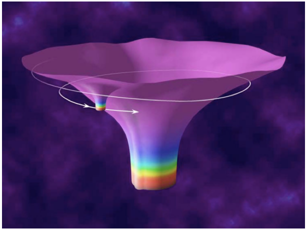
# Self Force Calculations for Binary Black Hole Inspirals



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BritGrav 2012 @ Southampton, 3rd-4th April 2012.

# Talk Outline

- **Motivation:** Black holes, astrophysics and the 2-body problem in relativity.
- **Orbital resonances on Kerr spacetime:** a key challenge.
- **Self-Force on Kerr:** with $m$-mode regularization and 2+1D evolution
- **Progress:** Circular orbits on Schw., first results on Kerr.
- **Low multipoles:** Energy, angular momentum and centre-of-mass.
- **Problem:** Gauge-mode instabilities and their mitigation.
- **Conclusion**.

# Motivation: Astrophysics I

- Supermassive BHs in galactic centres:

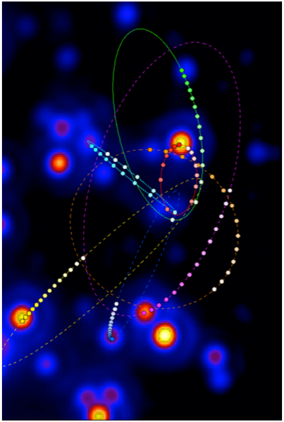


Figure: Orbits in Central Arcsec (Credit: Keck/UCLA)

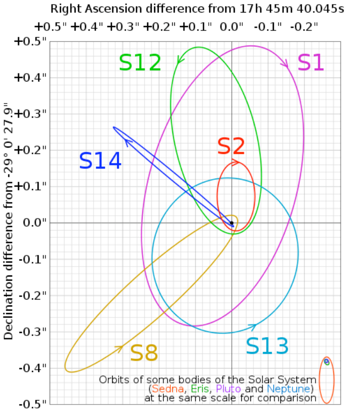
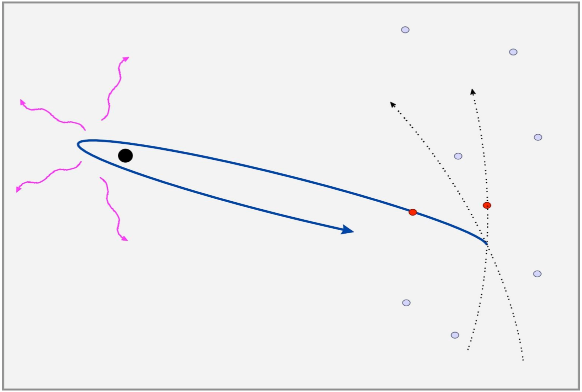



Figure: Eisenhauer *et al.*, Astrophys. J. **628**, 246 (2005)

# Motivation: Astrophysics II

- ‘Cusp’ population of BH and neutron stars in vicinity of SM BH.



# Motivation: Astrophysics III

- Strong but indirect evidence for existence of Gravitational Waves:

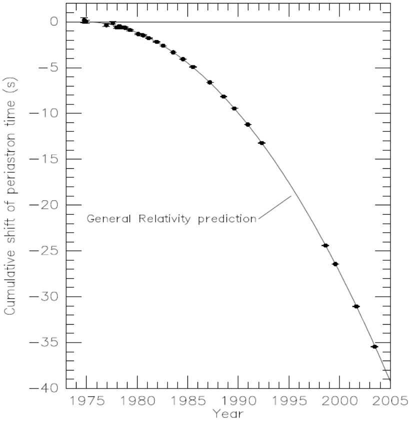
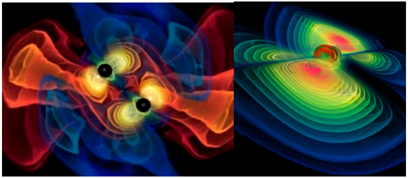
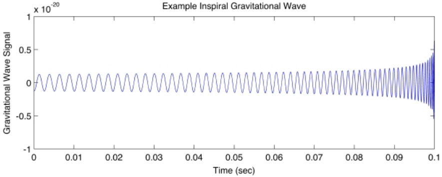


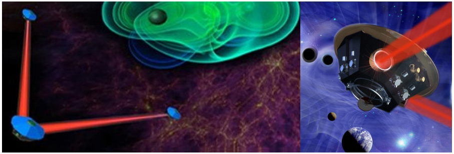

Figure: Three decades of data from the Hulse-Taylor binary pulsar.

# Motivation: Astrophysics IV

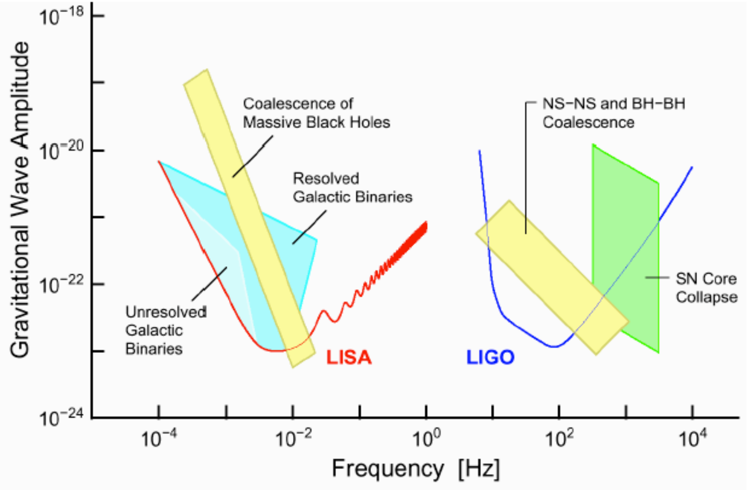
- Bodies in orbit emit GWs
- First GW detection possible within five years
- **2015:** Newly-upgraded ground-based detectors
- **2025:** Space-based mission: eLISA
- Key aim: to map spacetime near event horizons
- Birth of new field: **Multimessenger astronomy**





