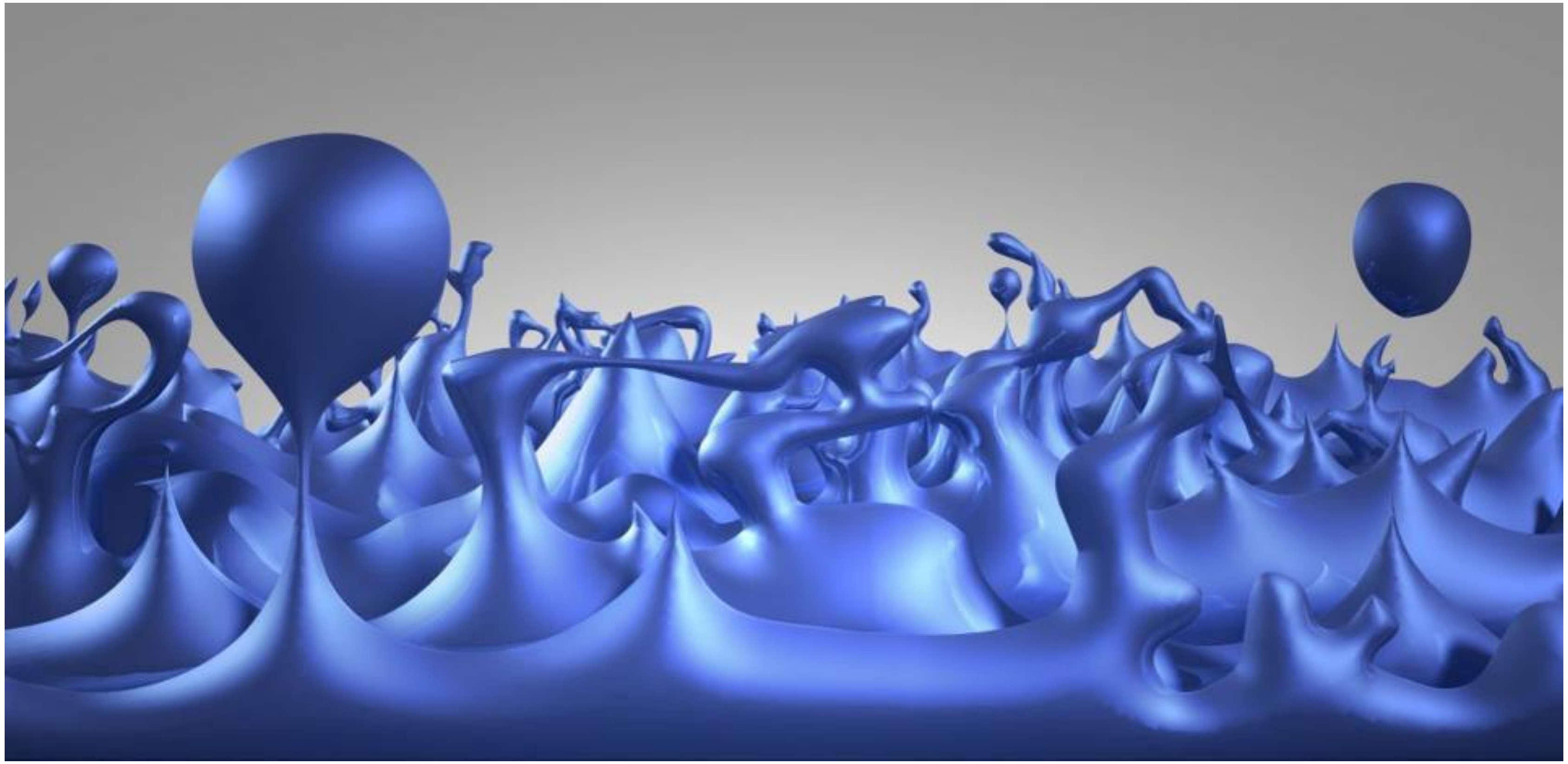
Do Coleman's Euclidean wormholes "exist"?

Thomas Van Riet – Leuven U.



Southampton 2021

Based on

- VR, arXiv:2004.08956

- Katmadas, Trigiante, Ruggeri, VR: 1812.05986 & Astesiano, Trigiante, Ruggeri, VR, arXiv: 2007.XXXX.

- Hertog, Truijen, VR, arXiv: 1811.12690 & Hertog, Maenaut, Tielemans, VR, *in progress*.

Old question: Can we define the ***Euclidean*** path integral with gravity at least semi-classically? Which saddles contribute? Do we include non-trivial topologies?

This talk; for concrete model we try to work towards some answers

It will link Coleman's wormholes to instantons in holography & string theory.

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2. Euclidean stability
3. Interpretation extremality
4. Holography
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General relativity action (M_p=1):

$$S = \int \sqrt{-g}\,(\mathcal{R} + \mathcal{L}_{\mathrm{matter}}) \qquad \& \qquad \mathcal{L}_{\mathrm{matter}} = -\tfrac{1}{12}(F_{\mu\nu\rho}F^{\mu\nu\rho})$$

Ansatz for solution:

$$ds^2 = f(\tau)^2 d\tau^2 + a^2(\tau)^2 d\Omega_3^2$$

$$F_3 = Q\epsilon_3$$

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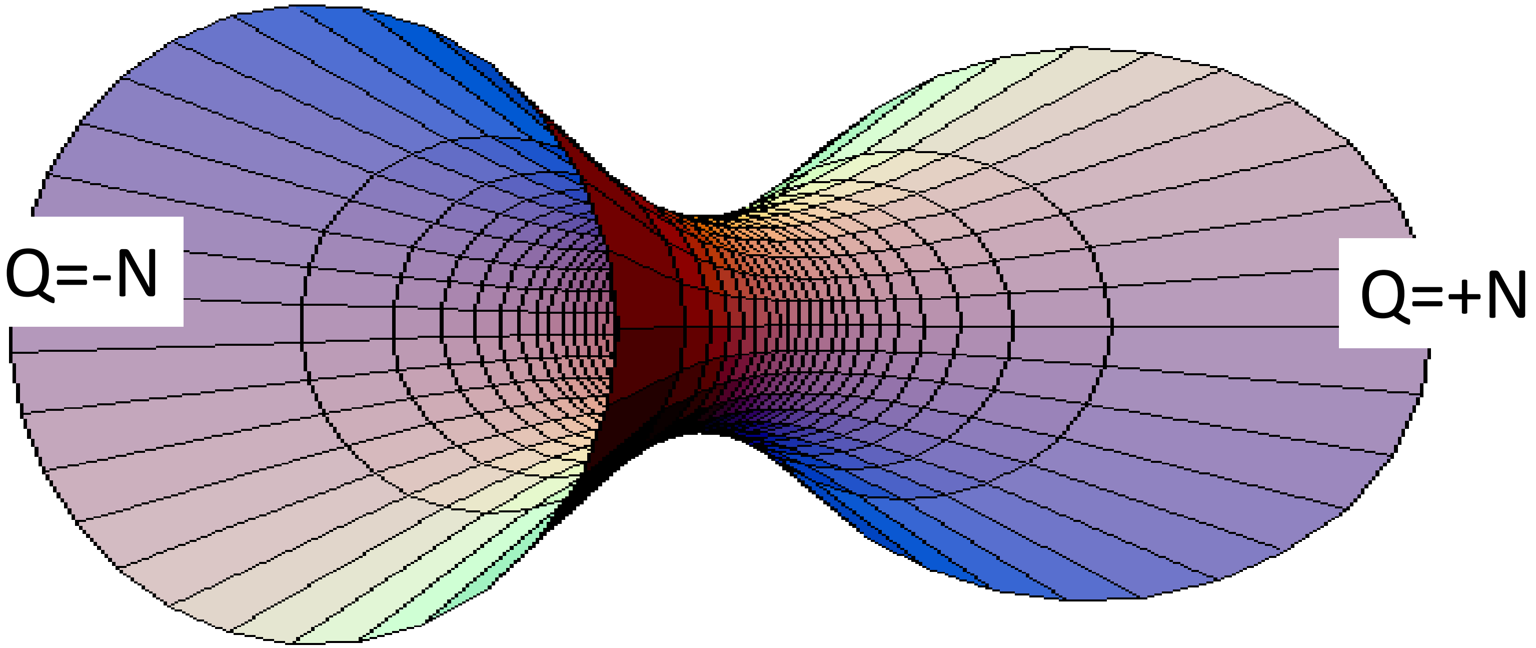
Wormhole? In gauge f=1, a(t) should grow, reach a minimum and then grow again.

