closer to himself than where the crab appears to be. How much closer?



The competition in Hampshire is organised by the School of Mathematical Sciences Outreach Team, University of Southampton

With kind acknowledgement to MEM (Mathematical Education on Merseyside) for providing these questions and the concept of the Challenge Competition. http://www.mathsmerseyside.org.uk/

1. Flicks for Four Families

Let a be the price of an adult ticket, c be the price of a child ticket, and s be the price of a senior ticket.

Reids: $2a + 2c = 18$ ①
McGhees: $s + 2a + c = 18.5$ ②
Griffiths: $s + 3a = 19.5$ ③

We solve these three equations for a, c, and s:
④ = ③ - ②: $a - c = 1$
① + 2*④: $4a = 20$, so $a = 5$

Substituting $a = 5$ into ① gives $c = 4$
Substituting $a = 5$ into ③ gives $s = 4.5$

So a = £5, c = £4, s = £4.50

The cost for the Lintons is $s + 2a + 3c = 4.5 + 2*5 + 3*4 =$ £26.50

2. I Can See Here from My House

$635^2 - 1 = 403{,}225 - 1 = 403{,}224 = 4*100{,}806$

$7^2 - 1 = 49 - 1 = 48 = 4*12$

We can write any odd number as $2n+1$ for some integer n
The square of the number is $(2n+1)*(2n+1) = 4n^2+4n+1$
Subtracting 1 gives $4n^2+4n = 4(n^2+n)$, which is divisible by 4

3. Spectacular!

Only Simon prefers green and red, so the other five students only have four favourite colours between them. At least one of them must settle for a non-favourite colour.

Since Bonnie doesn’t have an r in her name, it must be either Rachel or Georgia who swaps to either red or green. Rachel swapping wouldn’t help: it would only allow Oleg his choice of either silver or pink, but no other student wants those colours. So it must be Georgia who swaps.

If she chooses red, Simon gets green; if she chooses green, Simon gets red. The other two girls then get their only preferences: Rachel gets pink, Bonnie gets orange. This leaves Oleg with silver and Pete with blue.

4. Whatever the Weather

For 3 stations, call them A, B, C. First A calls C. Then A calls B, and B calls C (3 calls)

For 5 stations, call them A through E. First A calls E. Then A calls B, and C calls D. Then A calls C, and B calls D. Then A calls E. (6 calls)

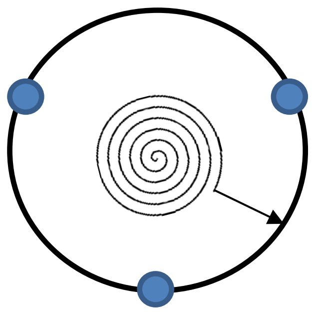
For 7 stations, A through G. A calls E, F, and G. Then A calls B, and C calls D. Then A calls C, and B calls D. Then A calls E, F, and G. (10 calls)

For n=4, 4 calls are needed. This is illustrated by the calls between A,B,C,D in the cases n=5 and n=7 above.
For n≥4, we can see from the n=5 and n=7 examples that A can make n-4 calls to get information about the stations after D, then A through D can make 4 calls to share the information about each other and about the stations after D, then A can make n-4 more calls to share all the information with the stations after D. So the total number of calls is
(n-4) + 4 + (n-4) = 2n - 4

5. Staying Out of the Spotlight

Andrew can escape.

Let r be the radius of the circle. Andrew first walks in a slowly expanding clockwise spiral until he reaches a distance r/2 from the circle's center. Assume that while he's walking in this path, Nicole makes the spotlights also move in a clockwise direction to "chase" him. He makes sure he precisely matches their movement such that he always keeps one spotlight directly behind him. To see that he can do this, note that if he were to walk in a small circle around the center, he can move around the small circle at least as fast as the spotlights can move around the big circle if the small circle has a radius smaller than r/2; this remains true for the spiral if the spiral is gradual enough.



What if Nicole decides not to always "chase" Andrew and moves the spotlights counterclockwise instead? Andrew then reverses direction to follow a counterclockwise outward spiral. In this way, he can always keep one spotlight directly behind him no matter how often Nicole changes direction.

When Andrew reaches a distance r/2 from the center, the spotlights will be arranged as in the figure. He then makes a dash for the edge of the circle. Can a spotlight get to his escape point before he does? The distance between the spotlights around the circle is $2\pi r/3$. So to reach Andrew's escape point, one of the spotlights has to traverse a distance $\pi r/3$. But in his dash, Andrew covers a distance r/2, meaning the spotlights can only move a distance r in that time. Since $\pi r/3 > r$, the spotlight can't make it to his escape point in time to catch him.