# Motivation: LISA?



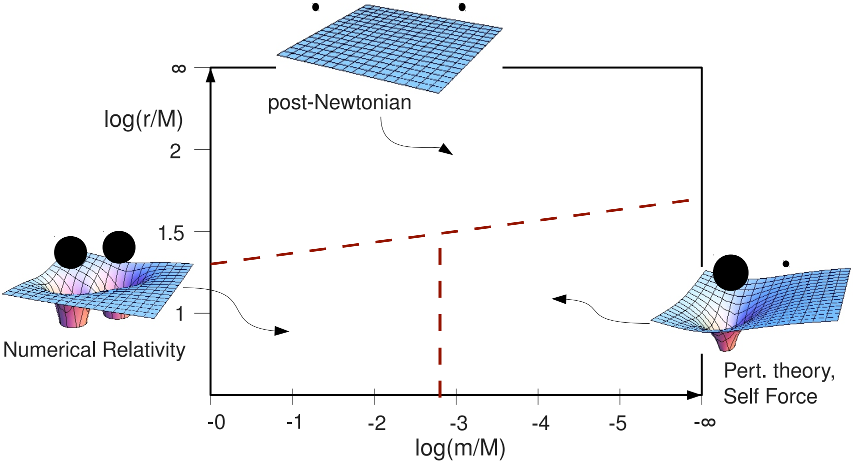
# Motivation: eLISA

## Rescoping exercise for ESA mission

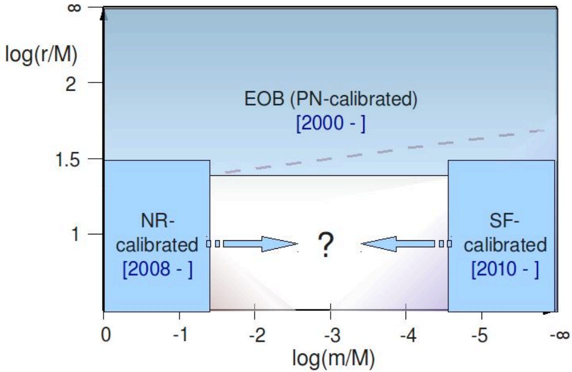
"The new [LISA] configuration should detect thousands of galactic binaries, tens of (super)massive black hole mergers out to a redshift of z=10 and tens of extreme mass ratio inspirals out to a redshift of 1.5 during its two year mission."

Karsten Danzmann, Aug 2011.

# Motivation: the general 2-body problem in relativity



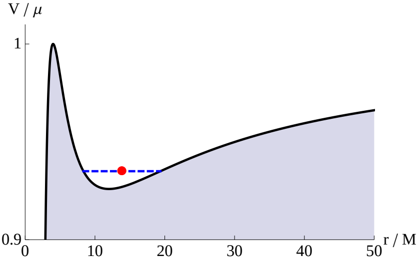
# Motivation: the general 2-body problem in relativity

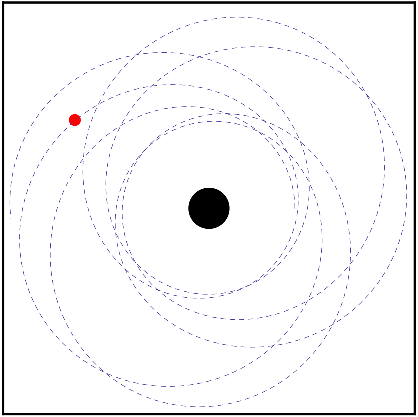



- Effective One-Body (EOB) model (Damour *et al.*) provides a possible analytic fitting framework

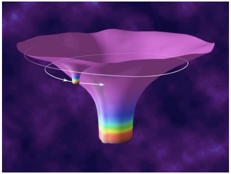
# Gravitational Self Force

- Test bodies ($\mu = 0$) follow geodesics on background spacetime
- Compact bodies ($\mu \neq 0$) are deflected away from test-body geodesics by effect of a 'self-force' $\mathcal{O}(\mu^2)$



# Gravitational Self Force



- **Mass ratio**: $M \gg \mu$ with $\eta \equiv \mu/M \sim 10^{-4} - 10^{-6}$.
- **Perturbation theory**: split into black-hole background + perturbation

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$$

- **Back-reaction**: $h_{\mu\nu} \sim \mathcal{O}(\mu)$ generates back-reaction at $\mathcal{O}(\mu^2)$
- **Self force** w.r.t. background spacetime, $F_\alpha^{\text{self}} \sim \mathcal{O}(\mu^2)$, leading to self-acceleration $a_\alpha \sim \mathcal{O}(\mu)$.
- **Key steps**: Regularization and gauge.