Other gauge is easier:

$$ds^2 = \left(1 + \frac{\tau^2}{\ell^2} - \frac{Q^2}{\tau^4}\right)^{-1} d\tau^2 + \tau^2 d\Omega_3^2$$

Where I allowed negative cc: $\Lambda = -6\ell^{-2}$

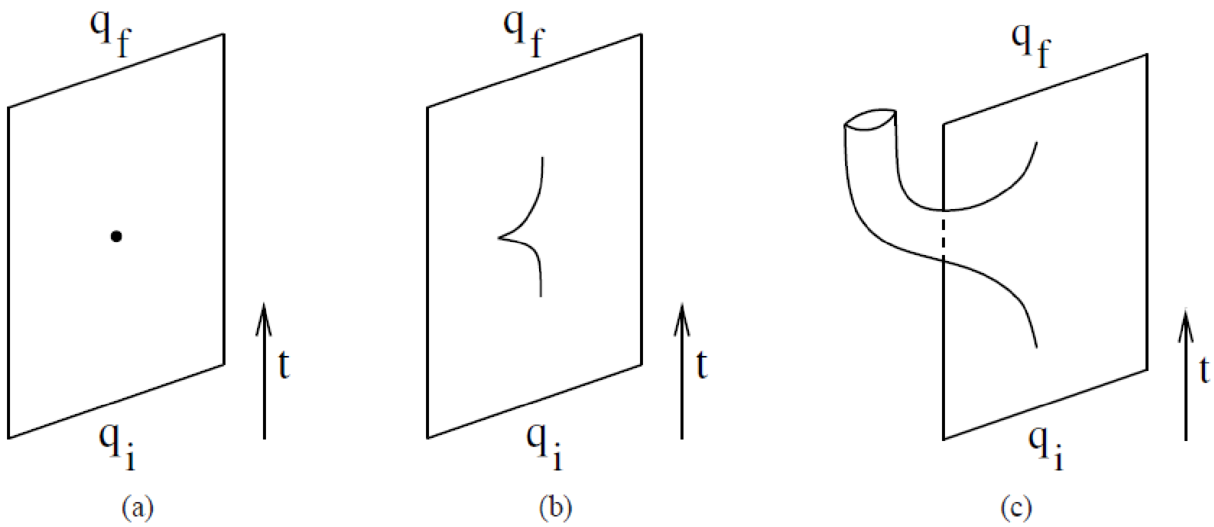


Wormhole is a dipole. There is no monopole axion charge, only locally at one side.

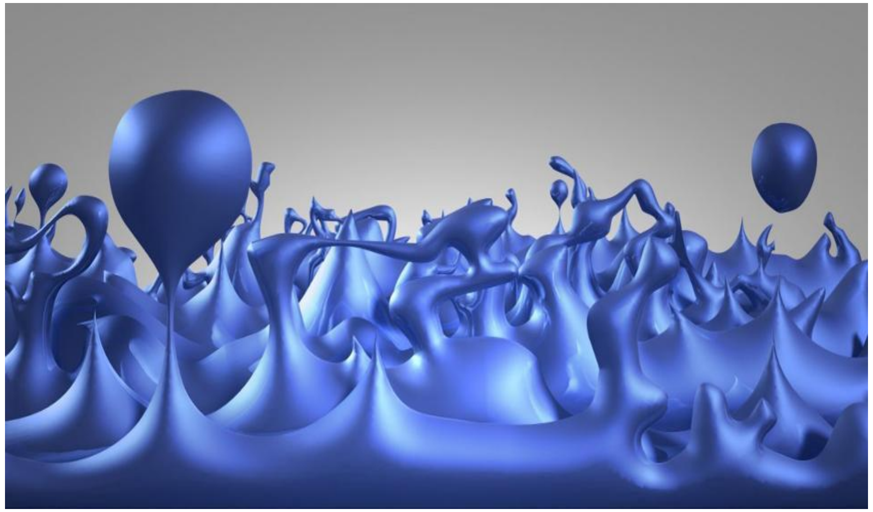
Finite action: $\boxed{S \sim |Q|}$

Very rich and long history in quantum gravity, prior to string theory. Recent revival in string theory due to Swampland discussions & holography. See [Hebecker, Mikhail, Soler 2018] for comprehensive review

Interpretation as tunneling instantons describing nucleation of baby universes → only if cut in half:



[Giddings/Strominger 1987, Lavrelashvili/Tinyakov/Rubakov 1988, Hawking 1987, …]



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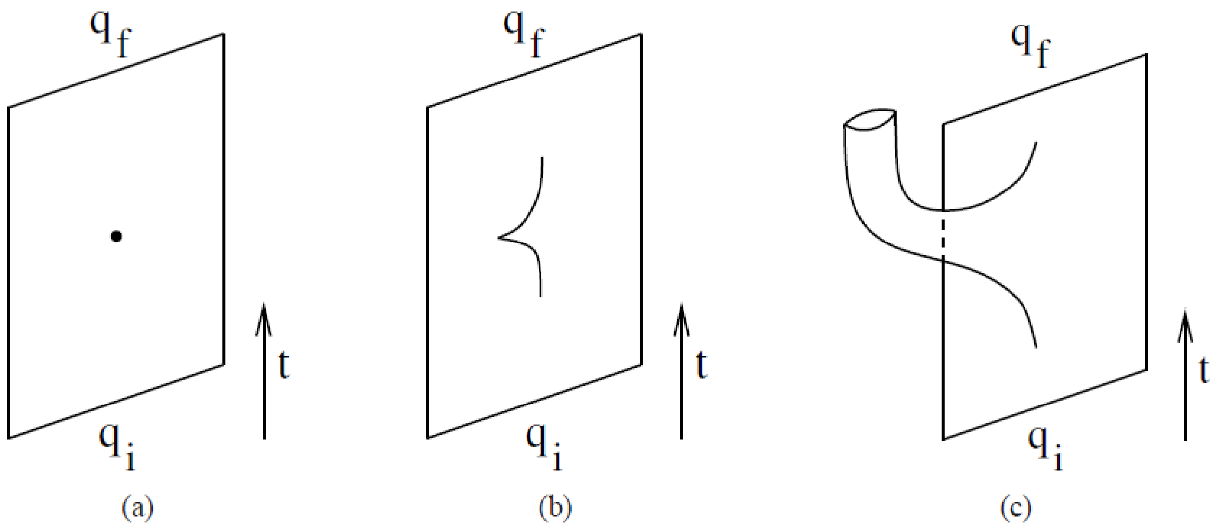

