School of
Mathematical
Sciences

UNIVERSITY OF
Southampton

Senior Challenge '21

Year 10 or below

Illustrations by Theo Chaddock

Rules

1) The Senior Challenge '21 should be attempted on your own time at home.
2) Your entry must be your own work, though of course you may ask for help on how to get started or for the meanings of unfamiliar words.
3) Entries without any working out at all or written on this sheet **will not be marked**.
4) It is possible to win a prize or certificate even if you have not completed all the questions, so hand in your entry even if it is not quite finished.
5) You must write **your name and school neatly on every page.**

Either you or your maths teacher needs to submit a pdf scan of your entry by 23rd April using one of the submission forms at
https://sites.google.com/view/southamptonmathchallenge/home
Teachers can submit batches of multiple students' entries, either as a combined pdf or multiple pdfs, using the batch submission form.

If you and your teacher do not have access to a scanner, you can scan your entry using a smart phone, following the instructions at the end of this document. Please use your phone's scan function rather than submitting high-resolution photos.

If you're unable to submit your entry electronically, please have your teacher or parents contact us at math4all@soton.ac.uk

A virtual Prize-Giving Evening will be held online in June.

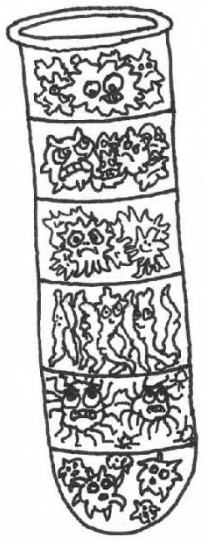
We hope that you enjoy the questions.

1. Flightless of Fancy

On the island of Spark, there is a unique flightless bird called the dontdont.
After many years of hunting, it has become endangered. Hunting is now banned, and the dontdont has no natural predators.
A captive-breeding programme has been a great success, and 24 captive-bred birds are released into the wild.
A year later, a random sample of 46 birds is captured and studied before being released again. It is found that 6 of the captured birds are ones which were captive-bred.
Assuming that no birds have suffered an unfortunate demise, estimate the total population of dontdonts now.

2. A Bacterial Buffet

A biologist keeps dangerous bacteria of six different kinds in a long glass tube separated into six chambers by membranes.
She has noticed that, if she removes a membrane, bacteria start fighting their neighbours.
Those which are in a minority always die, together with an equal number of the enemy. Equal colonies eat each other completely!
Originally, there were a million bacteria of the first kind, two million of the second, and so on, up to six million of the sixth kind.
Is it possible for her to open the membranes in turn in some order so that no bacteria survived? Explain your answer.



3. Going the Way of the Dinosaurs?

Colin the Conservationist has been investigating the IUCN Red Data list of species. Between 66 and 67 thousand species have been assessed and placed on the list. Your task is to work out the number of species in each of 7 categories.
There are 78 species which are “extinct in the wild”.
11 times that many have gone “extinct” since 1500AD.
Respectively 6 and 9 times as many species as have gone “extinct” are on the “critically endangered” and “endangered” lists.
24,570 species are listed as threatened, which means they are “vulnerable”, “endangered” or “critically endangered”.

The number of “near threatened” species is 468 more than a third of the number which are “vulnerable”.
The good news is that there are 7,098 more species on the “least concern” list than in all the other categories combined.

4. Dizzy Planets

Ellie is a keen astronomer. One day, whilst stargazing, she spots Mercury and Venus.
Being a keen astronomer, she knows that Venus takes 225 Earth days to go around the sun and Mercury takes 88 Earth days for the same journey.
Assuming circular orbits, and that Venus and Mercury are in alignment (relative to the sun) at 00:00 on 1st January 2021, how many times will they be aligned (V, M, S in that order) during that Earth year?
On which days will these alignments occur?

5. Numerous Nuts

In preparation for hibernation, Sue the squirrel buries nuts in five different stashes. One location has the same number of nuts as the other four combined. A second location has 2/19 of her nuts; the other three locations each have different fractions, but all with 1 as the numerator. What are the possibilities?

6. Ewe'll Love this Lawn!

Charlie's lawn covers an equilateral triangle of side length 20 metres. He doesn't enjoy mowing and so he borrows 3 sheep, tying one to a post at each corner, using ropes 10 metres long, so that they can graze the lawn for him.

He decides to create a circular flower bed in the centre of the lawn. What is the diameter of the largest flower bed he could have that is still safe from the sheep?

How much lawn will he actually need to mow? Give your answers to 3dp.

7. A Bear Does Go Out of the Woods

A sleuth (group) of bears is in the woods. They find themselves 200 metres from the nearest point, P, on a straight stretch of river which runs past the edge of the woods. Their den (D) is 800 metres downstream along the river from P.