# Gravitational Self Force: Dissipative and Conservative

- Dissipative part $F_\alpha^{\rm diss} \Rightarrow$ secular loss of energy and angular momentum.
- Conservative part $F_\alpha^{\rm cons} \Rightarrow$ shift in orbital parameters, periodic.
- Conservative and dissipative parts of perturbation

$$
\begin{aligned}
h^R_{\rm cons} &= \frac{1}{2}\left(h^R_{\rm ret} + h^R_{\rm adv}\right) = \frac{1}{2}\left(h_{\rm ret} + h_{\rm adv} - 2h^S\right) \\
h^R_{\rm diss} &= \frac{1}{2}\left(h^R_{\rm ret} - h^R_{\rm adv}\right) = \frac{1}{2}\left(h_{\rm ret} - h_{\rm adv}\right)
\end{aligned}
$$

- Dissipative part does not need regularization, get from (e.g.) energy balance arguments.
- Conservative part requires careful regularization.

# Application: Resonances on Kerr (I)

- Two distinct timescales: $\tau_{\rm orb} \sim M \ll \tau_{rad} \sim M/\eta$
- Second-order GSF needed for $x \sim \mathcal{O}(\eta^0)$, as $x \sim (\eta a_0 + \eta^2 a_1)t^2$ where $t_{rad} \sim 1/\eta$.
- Two-timescale expansion using action-angle variables [Hinderer & Flanagan (2010)]:
- Action : 'constants' of motion : $J_\nu = \left(E/\mu, L_z/\mu, Q/\mu^2\right)$
- Angle : 'phase' variables $q_\alpha = (q_t, q_r, q_\theta, q_\phi)$.
- Frequencies $\omega_\alpha(J) = (\omega_r, \omega_\theta, \omega_\phi)$
- Generic orbits on Kerr are ergodic (space-filling)
- $q_r \to q_r + 2\pi$ as orbit goes $r = r_{\rm min} \to r_{\rm max} \to r_{\rm min}$ with period $\tau_r = 2\pi/\omega_r$.
- Isometries of Kerr $\Rightarrow$ $(q_t, q_\phi)$ 'irrelevant', $(q_r, q_\theta)$ 'relevant' params

# Application: Resonances on Kerr (II)

1. Geodesic approximation ($\eta = 0$):

$$\begin{aligned} \frac{dq_\alpha}{d\tau} &= \omega_\alpha(J) \\ \frac{dJ_\nu}{d\tau} &= 0 \end{aligned}$$

Solution :

$$\begin{aligned} q_\alpha(\tau, \eta = 0) &= \omega_\alpha \tau && (1) \\ J_\nu(\tau, \eta = 0) &= \text{const.} && (2) \end{aligned}$$

Timescale : unchanging

# Application: Resonances on Kerr (III)

2. Adiabatic approximation:

$$\begin{aligned} \frac{dq_\alpha}{d\tau} &= \omega_\alpha(J) \\ \frac{dJ_\nu}{d\tau} &= \eta \left\langle G_\nu^{(1)}(q_r, q_\theta, J) \right\rangle \end{aligned}$$

Solution :

$$\begin{aligned} q_\alpha(\tau, \eta) &= \eta^{-1} \hat{q}(\eta\tau) \\ J_\nu(\tau, \eta) &= \hat{J}(\eta\tau) \end{aligned}$$

Timescale : $\tau_{rad.reac.} \sim \eta^{-1}$

# Application: Resonances on Kerr (IV)

3. Post-adiabatic approximation:

$$\begin{aligned}\frac{dq_\alpha}{d\tau} &= \omega_\alpha(J) + \eta g_\alpha^{(1)}(q_r, q_\theta, J) + \mathcal{O}(\eta^2)\\ \frac{dJ_\nu}{d\tau} &= \eta G_\nu^{(1)}(q_r, q_\theta, J) + \eta^2 G_\nu^{(2)}(q_r, q_\theta, J) + \mathcal{O}(\eta^3).\end{aligned}$$

Two timescales : $\sim \eta^{-1}$ (secular) and $\sim 1$ (oscillatory).

# Application: Resonances on Kerr (V)

Key question: Is adiabatic approximation justified? Consider Fourier decomposition

$$G_\nu^{(1)}(q_r, q_\theta, J) = \sum_{k_r, k_\theta} G^{(1)}_{\nu k_r, k_\theta}(J) e^{i(k_r q_r + k_\theta q_\theta)}$$

and $q_r = \omega_r \tau + \dot{\omega}_r \tau^2 + \ldots$, $q_\theta = \omega_\theta \tau + \dot{\omega}_\theta \tau^2 + \ldots$

$$k_r q_r + k_\theta q_\theta = (k_r \omega_r + k_\theta \omega_\theta)\, \tau + (k_r \dot{\omega}_r + k_\theta \dot{\omega}_\theta)\, \tau^2 + \ldots$$

Cannot neglect higher Fourier components when resonance condition is satisfied:

$$k_r \omega_r + k_\theta \omega_\theta = 0 \quad \Rightarrow \quad \omega_r / \omega_\theta = \text{integer ratio}$$

# Application: Resonances on Kerr (VI)

- Duration of resonance set by $(k_r\dot{\omega}_r + k_\theta\dot{\omega}_\theta)\,\tau^2 \sim 1$, i.e.