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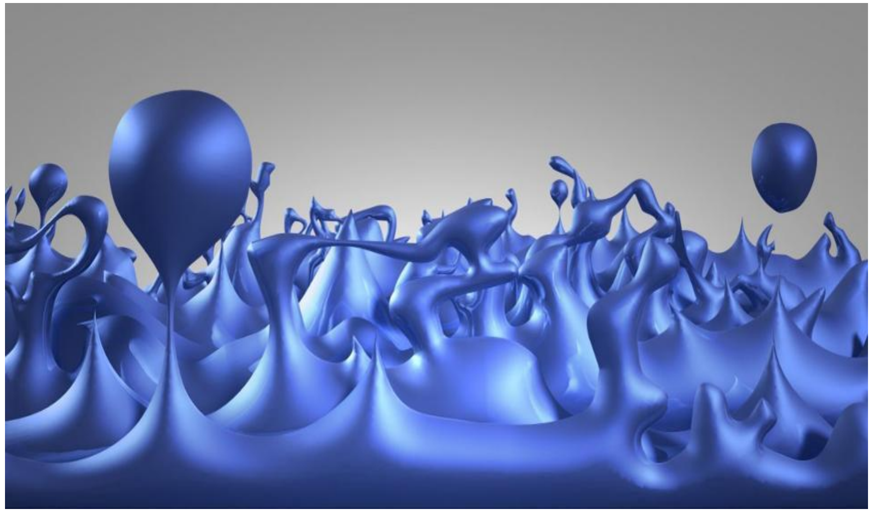

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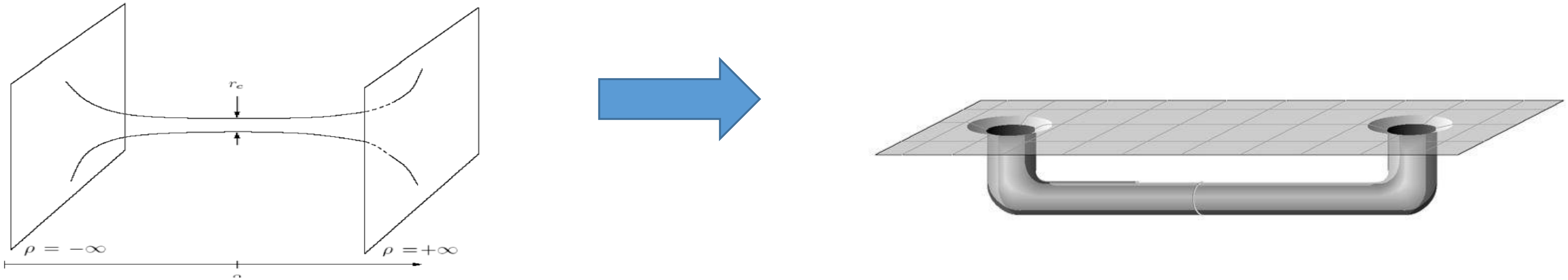
An observer detects a violation of axion charge conservation. (Not surprising since it is global symmetry.) Related phenomenom of ***non-unitarity***.

If one glues the two boundaries into one space-time:



then wormholes represent a breakdown of (macroscopic) locality : the effective action would include operators of the form

$$S_{WH} = -\frac{1}{2}\sum_{IJ}\int d^Dx\, d^Dy\, \mathcal{O}_I(x) C_{IJ} \mathcal{O}_J(y)$$

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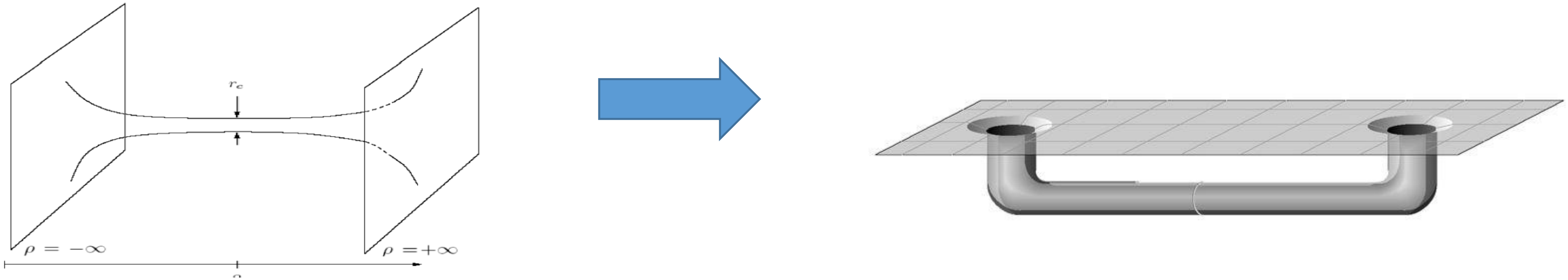

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[Coleman 1998]: Not really since

$$e^{-S_{WH}} = \int d\alpha_I\, e^{-\frac{1}{2}\alpha_I (C^{-1})_{IJ}\alpha_J}\, e^{-\int d^D x \sum_I \alpha_I \mathcal{O}_I(x)} \, .$$

ENSEMBLES

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Interpretation of instantons depends on **stability**

Perform Gaussian approximation around saddle point:

$$Z = e^{-S[\Phi_0]} \int \mathrm{D}\phi \, e^{-\delta^2 S[\Phi_0,\phi] + \mathcal{O}(\phi^3)} \qquad \delta^2 S = \tfrac{1}{2} \int \phi \hat{\mathrm{M}} \phi$$

Solve eigenvalue problem: $$\frac{1}{X} \hat{\mathrm{M}} \phi_n = \lambda_n \phi_n \, , \qquad \int X \, \phi_n \phi_m = \delta_{nm}$$

To find: $$Z \sim e^{-S[\Phi_0]} \int \Pi_n \mathrm{d}z_n \, e^{-\frac{1}{2} \sum_n \lambda_n z_n^2} \sim \frac{e^{-S[\Phi_0]}}{\sqrt{\Pi_n \lambda_n}} \, .$$

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*Coleman: in QM & QFT we have standard instantons (all eigenvalues positive) or "bounces" with **one** negative eigenvalue. The latter describe tunneling amplitudes. **Multiple** negative eigenvalues means instanton is spurious.*

Hodge duality allows us to write action in terms of a pseudo scalar χ

$$S = \int \star R + \tfrac{1}{2}(-1)^t \star d\chi \wedge d\chi \,.$$

Note that hodge duality flips sign kinetic term in Euclidean signature.

So which action is fundamental? The one with positive or negative kinetic term ?

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So which action is fundamental? The one with positive or negative kinetic term ?

→ *Depends on the purpose! If we go to Euclidean signature to compute an instanton effect, it will be the boundary conditions in the path integral who decide. Boundary conditions depend on which effect one wishes to compute (which matrix element).*

We want matrix elements from charge eigenstates = momentum eigenstates

$$|\Pi\rangle = |Q\rangle$$

→ Negative sign action!

- Previous literature = not so careful about boundary conditions, found there is possibly one or no negative eigenmodes [Rubakov 1989, Kim&Lee&Myung 1997, Kim&Kim&Hetrick2003, Alonso&Urbano 2017].

- [Hertog, Truijen, VR 2018] Computations did not use the right gauge-invariant variables + Interpretation as path integral for axion-charge transitions is crucial (boundary conditions).