They move at the speed of their slowest member, Edward, who can swim along the river at 5km/h, but ambles through the woods at only 3km/h.

Let x be the distance in km from P to the point where they aim to meet the river.

Plot a graph for x (x-axis) against the time t (y-axis) taken in minutes to reach D.

Use your graph to determine how far from D the bears should aim to meet the river in order to get there as fast as possible.

The competition in Hampshire is organised by the School of Mathematical Sciences Outreach Team, University of Southampton

We kindly acknowledge MEM (Mathematical Education on Merseyside) for providing these questions and the concept of the Challenge Competition.
http://www.mathsmerseyside.org.uk/

Senior Challenge '21 Solutions

1. **Flightless of Fancy**
At $t = 0$, 24 captive birds are released, after which they produce offspring over the year. After 1 year, 46 dontdonts are recaptured, of which 6 are of the original captive birds. I.e., $\frac{6}{46}$ of the sample are the original captives. So, using the sample as an estimate of the current population of dontdonts, the ratio of "original captives : total population" should be the same for the sample and the population, *N*. I.e., $$6:46 = 24:N$$ Solving gives: $$\frac{6}{46} = \frac{24}{N}$$ $$N = 24 \times \frac{46}{6}$$ $$= \mathbf{184}$$
2. **A Bacterial Buffet**
Each time a membrane is removed, an even number of millions of bacteria are destroyed. Since there are 21 million bacteria, there will always be at least 1 million left at the end. **Therefore it is not possible for all the bacteria to be destroyed.**

3. Going the Way of the Dinosaurs?

Let:

EW = Extinct in the wild
EX = Extinct
CE = Critically endangered
EN = Endangered
VU = Vulnerable
NT = Near threatened
LC = Least concern

Given:

EW = **78**
EX = 11 × EW = 11 × 78 = **858**
CE = 6 × EX = 6 × 858 = **5148**
EN = 9 × EX = 9 × 858 = **7722**
VU + EN + CE = 24570
⇒ VU = 24570 – 5148 – 7722 = **11700**
NT = 1/3 × VU + 468 = (1/3)×11700 + 468 = **4368**
LC = EW + EX + CE + EN + VU + NT + 7098 = **36972**

4. Dizzy Planets

Let:
$\Delta\theta_{\mathrm{v}}$ = be the angle traversed by Venus from its starting point.
$\Delta\theta_{\mathrm{m}}$ = be the angle traversed by Mercury from its starting point.

Given:
ω_{v} = the angular speed (angle traversed in a given time, $\frac{\Delta\theta_{\mathrm{v}}}{\Delta t}$) of Venus = $\frac{360^\circ}{225\ \mathrm{Ed}}$,
ω_{m} = the angular speed (angle traversed in a given time, $\frac{\Delta\theta_{\mathrm{m}}}{\Delta t}$) of Mercury = $\frac{360^\circ}{88\ \mathrm{Ed}}$,
where Ed = Earth days.

Since Mercury travels faster, the planets will be out of alignment immediately after they start – until Mercury catches up Venus again from behind after travelling additional complete cycles. In other words, when they are aligned again, they will be displaced by the same angle from the starting point, only that Mercury would have traversed an extra 360˚.

In fact, they will align whenever the angle-travelled by Mercury is integer-multiples of 360˚ more than the angle-travelled by Venus.

Putting this as an equation, we want:

$$\Delta\theta_{\mathrm{m}} = \Delta\theta_{\mathrm{v}} + 360^\circ\, n$$

where *n* can be any integers.

Rearranging with what we are given:

$$(\omega_{\mathrm{m}} \times \Delta t) = (\omega_{\mathrm{v}} \times \Delta t) + 360^\circ\, n \qquad (1)$$

$$(\omega_{\mathrm{m}} - \omega_{\mathrm{v}})\, \Delta t = 360^\circ\, n \qquad (2)$$

$$\Delta t = \frac{360^\circ\, n}{\omega_{\mathrm{m}} - \omega_{\mathrm{v}}} \qquad (3)$$

$$= \frac{360^\circ\, n}{\left(\frac{360^\circ}{88\ \mathrm{Ed}}\right) - \left(\frac{360^\circ}{225\ \mathrm{Ed}}\right)} \qquad (4)$$

$$= \frac{n}{\left(\frac{1}{88}\right) - \left(\frac{1}{225}\right)}\ \mathrm{Ed} \qquad (5)$$

$$\Delta t = 144.5\, n\ \mathrm{Ed} \qquad (6)$$

So, when *n* = 0, they are aligned in the beginning as $\Delta t = 0$.
When *n* = 1, they are aligned again with Mercury travelling 1 extra cycle, with $\Delta t = 144.5\ \mathrm{Ed}$.
When *n* = 2, they are aligned again with Mercury travelling 1 extra cycle, with $\Delta t = 289.1\ \mathrm{Ed}$.
When *n* = 3, they are aligned again with Mercury travelling 1 extra cycle, with $\Delta t = 433.6\ \mathrm{Ed}$, which is more than a year from the beginning.