6. Say What You See

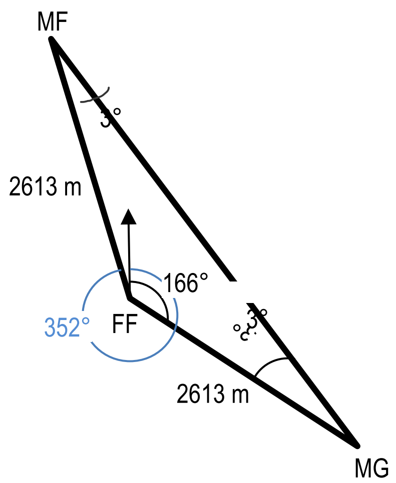
Ellen is wearing a green hat.

How does Ellen know this? Lilly can see both Shaun and Ellen, so if both Shaun and Ellen wore red hats, Lily would realise she was wearing green and announce it.

Since Lily remains silent, Shaun can deduce that he and/or Ellen is wearing green. If Shaun could see that Ellen's hat was red, he would know for certain that he was wearing green and announce it.

Ellen can follow the above reasoning, so she knows that Shaun would announce his hat was green if her hat was red. Therefore, since he remains silent, she knows her hat is green.

7. Clwydian Conundrum



The angle ∠MF-FF-MG is
360° - (352°-166°) = 174°.
This makes the other two angles in the triangle 3°.

We can use the sine rule:

$$\frac{\text{length}(MF - MG)}{\sin(174°)} = \frac{2613}{\sin(3°)}$$

Or we can use the cosine rule:

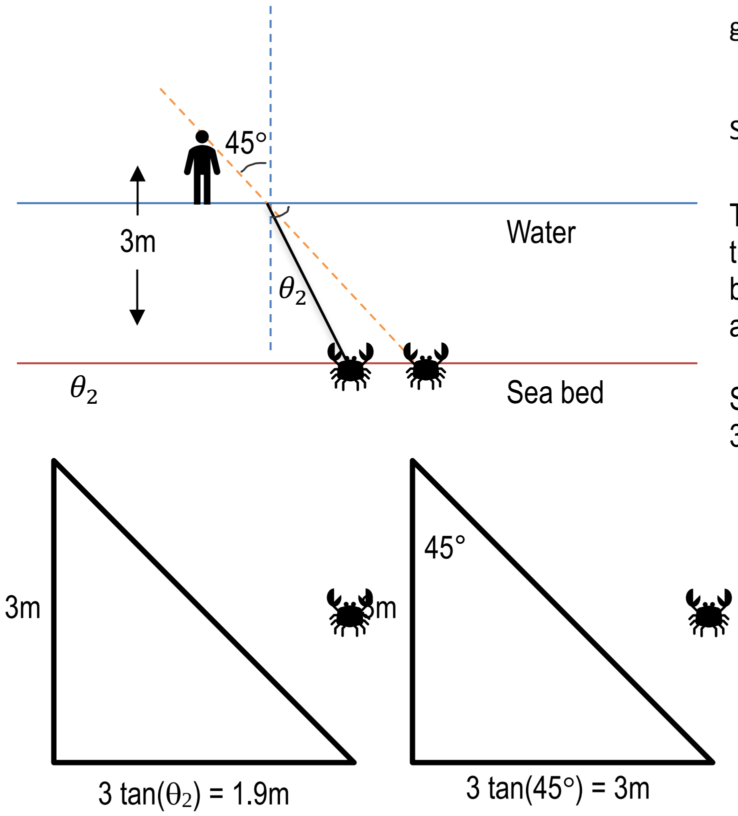
$$(length(MF - MG))^2 = 2613^2 + 2613^2 - 2 \times 2613 \times 2613 \times \cos(174°)$$

Either way, we obtain $\text{length}(MF - MG) = 5219$ m.

We could also find this result using basic trig and Pythagoras on the two right-angled triangles with the sides MF-FF and FF-MG as their hypotenuses.

The furthest away visible object is the Sun.

8. Catching Crabs



With $n_1 = 1$, $\theta_1 = 45°$, and $n_2 = 4/3$, we get

$$\sin 45° = \frac{4}{3} \sin \theta_2$$

So $\theta_2 = 32°$ (to 2 significant digits)

The triangle below and to the left shows the crab's true position. The triangle below and to the right shows the crab's apparent position.

So the crab is
3m - 1.9m = 1.1m closer than it appears to be.