Taking this into account gives well-behaved quadratic action. *No conformal factor problem (no Hawking-Perry rotation)*! Reason: homogenous modes non-dynamical.

Also, more important, it reveals multiple instabilities!

Use formalism of cosmological perturbation theory [Gratton-Turok 1999]

$$\mathrm{d}s^2 = b^2\Big(-(1+A)^2\mathrm{d}\eta^2 + \partial_i B\,\mathrm{d}x^i\mathrm{d}\rho + [(1-2\psi)\gamma_{ij} + \partial_i\partial_j E]\mathrm{d}x^i\mathrm{d}x^j\Big)\ ,$$

We focus on scalar perturbations and use the following gauge invariant observable:

$$\mathcal{X} = \psi + \frac{b'}{b\chi'}\delta\chi$$

After a mode decomposition and lengthy computation (software) :

$$S_2 = \frac{\mathrm{Vol}(S^3)}{\kappa^2}\int \mathrm{d}\rho\big(A_n\dot{\mathcal{X}}_n^2 - B_n\mathcal{X}_n^2\big)$$

With A_n, B_n certain functions of Euclidean time.

Crucially we need the quadratic action for the momenta instead! Formal manipulation;

$$S_2 = \frac{\mathrm{Vol}(S^3)}{\kappa^2} \int \mathrm{d}\rho \left(-B_n^{-1} (\dot{\Pi}^n_{\mathcal{X}})^2 + A_n^{-1} (\Pi^n_{\mathcal{X}})^2\right).$$

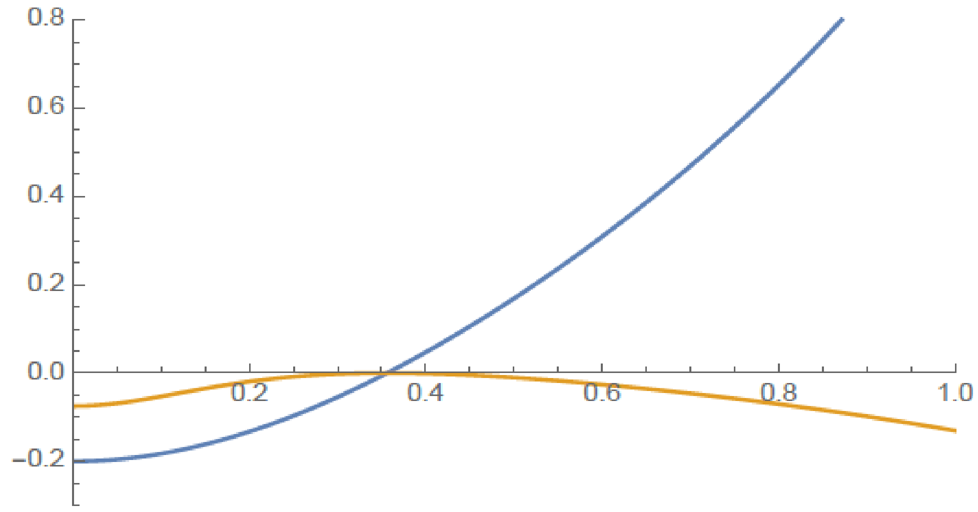
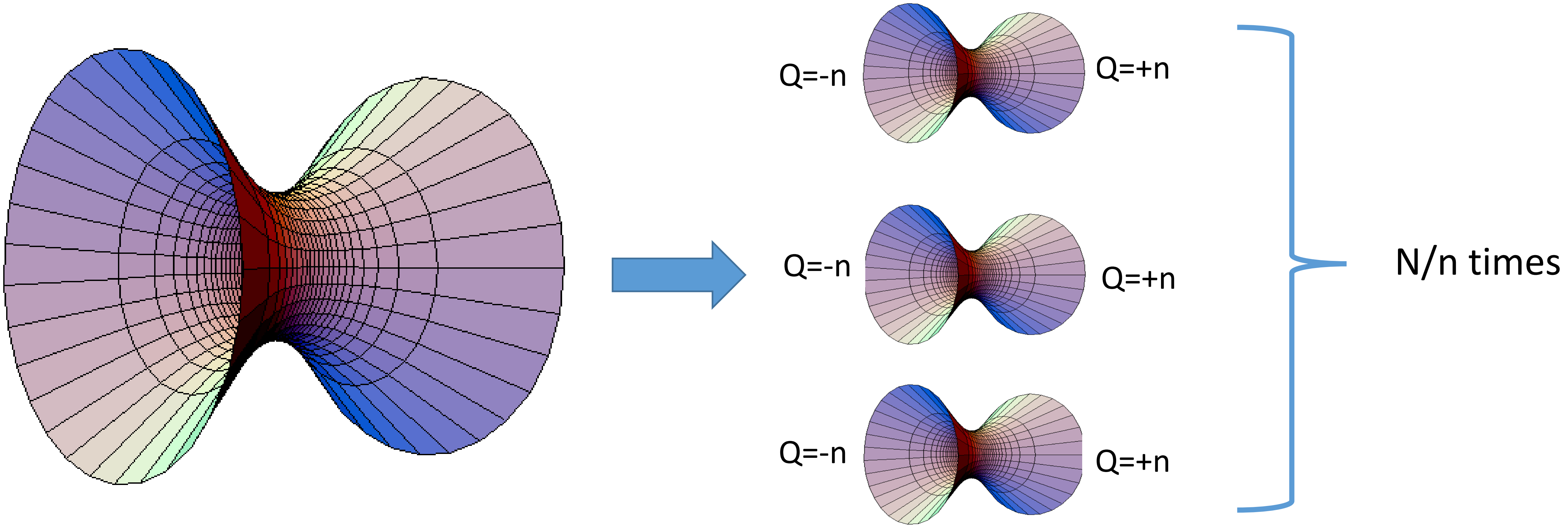


FIG. 1: *The coefficients* A_n^{-1} *(blue) and* B_n^{-1} *(orange) entering in the action for perturbations about axion wormholes, shown here for* $n = 3$ *(and with* $c = 1$*).*

Kinetic term positive. Potential bounded from below and negative only near neck. But enough to find square integrable test functions that lower the total action. Only for n>2.

Infinitely many modes lower the action. All centered close to the neck and probe the non-trivial topology. For very small wormholes those modes become ***sub-planckian***.

→ Macroscopic wormholes do not contribute. There is a lower action saddle with same boundary conditions? Which one? → wormhole fragments into smaller wormholes.



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Recall notion of **extremality** in GR. Charged black holes.

The metric:

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_2^2$$

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r}$$

- *Sub-extremal* : Q < M (eg Schwarzschild)
- *Extremal* : Q = M (Inner and outer horizon coincide)
- *Super-extremal*: Q > M (Naked singularity)

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Extremal solution allows a generalization to multiple centers:

$$ds^2 = -h^{-2}(\vec{x})dt^2 + h^2(\vec{x})\left(d\vec{x}^2\right)$$

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⟺ Gravitational attraction equals coulomb repulsion.