∴ The planets will have **2 further alignments**, on the 145th day and 290th day, which is **on 25th May and 17th October**.

*Additional note:
Formally, the “amount of angle traversed” is known as *angular displacement*, in comparison with the concept of *linear displacement*, which refers to the amount of linear distance traversed.

5. Numerous Nuts

Let, A, B, C, D, E be the number of nuts for each pile, and T be the total number of nuts.

Given:
A = B + C + D + E (1)
B = (2/19) × T (2)
C = (1/*x*) × T (3)
D = (1/y) × T (4)
E = (1/z) × T (5)
where *x, y, z* are integers.

(1) implies A = (½)T.
This implies A + B = $\left(\frac{23}{38}\right)$ T , and the remaining C + D + E = $\left(\frac{15}{38}\right)$ T .

If one fraction is $\frac{1}{3}$, the other two fractions must add up to $\left(\frac{7}{114}\right)$. So, it can be:

- $\frac{1}{114}$ and $\frac{6}{114} \equiv \frac{1}{19}$, or
- $\frac{1}{18}$ and $\frac{1}{171}$

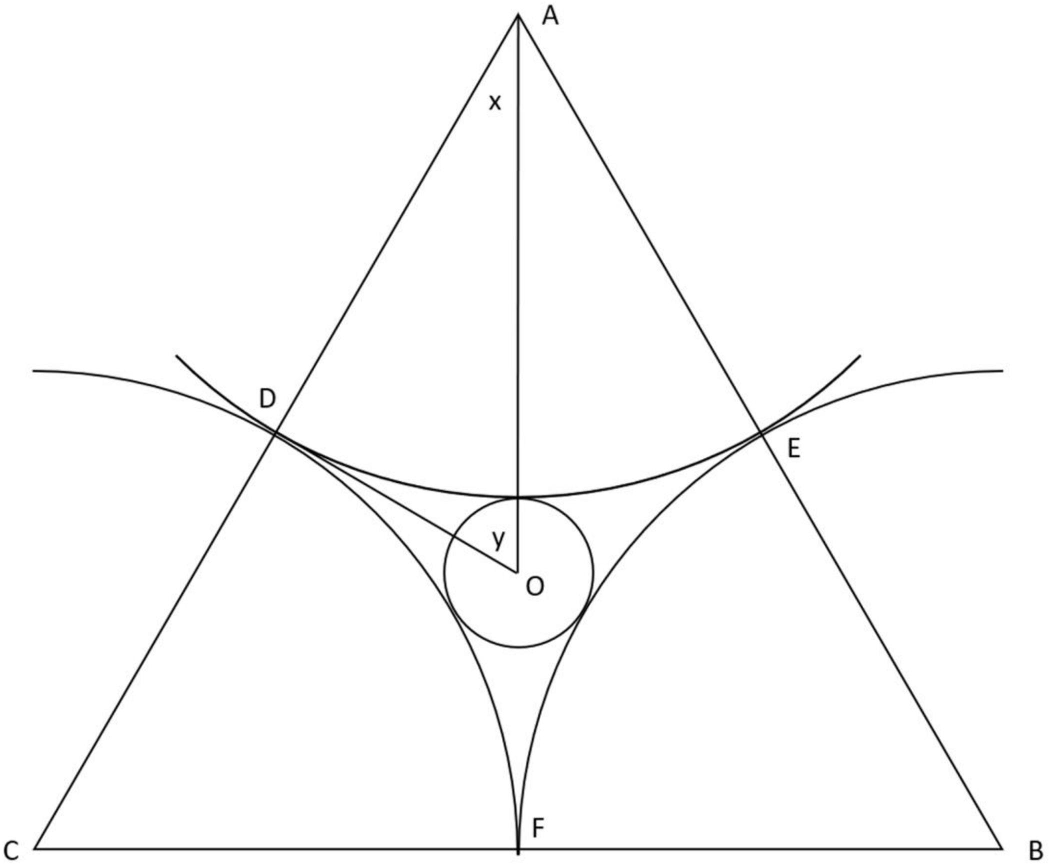
If one fraction is not $\frac{1}{3}$, the only possibility is $\frac{1}{4} + \frac{1}{7} + \frac{1}{532}$.