$$\tau_{\rm res} \sim 1/\sqrt{p\eta}$$

  where $p \equiv |k_r| + |k_\theta|$.
- Net change in 'constants' of motion is

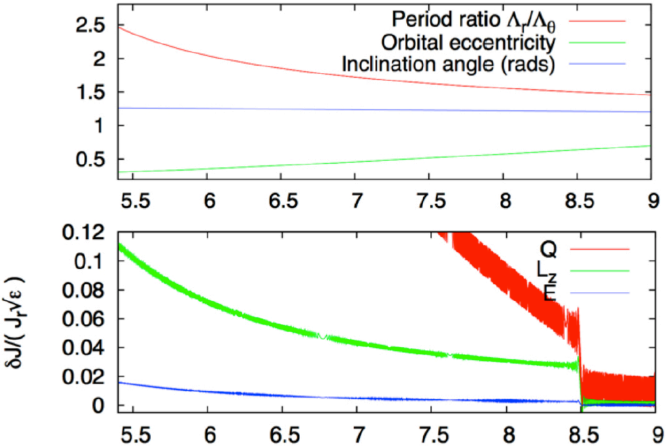
$$\Delta J \sim \sqrt{\eta/p}$$

- Net change in phase is

$$\Delta q \sim 1/\sqrt{\eta p}$$

- Need to compute full 1st-order and dissipative part of 2nd-order GSF on Kerr.
- Without complete knowledge, a resonance effectively resets the phase.

# Application: Resonances on Kerr (VII)




- Credit: Hinderer & Flanagan, arXiv:1009.4923.

# Gravitational Self Force: Formulation

- **Linearized Einstein Eqs:** Ten linear second-order equations with $\delta$-fn source:

$$\Box\bar{h}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} + \hat{\mathcal{B}}_{\mu\nu}[\bar{h}_{\alpha\beta}] = -16\pi T_{\mu\nu} \ \propto \ \delta^4\left[x - z(\tau)\right]$$

- **Gauge choice:** Lorenz-gauge $\bar{h}^{;\nu}_{\mu\nu} = 0$ gives 'symmetric' singularity $h_{\mu\nu} \sim u_\mu u_\nu / r$, and $\hat{\mathcal{B}}_{\mu\nu} = 0 \Rightarrow$ hyperbolic wave equations.

- **Regularization:** split into 'S' and 'R' $h_{\mu\nu} = h^{(S)}_{\mu\nu} + \bar{h}^{(R)}_{\mu\nu}$ [Symmetric/Singular + Radiative/Regular parts]

$$\Box\bar{h}^{(R)}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}^{(R)}_{\alpha\beta} = S^{\text{eff}}_{\mu\nu}$$

- **Self-force:** found from gradient of regularized perturbation

$$F^{\alpha}_{\text{self}} = k^{\alpha\beta\mu\nu}\nabla_{\beta}\bar{h}^{(R)}_{\mu\nu}$$

# Gravitational Self Force: Formulation

- Schw. $\Rightarrow$ separability of equations $\Rightarrow$ $l$-mode regularization $\Rightarrow$ easy!
  - decompose $\bar{h}_{ab}$ in tensor spherical harmonics $Y_{ab}^{lm(i)}$
  - use Lorenz gauge $\nabla^b \bar{h}_{ab} = 0$ with gauge constraint damping
  - solve 1+1D in time domain, or ODEs in freq. domain
  - apply $l$-mode regularization:

$$F_\mu^{\rm self} = \sum_{\ell=0}^{\infty} \left[ F_\mu^{\ell,{\rm ret}} - A(l+1/2) - B - C/(l+1/2) \right] - D$$

# Gravitational Self Force: Formulation

- Kerr $\Rightarrow$ lack of separability ... hard choices ...
  - Teukolksy variables $\Psi_0, \Psi_4$ ...spin-weighted spheroidal harmonics ... metric reconstruction in radiation gauge [Chrzanowski '77] $\rightarrow$ Lorenz gauge? $l = 0, 1$ modes?
  - Hertz potential approach under development by Friedman *et al.*
  - tensor spheroidal harmonics ... [don't exist?]
  - Full 3+1D approach ... expensive!
  - $m$-mode + 2+1D evolution ... practical compromise.
- Proof-of-principle for $m$-mode recently established with scalar-field toy model for circular orbits on Kerr [Dolan & Barack 2011]

$$\Phi_{\mathcal{R}} = \sum_{m=-\infty}^{\infty} \Phi_{\mathcal{R}}^{(m)} e^{im\varphi}, \quad F_\mu^{(m)} = q\partial_r \Phi_{\mathcal{R}}^{(m)}, \quad F_\mu = \sum_{m=-\infty}^{\infty} F_\mu^{(m)}$$

# Gravitational Self Force: Formulation

- Linearized equations:

$$\Delta_L \bar{h}_{ab} \equiv \nabla^c \nabla_c \bar{h}_{ab} + 2\, R^c{}_a{}^d{}_b\, \bar{h}_{cd} + g_{ab} \mathcal{Z}^c_{;c} - \mathcal{Z}_{a;b} - \mathcal{Z}_{b;a} = -16\pi T_{ab}$$

  where

$$\mathcal{Z}^b \equiv \nabla_a \bar{h}^{ab}$$

- Mixed hyperbolic-elliptic type equations.
- Impose Lorenz gauge constraints $\mathcal{Z}_a = 0 \Rightarrow \Box \mathcal{Z}_a = 0$.
- Z4 system: add constraints to linearized equations

$$\Delta_L \bar{h}_{ab} \to \Delta_L \bar{h}_{ab} + \mathcal{Z}_{a;b} + \mathcal{Z}_{b;a} - g_{ab} \mathcal{Z}^c{}_{;c}$$