Over-extremal solutions have stronger repulsion than attraction. All charged particles in the Standard Model! But they cannot be seen as black holes. *Microscopic versus macroscopic*. **Same applies to instantons?**

You need some extra bells and whistles. Inspiration from reducing black hole in D+1 dimensions "over time" to instanton in D. The reduction of vector potential gives axion, size of extra dimension gives "saxion":

$$G_{ij}\partial\phi^i\partial\phi^j = (\partial\phi)^2 - e^{b\phi}(\partial\chi)^2$$

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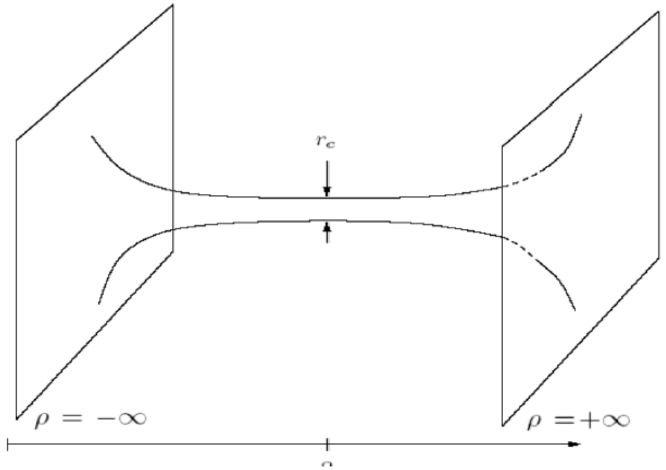
Solutions?

$$ds^2 = \left(1 + \frac{\tau^2}{\ell^2} + \frac{c}{12\tau^4}\right)^{-1} d\tau^2 + \tau^2 d\Omega_3^2\,.$$

Works for ANY sigma model.

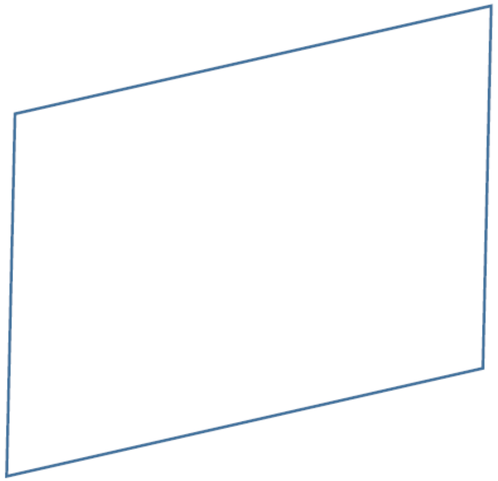
$$\frac{\mathrm{d}^2}{\mathrm{d}h^2}\phi^i + \Gamma^i_{jk}\frac{\mathrm{d}}{\mathrm{d}h}\phi^j\frac{\mathrm{d}}{\mathrm{d}h}\phi^k = 0 \quad \Rightarrow \quad G_{ij}\frac{\mathrm{d}}{\mathrm{d}h}\phi^i\frac{\mathrm{d}}{\mathrm{d}h}\phi^j = c$$

“Over-extremal” c < 0



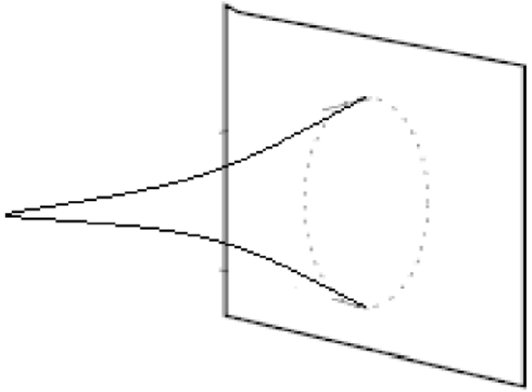
``Time-like” geodesics

“Extremal” c = 0 :



``Light-like’’ geodesics

“Under-extremal” c > 0 :



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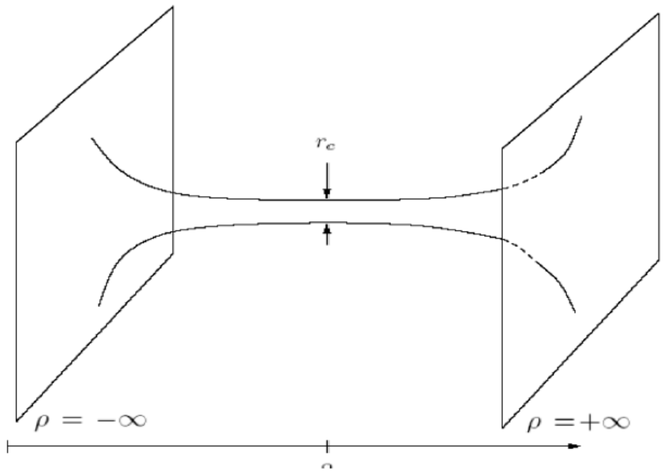

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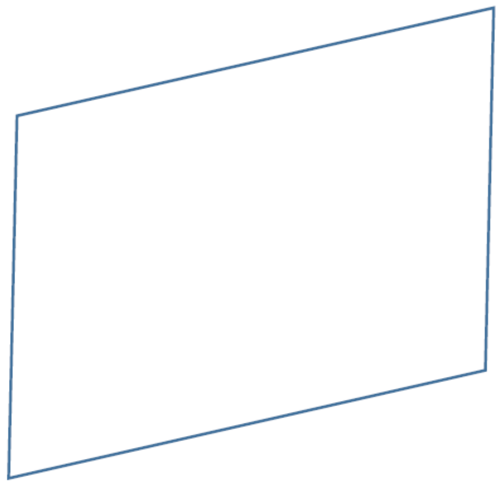

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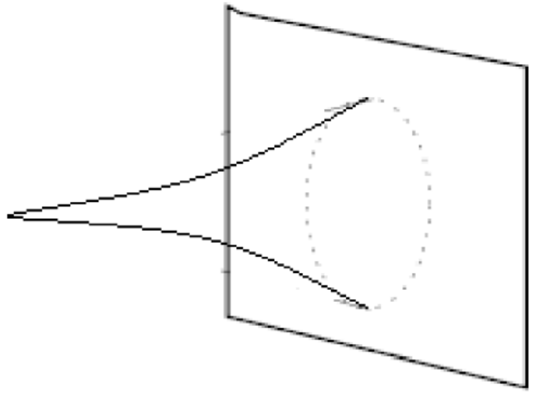

``Space-like” geodesics

→ With the “saxion” we can introduce notion of extremality.

This way embedding in string theory/holography can be made very explicit. [Hertog, Trigiante, VR 2017]

Extremality for instantons, how exactly? [VR, 2019, 2020]

- On-shell action (or direct dimensional reduction of black holes)

$$S \sim |Q| e^{-b\phi(\infty)/2} \sqrt{1 + \frac{c}{Q^2} e^{b\phi(\infty)}},$$

- c=0 allows multi-center extension.
- Probe extremal instantons show "repulsion" away from over-extremal instantons. Wormholes have "positive binding energy".