∴ There are 3 possibilities:

A	B	C	D	E
1/2	2/19	1/3	1/19	1/114
1/2	2/19	1/3	1/18	1/171
1/2	2/19	1/4	1/7	1/532

6. Ewe'll Love this Lawn!

Sketching out the layout



The sheep graze the areas covered by the sectors ADE, BFE, and CDF.

Triangle ADO has a right angle at D, and angles x and y are 30˚ and 60˚ respectively.
This means that the sides DO and AO are in ration 1:2.
Since the triangle ABC has sides of length 20, the **height of the triangle is $\mathbf{10\sqrt{3}}$.**

AO and DO add up to the height, so **AO = $\mathbf{2/3 \times 10\sqrt{3} = 20/\sqrt{3}}$**.
The radius of the flowerbed in the middle is therefore $20/\sqrt{3} - 10 \approx 1.547$, and hence the diameter is $1.547 \times 2 = \mathbf{3.094\ m}$.

The area of the triangle = $20 \times 10\sqrt{3} \div 2 = 100\sqrt{3} \approx \mathbf{173.205\ m^2}$.
The area grazed by the sheep is $(\pi(10)^2 \div 6) \times 3 = 50\pi \approx \mathbf{15.708\ m^2}$.
The area of the flowerbed is $\pi\left(20/\sqrt{3} - 10\right)^2 \approx \mathbf{7.519\ m^2}$.

So the area to be mowed is $100\sqrt{3} - 50\pi - \pi\left(20/\sqrt{3} - 10\right)^2 \approx \mathbf{8.607\ m^2}$.

7. Does a Bear Go Out of the Woods?

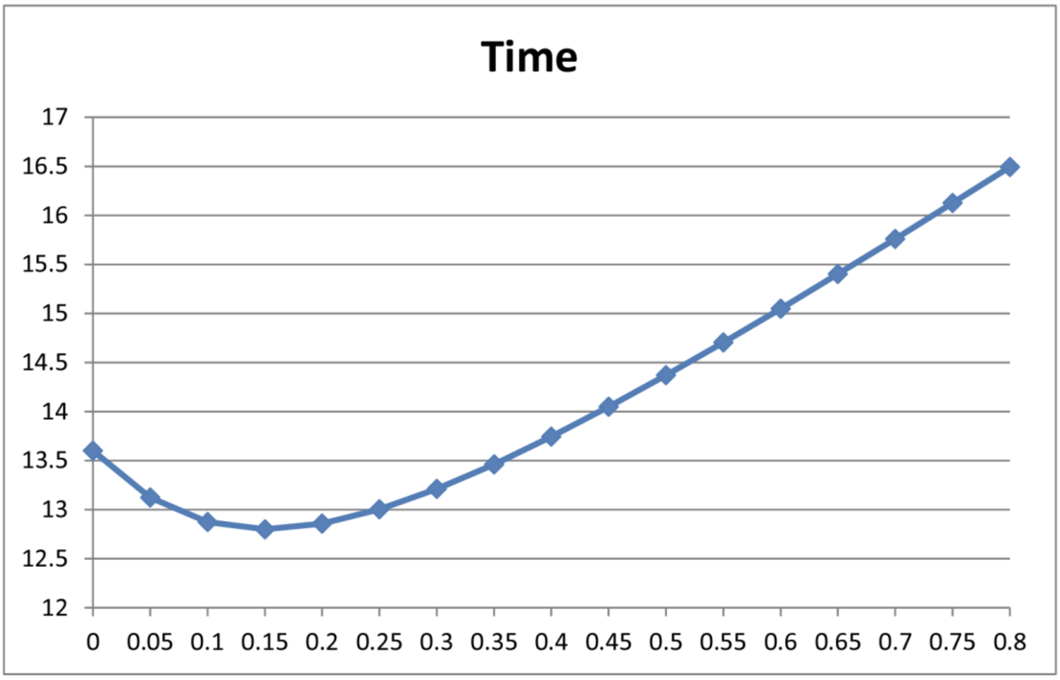
Given:
The distance from their location to P = 0.2 km.
The distance from P to where they emerge is *x* km.

The third side of the right-angled triangle (hypotenuses) is thus $\sqrt{0.2^2 + x^2}$ km.
(I.e., the distance which they amble through the woods).
The distance swum along the river is 0.8 – x.

The total time in minutes is thus

$$60 \times \frac{\sqrt{0.2^2 + x^2}}{3} + 60 \times \frac{0.8 - x}{5}.$$

This simplifies to $20\sqrt{0.2^2 + x^2} + 9.6 - 12x$ minutes.
This yields the following graph:



This shows an optimal distance x as 0.15 km at a time t of 12.8 minutes.

Thus they need to hit the river **0.65 km** from their den D.