- How to enforce constraints? Gauge-constraint damping [Gundlach *et al.* '05]

$$\nabla^c \nabla_c \bar{h}_{ab} + 2\, R^c{}_a{}^d{}_b\, \bar{h}_{cd} + n_a \mathcal{Z}_b + n_b \mathcal{Z}_a = -16\pi T_{ab}.$$

# GSF on Kerr

- $m$-mode decomposition:

$$\bar{h}_{ab} = \alpha_{ab}(r,\theta) u_{ab}(r,\theta,t) e^{im\phi}, \qquad \text{(no sum)}$$

- 10 wave equations:

$$\Box_{sc} u_{ab} + \mathcal{M}_{ab}(u_{cd,t}, u_{cd,r_*}, u_{cd,\theta}, u_{cd}) = S_{ab}$$

# 2+1D Wave Equations (Schw.)

$$f\Box_{sc}u_{ab} + \mathcal{M}_{ab}(\dot{u}_{cd,t}, u_{cd,r*}, u_{cd,\theta}, u_{cd}) = 0$$

$$\begin{aligned}
\mathcal{M}_{00} &= \frac{2\left(2r^2(\dot{u}_{01} - u'_{00}) + u_{00} - u_{11}\right)}{r^4} + \frac{4f\,(u_{00} - u_{11})}{r^3} + \frac{2f^2\,(u_{22} + u_{33})}{r^3} \\
\mathcal{M}_{01} &= -\frac{2f^2\,(\cos\theta u_{02} + imu_{03})}{r^2\sin\theta} + \frac{2(\dot{u}_{00} + \dot{u}_{11} - 2u'_{01})}{r^2} - \frac{2f^2(u_{01} + \partial_\theta u_{02})}{r^2} \\
\mathcal{M}_{02} &= -\frac{f(u_{02} + 2im\cos\theta u_{03})}{r^2\sin^2\theta} + \frac{2(\dot{u}_{12} - u'_{02})}{r^2} + \frac{f[(4+r)u_{02} + 2r\partial_\theta u_{01}]}{r^3} - \frac{f^2 u_{02}}{r^2} \\
\mathcal{M}_{03} &= -\frac{f(u_{03} - 2im\cos\theta u_{02})}{r^2\sin^2\theta} + \frac{2fimu_{01}}{r^2\sin\theta} + \frac{2(\dot{u}_{13} - u'_{03})}{r^2} + \frac{f(4+r)u_{03}}{r^3} - \frac{f^2 u_{03}}{r^2} \\
\mathcal{M}_{11} &= -\frac{4f^2(\cos\theta u_{12} + imu_{13})}{r^2\sin\theta} + \frac{2[2r^2(\dot{u}_{01} - u'_{11}) + u_{11} - u_{00}]}{r^4} - \frac{4f(u_{00} - u_{11})}{r^3} \\
&\quad - \frac{2f^2(2ru_{11} + u_{22} + u_{33} + 2r\partial_\theta u_{12})}{r^3} + \frac{2f^3(u_{22} + u_{33})}{r^2} \\
\mathcal{M}_{12} &= -\frac{f(u_{12} + 2im\cos\theta u_{13})}{r^2\sin^2\theta} - \frac{2f^2[\cos\theta(u_{22} - u_{33}) + imu_{23}]}{r^2\sin\theta} + \frac{2(\dot{u}_{02} - u'_{12})}{r^2} \\
&\quad + \frac{f[(4+r)u_{12} + 2r\partial_\theta u_{11}]}{r^3} - \frac{f^2(5u_{12} + 2\partial_\theta u_{22})}{r^2}
\end{aligned}$$

# 2+1D Wave Equations (Schw.)

$$f\Box_{sc}u_{ab} + \mathcal{M}_{ab}(\dot{u}_{cd,t}, u_{cd,r*}, u_{cd,\theta}, u_{cd}) = 0$$

$$\begin{aligned}
\mathcal{M}_{13} &= -\frac{f(u_{13} - 2im\cos\theta u_{12})}{r^2\sin^2\theta} - \frac{2f[2f\cos\theta u_{23} + im(fu_{33} - u_{11})]}{r^2\sin\theta} + \frac{2(\dot{u}_{03} - u'_{13})}{r^2} \\
&\quad + \frac{f(4+r)u_{13}}{r^3} - \frac{f^2(5u_{13} + 2\partial_\theta u_{23})}{r^2} \\
\mathcal{M}_{22} &= -\frac{2f[u_{22} - u_{33} + 2im\cos\theta u_{23}]}{r^2\sin^2\theta} + \frac{2(u_{00} - u_{11})}{r^3} + \frac{2f(u_{11} + u_{22} + 2\partial_\theta u_{12})}{r^2} \\
&\quad - \frac{2f^2(u_{22} + u_{33})}{r^2} \\
\mathcal{M}_{23} &= -\frac{2f[2u_{23} - im\cos\theta(u_{22} - u_{33})]}{r^2\sin^2\theta} - \frac{2f(\cos\theta u_{13} - imu_{12})}{r^2\sin\theta} + \frac{2f(u_{23} + \partial_\theta u_{13})}{r^2} \\
\mathcal{M}_{33} &= \frac{2f(u_{22} - u_{33} + 2im\cos\theta u_{23})}{r^2\sin^2\theta} + \frac{4f(\cos\theta u_{12} + imu_{13})}{r^2\sin\theta} + \frac{2(u_{00} - u_{11})}{r^3} \\
&\quad + \frac{2f(u_{11} + u_{33})}{r^2} - \frac{2f^2(u_{22} + u_{33})}{r^2}.
\end{aligned}$$