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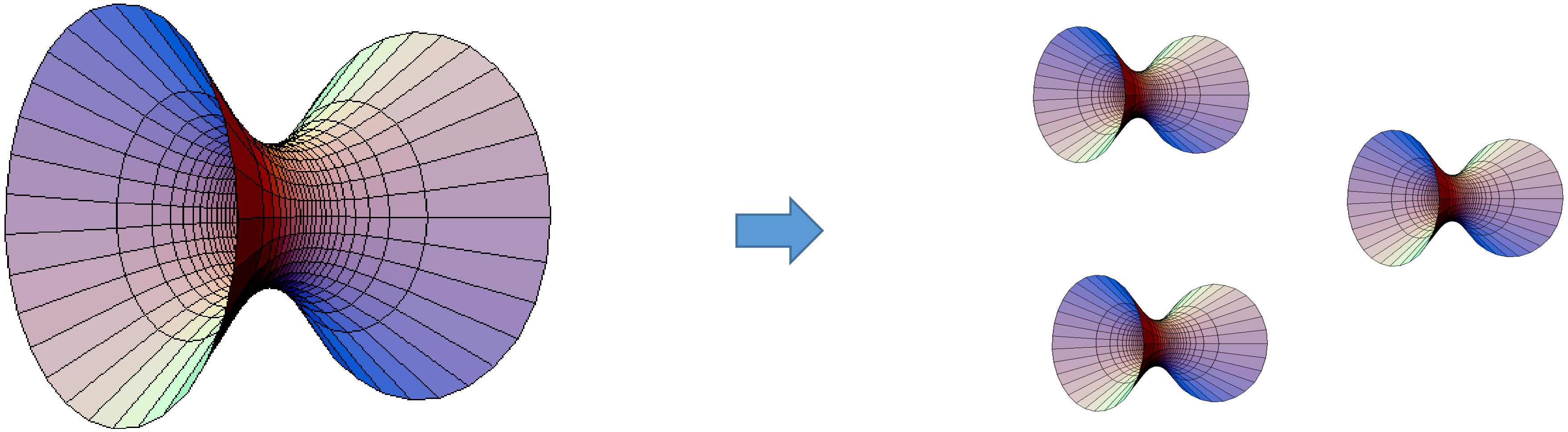
Over-extremal black holes unphysical. Not over-extremal particles. What about over-extremal instantons (Colemans wormholes)? There is no naked singularity to warn us.

It is the instability in the path integral that makes them unphysical. Instability is in non-homogenous sector: signals fragment into smaller pieces to lower action.

Just like super-extremal "black holes" shatter into super-extremal particles that cannot decay anymore. Microscopic over-extremal instantons are physical?

Wormhole fragmentation



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Coleman wormholes have no support from AdS/CFT [Arkani-Hamed/ Orgera/ Polchinski 2007, Maldacena/ Maoz 2004] Dual field theory has no sign of Coleman's α parameters.

Can we say something new? Regard the wormholes as one of the 3 instanton classes by adding saxions (moduli).

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Can we say something new? Regard the wormholes as one of the 3 instanton classes by adding saxions (moduli).

- Moduli inside AdS are coupling constants for exactly marginal operators in the dual field theory: they label the family of CFT's = *conformal manifold.*

- Metric Gij on moduli space corresponds to the `Zamolodchikov' metric gij defined by the two-point functions:

$$g_{ij}(\varphi) = x^{2\Delta} \langle O_i(x) O_j(0) \rangle_{S[\varphi]}$$

- **Holography suggest that some *geodesic curves on the conformal manifold correspond to instantons of the CFT*.**

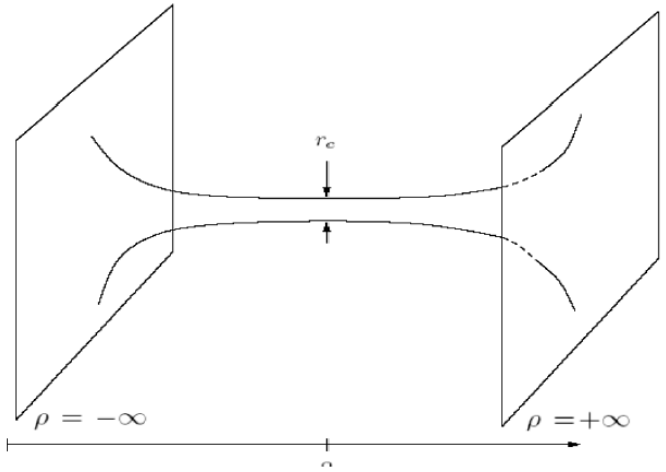
We [Katmadas, Ruggeri, Trigiante, VR, 2018] studied $\mathrm{AdS}_5 \times \mathrm{S}^5/\mathbb{Z}_\mathrm{k}$

Dual theory is N=2 "necklace quiver CFT" [Kachru, Silverstein '98] and has k gauge nodes → hence k complex couplings (k theta-angles), which form the *conformal manifold*.

$$\mathcal{L} \supset \sum_{\alpha=0}^{k-1} \left(-\frac{1}{4g_\alpha^2} \mathrm{Tr}[F_\alpha^2] - i\frac{\theta_\alpha}{32\pi^2} \mathrm{Tr}[F_\alpha \wedge F_\alpha] \right)$$

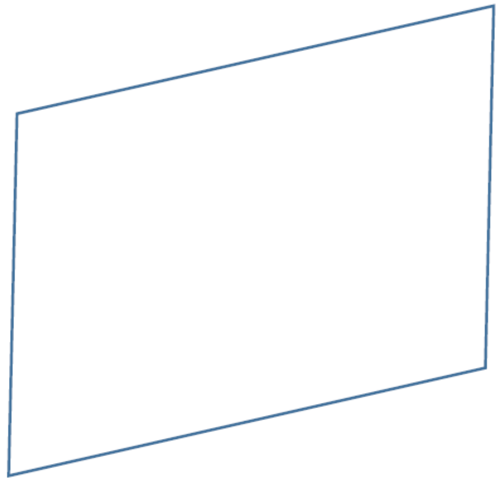
moduli space $\frac{\mathrm{SU}(1,k)}{\mathrm{S}[\mathrm{U}(1)\times\mathrm{U}(\mathrm{k})]} \Longrightarrow \frac{\mathrm{SL}(k+1,\mathbb{R})}{\mathrm{GL}(k,\mathbb{R})}\,.$ 2k real scalars.

“Over-extremal” c < 0



``Time-like” geodesics

“Extremal” c = 0 :



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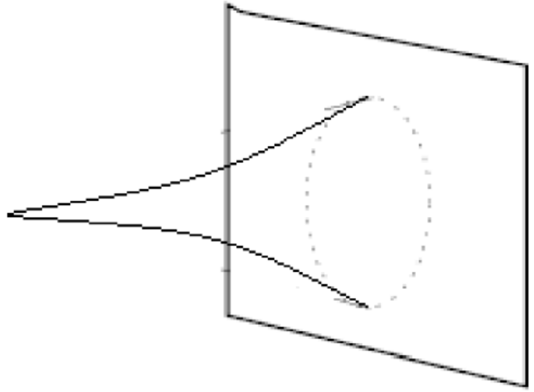


``Space-like” geodesics

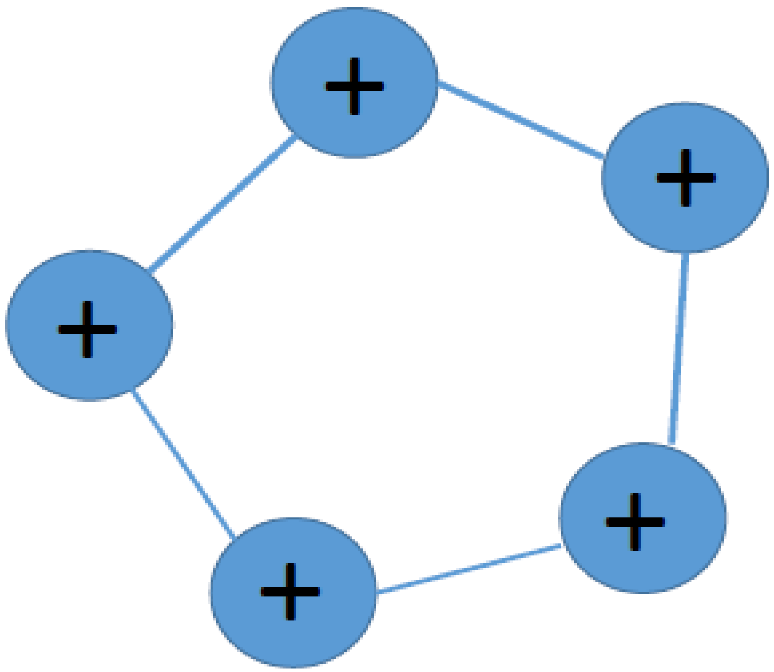
Main results are

- **<u>SUSY solutions</u>** match SUSY gauge theory instantons. (One point functions & on-shell actions)

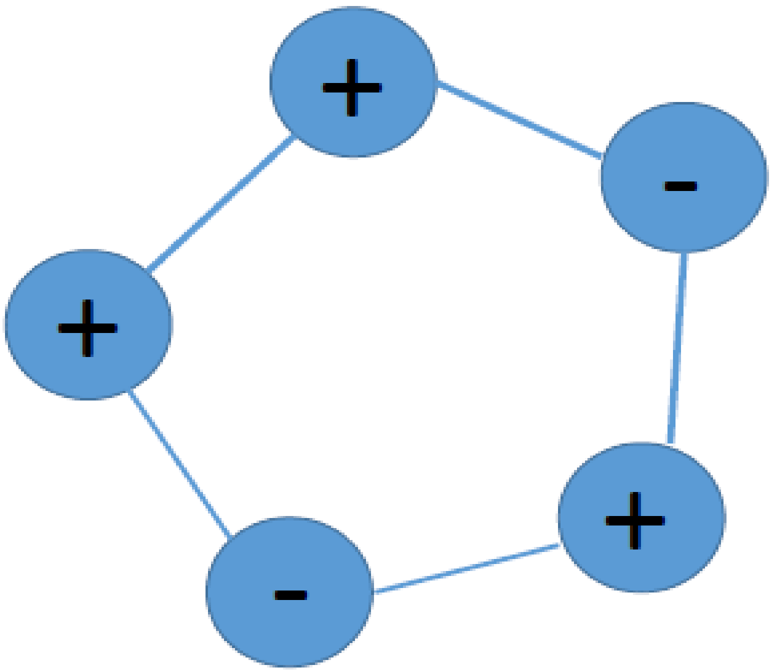
- **<u>non-SUSY solutions but extremal</u>**: Some of them can be interpreted and match so called "quasi-instantons" [Imaanpur 2008]. *These are solutions which are self-dual in each separate gauge node, but orientations differ from node to node. Very simple way of SUSY-breaking!*

$$\mathrm{Tr}[F_\alpha^2] = \mathrm{sign}(N_\alpha)\mathrm{Tr}[F_\alpha \wedge F_\alpha]$$

Potryagin index = axion charge quantum

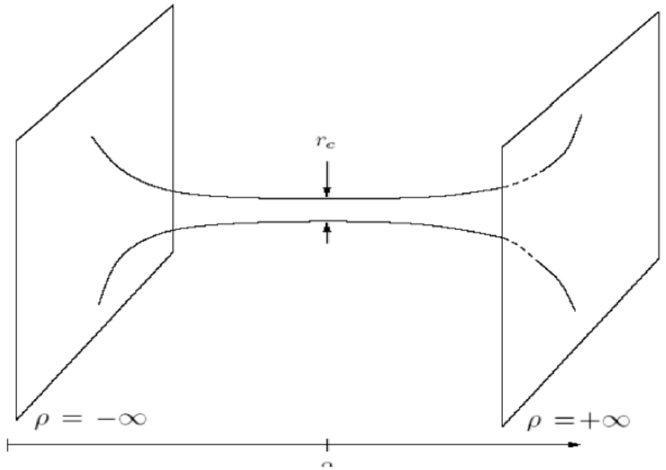


SUSY



Non-SUSY but extremal

“Over-extremal” $c < 0$



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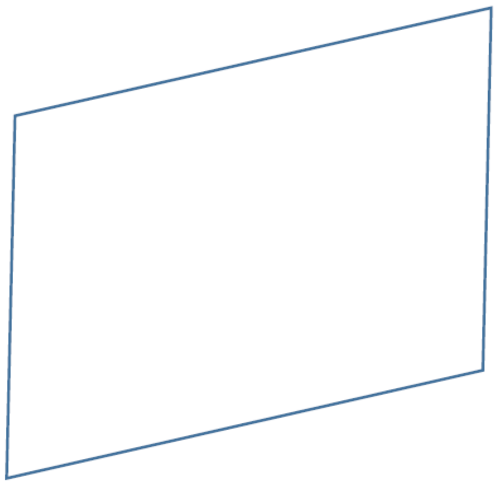

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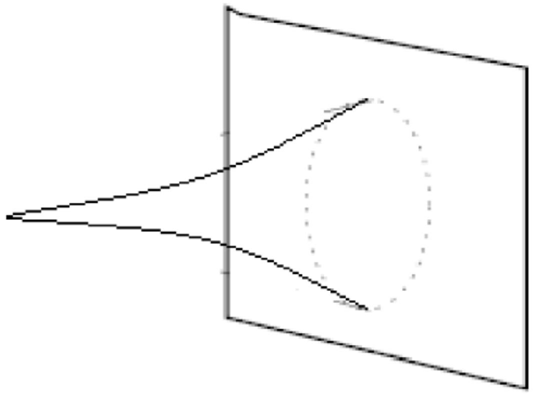

``Space-like” geodesics

- Solution is singular, but singularity seems ok?
- Suggestion for holographic dual from computing one point functions & action.

non-self dual YM instantons…

[Bergshoeff, Collinucci, Ploegh, Vandoren, VR 2005]

$$A_\mu^{\mathrm{SU(N)}} = \begin{pmatrix} A_\mu^{\mathrm{SU}(2)} & 0 & \cdots & 0 \\ 0 & A_\mu^{\mathrm{SU}(2)} & & 0 \\ \vdots & & \ddots & \\ 0 & & & \overline{A}_\mu^{\mathrm{SU}(2)} \end{pmatrix} .$$

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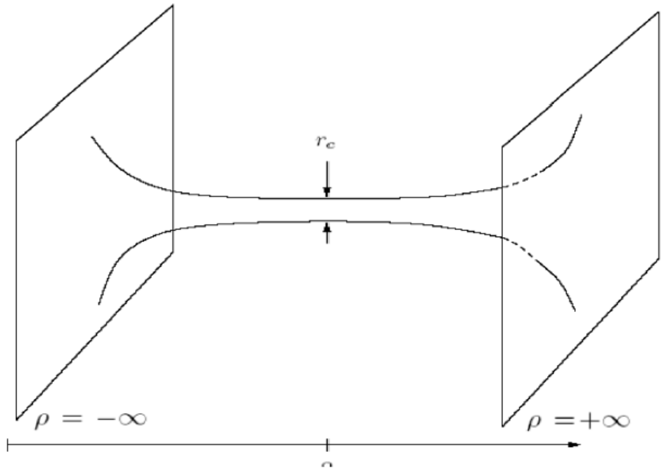