# Puncture scheme

- Barack, Golbourn & Sago (2007) give a 2nd-order puncture formulation:

$$\bar{h}^P_{ab}(x) = \frac{\mu}{\epsilon_P^{[2]}}\chi_{ab}, \qquad \chi_{ab} = \left[u_a u_b + (\Gamma^c_{ad} u_b + \Gamma^c_{bd} u_a) u_c \delta x^d\right]_{x=\bar{x}}$$

- For circular orbits in equatorial plane, this reduces to

$$\begin{aligned}
\chi_{00} &= C_{00} + D_{00}\delta r \\
\chi_{01} &= D_{01}\sin\delta\phi \\
\chi_{03} &= C_{03} + D_{03}\delta r \\
\chi_{13} &= D_{13}\sin\delta\phi \\
\chi_{33} &= C_{33} + D_{33}\delta r
\end{aligned}$$

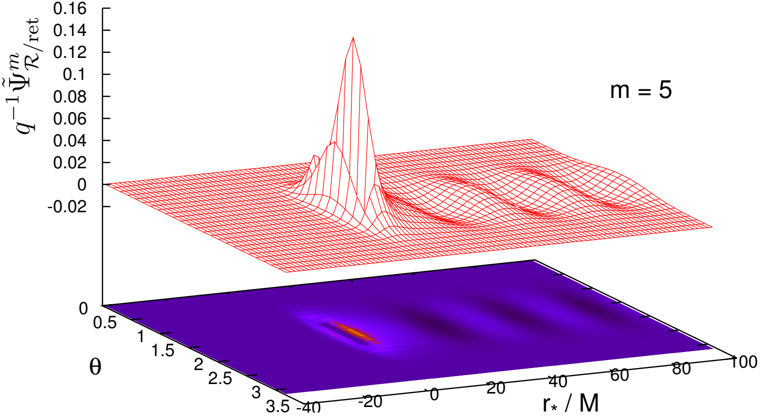
# Puncture scheme

- Effective source: $S^{\text{eff}}_{ab} = -16\pi T_{ab} - \left(\Box \bar{h}^{P}_{ab} + 2R^{c}{}_{a}{}^{d}{}_{b}\bar{h}^{P}_{cd}\right)$
- $m$-mode decomposition: $\bar{h}^{P(m)}_{ab}$ and $S^{\text{eff}(m)}_{ab}$
- Puncture and source found in terms of 'symmetric' elliptic integrals $I^m_1, \ldots, I^m_5$ ...
- ...and antisymmetric integrals $J^m_1, \ldots, J^m_5$ ...

$$\begin{aligned}
\int_{-\pi}^{\pi} \epsilon_P^{-3} \sin\delta\phi \, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i}{B^{3/2}\rho}\left[q^m_{1K}K(i/\rho) + \rho^2 q^m_{1E}E(i/\rho)\right] \\
\int_{-\pi}^{\pi} \epsilon_P^{-3} \sin\delta\phi \, \cos\delta\phi \, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i\gamma}{B^{3/2}}\left[q^m_{2K}K(\gamma) + q^m_{2E}E(\gamma)\right] \\
\int_{-\pi}^{\pi} \epsilon_P^{-5} \sin\delta\phi \, \cos^2(\delta\phi/2) \, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i\gamma}{B^{5/2}}\left[q^m_{3K}K(\gamma) + \rho^{-2} q^m_{3E}E(\gamma)\right] \\
\int_{-\pi}^{\pi} \epsilon_P^{-5} \sin\delta\phi \, \sin^2(\delta\phi) \, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i}{B^{5/2}\rho}\left[q^m_{4K}K(i/\rho) + \rho^2 q^m_{4E}E(i/\rho)\right] \\
\int_{-\pi}^{\pi} \epsilon_P^{-5} \sin\delta\phi \, \sin^2(\delta\phi/2) \, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i\gamma^2}{B^{5/2}\rho}\left[q^m_{5K}K(i/\rho) + \rho^2 q^m_{5E}E(i/\rho)\right]
\end{aligned}$$

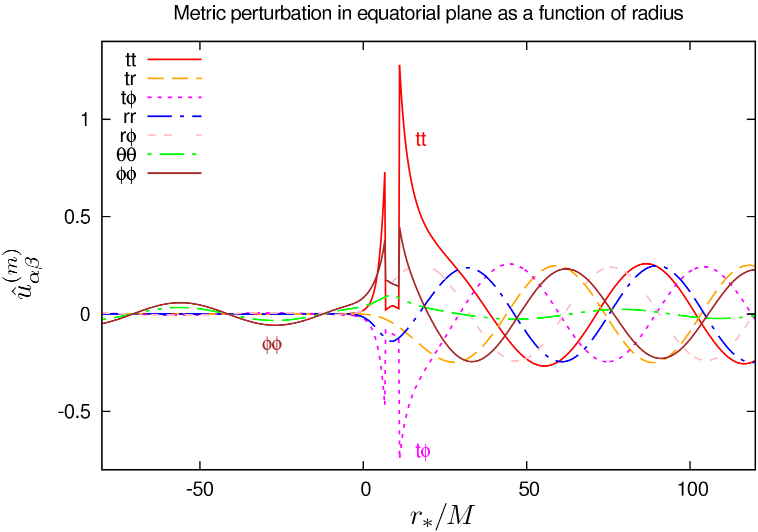
- Wardell and co. developing a 4th-order scheme

# Example data



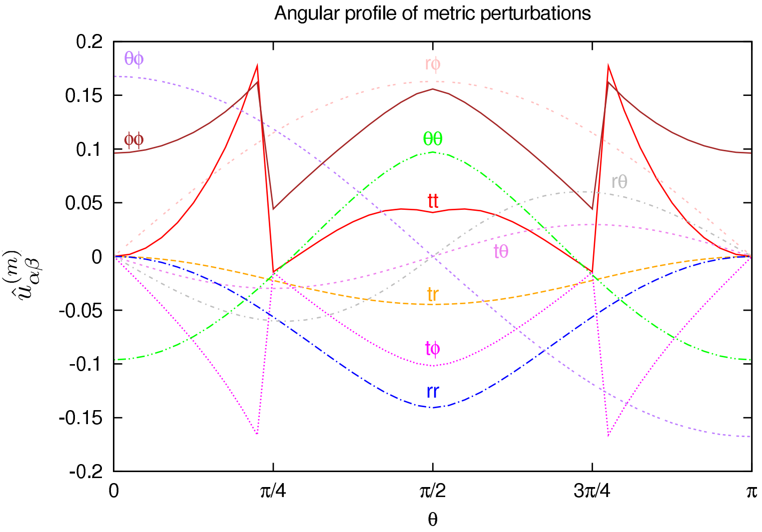
# Example data: $m = 2$ mode of metric perturbation

- Slice (i): $\theta = \pi/2, t = t_f$



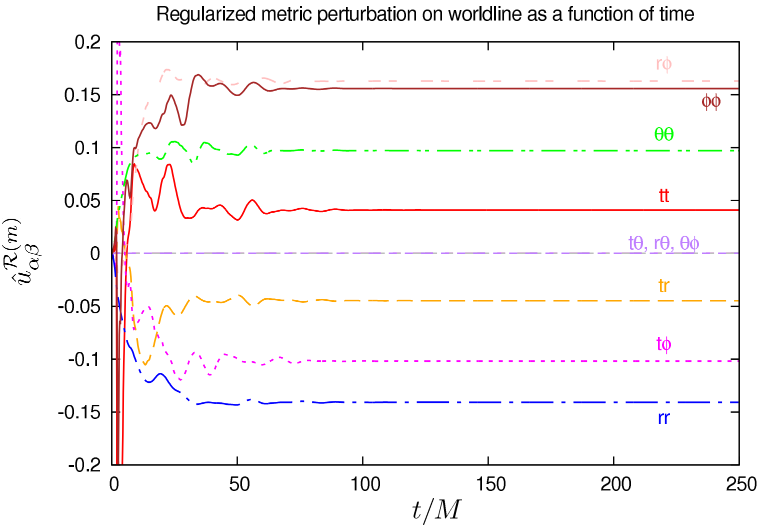
# Example data: $m = 2$ mode of metric perturbation

- Slice (ii): $r = r_0, t = t_f$



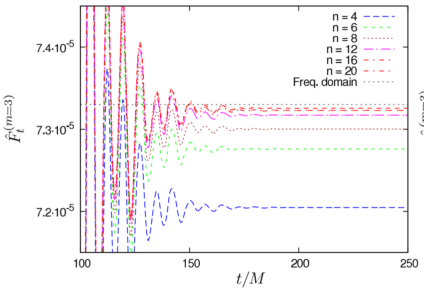
# Example data: $m = 2$ mode of metric perturbation

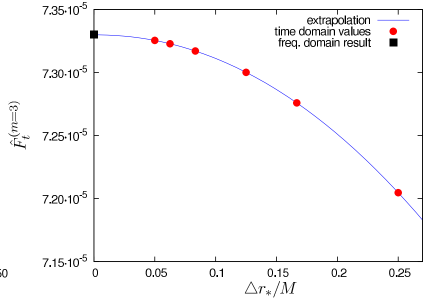
- Slice (iii): $\theta = \pi/2, r = r_0$



# Results: Richardson Extrapolation

- Numerical data depends on grid resolution, but scaling is well-understood ⇒ extrapolate to infinite resolution.