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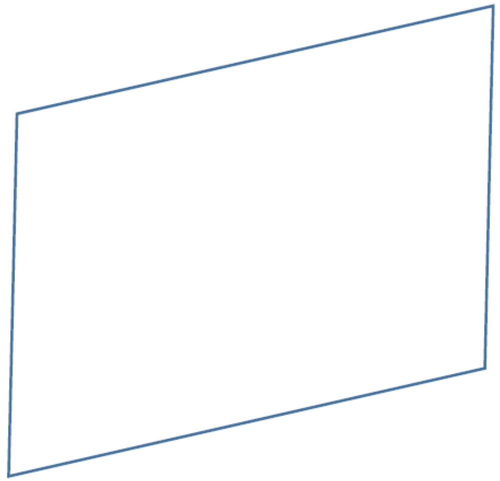
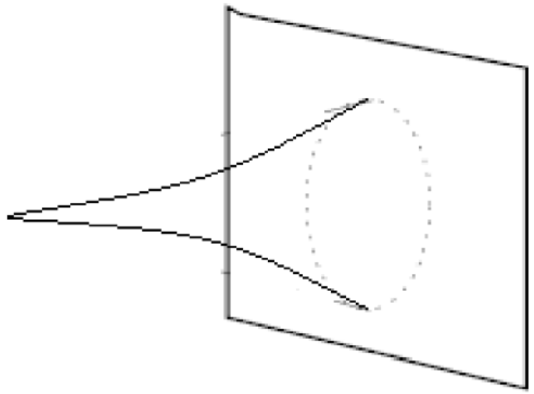


``Light-like” geodesics

“Under-extremal” c > 0 :



``Space-like” geodesics

- First examples of smooth Euclidean axion wormholes in AdS! Despite claim in [Arkani-Hamed &Orgera & Polchinski 2007], **no** smooth examples in D1-D5 system (AdS3xS3xT4 or AdS3xS3xK3) [Astesiano, Trigiante, Ruggeri, VR, in progress.]

- Our explicit embedding provides another paradox: holographic one-point function give violation of positivity [Katmadas, Ruggeri, Trigiante, VR, 2018]:

$$|\mathrm{Tr}[F_\alpha^2]| < |\mathrm{Tr}[F_\alpha \wedge F_\alpha]| \,.$$

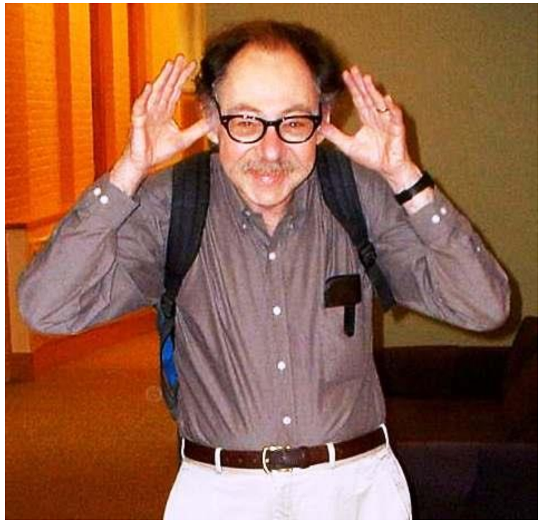
- Field theories without gravity do not allow a notion of super-extremality. BPS bounds cannot be violated. It requires gravity. But AdS gravity = CFT.

→ **Again evidence for spurious nature of wormholes.**

1. Euclidean Wormholes à la Coleman
2. Euclidean stability
3. Interpretation extremality
4. Holography
5. Conclusion

Summary

Are Coleman's Euclidean wormholes real?



Probably not as they are self-repulsive in a Euclidean sense. So macroscopically sized wormholes (bigger than Planck scale) won’t contribute in the path integral. We gave direct evidence through computation. And then a GR-like interpretation plus evidence from holography.

(S. Coleman 1937-2007)

EXTRA

Criticism?

- Kinetic term has a zero. Well-behaved self-adjoint operator only required when computing determinants in case of stability. No reason for “bad instantons” to have nice operator.

- Rotation 1: We Wickrotated cosmological (Lorentzian) perturbation theory. Is there a catch?

- Rotation 2: The axion is known to Wickrotate, but we Wickrotated the gauge invariant combination

$$\chi \to i\chi \qquad \xrightarrow{??} \qquad \mathcal{X} \to i\mathcal{X}$$

Fluctuation theory directly in Euclidean space with form field having normal kinetic term can be shown to yield identical results. [Hertog, Maenaut, Tielemans, VR, in progress]

A relevant technical remark

Consider the 2-form action for the axion A_2 , (with 3 form fieldstrength)

$$S[A] = \int \star R - \frac{1}{2} \star F_3 \wedge F_3$$

We can forget that F_3 comes from A_2 and treat F_3 as fundamental. But then we need a Lagrange multiplier χ that reminds us that F_3 is closed and hence locally exact:

$$S[F_3, \chi] = \int \star R - \frac{1}{2} \star F_3 \wedge F_3 + dF_p \wedge \chi$$

The EOM for χ indeed gives $dF_3=0$. After partial integration, dropping boundary term, we have:

$$S[F_3, \chi] = \int \star R - \frac{1}{2} \star F_3 \wedge F_3 + F_3 \wedge d\chi$$

The EOM for F_3 now is: $\star F_3 = -d\chi$

Boundary conditions

We want matrix elements from charge eigenstates = momentum eigenstates

$$|\Pi\rangle = |Q\rangle$$

So we wish to evaluate

$$K \equiv \langle \Pi_F | \exp(-HT) | \Pi_I \rangle \qquad |\Pi\rangle = \int \mathrm{d}[\chi]\, e^{i\int_\Sigma \chi \Pi} |\chi\rangle$$

$$\Rightarrow \quad K = \int d[\chi] \mathrm{e}^{-\frac{1}{\hbar}\int L[\chi,\partial\chi]} \mathrm{e}^{-i\int_{\Sigma_F} \Pi_F \chi + i\int_{\Sigma_I} \Pi_I \chi} \,.$$

NO BC

Saddles obey: $\quad \Box\chi = 0 \qquad (\star \mathrm{d}\chi - i\Pi)|_{\Sigma_{I,F}} = 0$

→ Dirichlet boundary for momentum.

→"Euclidean free field action with wrong sign kinetic term"

Equivalent to

$$Z = \int_{bc} d[F]d[\chi]\mathrm{e}^{-\frac{1}{\hbar}\int \star F\wedge F + i\chi \mathrm{d}F}$$

$$\boxed{F_{I,F} = \star\Pi_{I,F}}$$

$$
\begin{aligned}
Z &= \int \mathcal{D}q\ exp\left[-\int dt\left(-\frac{A}{2}\dot{q}^2 + \frac{B}{2}q^2\right)\right] \\
&= \int \mathcal{D}q\mathcal{D}p\ exp\left[-\int dt\left(\frac{A^{-1}}{2}(p - A\dot{q}^2) - \frac{A}{2}\dot{q}^2 + \frac{B}{2}q^2\right)\right] \\
&= \int \mathcal{D}q\mathcal{D}p\ exp\left[-\int dt\left(\frac{A^{-1}}{2}p^2 + \dot{p}q + \frac{B}{2}q^2\right)\right] \\
&= \int \mathcal{D}q\mathcal{D}p\ exp\left[-\int dt\left(\frac{A^{-1}}{2}p^2 + \frac{B}{2}(q + B^{-1}\dot{p})^2 - \frac{B^{-1}}{2}\dot{p}^2\right)\right] \\
&= \int \mathcal{D}p\ exp\left[-\int dt\left(-\frac{B^{-1}}{2}\dot{p}^2 + \frac{A^{-1}}{2}p^2\right)\right]
\end{aligned}
$$