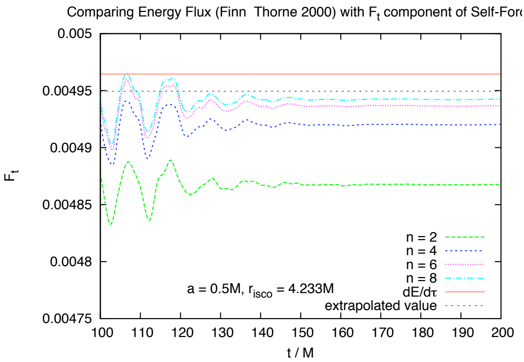



# Results: GSF for circular orbits on Schwarzschild

| Time-domain GSF results | | | |
|---|---|---|---|
| | $\tilde{H} \equiv \frac{1}{2} h_{\alpha\beta} u^\alpha u^\beta$ | $(M^2/\mu^2) F_t^{\rm self}$ | $(M^2/\mu^2) F_r^{\rm self}$ |
| $r_0 = 6M$ | $-5.2355(6)$<br>$-5.23602$ $\times 10^{-1}$ | $1.3299(1)$<br>$1.32984$ $\times 10^{-3}$ | $3.6695(7)$<br>$3.66992$ $\times 10$ |
| $r_0 = 7M$ | $-4.0314(4)$<br>$-4.03177$ $\times 10^{-1}$ | $5.293(2)$<br>$5.29358$ $\times 10^{-4}$ | $3.0096(3)$<br>$3.00985$ $\times 10$ |
| $r_0 = 8M$ | $-3.3022(3)$<br>$-3.30239$ $\times 10^{-1}$ | $2.482(2)$<br>$2.48055$ $\times 10^{-4}$ | $2.4475(4)$<br>$2.44769$ $\times 10$ |
| $r_0 = 10M$ | $-2.4462(6)$<br>$-2.44630$ $\times 10^{-1}$ | $7.348(7)^*$<br>$7.35254$ $\times 10^{-5}$ | $1.6736(2)$<br>$1.67369$ $\times 10$ |
| $r_0 = 14M$ | $-1.6270(4)$<br>$-1.62705$ $\times 10^{-1}$ | $1.2583(9)^*$<br>$1.25872$ $\times 10^{-5}$ | $9.0685(3)$<br>$9.06858$ $\times 10$ |
| $r_0 = 20M$ | $-1.0889(3)$<br>$-1.08893$ $\times 10^{-1}$ | $2.033(4)^*$<br>$2.02994$ $\times 10^{-6}$ | $4.620(1)$<br>$4.61896$ $\times 10$ |

# First validations of Kerr code

## Test of Energy Balance for $m = 2$ Mode




- Non-smooth 2nd-order puncture ⇒ $x^2 \ln x$ convergence.
- 0.3% disagreement here ... because Finn & Thorne (2000) give $\dot{E}_\infty$, whereas $F_t = \dot{E}_\infty + \dot{E}_{hor}$.
- Superradiance ⇒ $\dot{E}_{hor}$ is negative for $m = 2$ (but small).
- Full result in excellent agreement

# Low Multipoles

$m = 0$ and $m = 1$ modes require careful treatment:

- Contain non-radiative d.o.f: energy, angular momentum and centre-of-mass
- Exhibit gauge-mode instabilities

# Conserved quantities in non-radiative multipoles (I)

- Linearized equation: $\hat{\mathcal{W}}_{ab}[\bar{h}_{cd}] = -16\pi T_{ab}$
- Symmetries: Background spacetime has Killing vectors $X_a$ : $X_{(a;b)} = 0$
- Stress-energy is conserved, $\nabla_a T^{ab} = 0$, so we can construct a conserved vector:

$$j^a \equiv T^{ab} X_b \quad \Rightarrow \quad j^a{}_{;a} = 0.$$

- The vector $j_a = (-16\pi)^{-1} \mathcal{W}_{ab} X^b$ can be written

$$j^a = \nabla_b F^{ab}, \qquad \text{where} \quad F_{ab} = -F_{ba}$$

- i.e. the divergence of an antisymmetric tensor (2-form) $F_{ab}$,

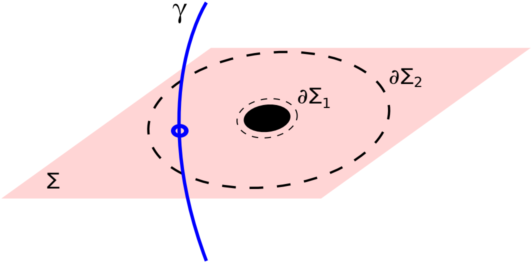
$$\boxed{(-8\pi) F_{ab} = X^c \bar{h}_{c[a;b]} + X^c{}_{;[a} \bar{h}_{b]c} + X_{[a} \mathcal{Z}_{b]}}$$

- Apply Stokes' theorem $\Rightarrow$ Conserved integrals on two-surfaces

# Conserved quantities in non-radiative multipoles (II)

- Stokes' theorem ($j^a = F^{ab}{}_{;b}$):

$$\int_{\Sigma} F^{ab}{}_{;b} d\Sigma_a = \left[ \frac{1}{2} \int_{\partial\Sigma} F^{ab} d\Sigma_{ab} \right]_{\partial\Sigma_1}^{\partial\Sigma_2}$$



# Conservation Law (III)

- Integrate on constant-$t$ hypersurfaces, on concentric spheres:
- $X_a^{(t)} \Rightarrow$ Energy $\mathcal{E}$, $\quad X_a^{(\phi)} \Rightarrow$ Ang. Mom. $\mathcal{L}_z$ in perturbation
- Ang. mom. in $l=1$ odd-parity sector, energy is in monopole ($l=0$),

$$4\pi \left[ r^2 F_{01}^{(t)} \right]_{r_1}^{r_2} = \begin{cases} \mathcal{E} \equiv -u_t, & r_1 < r_0 < r_2, \\ 0, & \text{otherwise.} \end{cases}$$

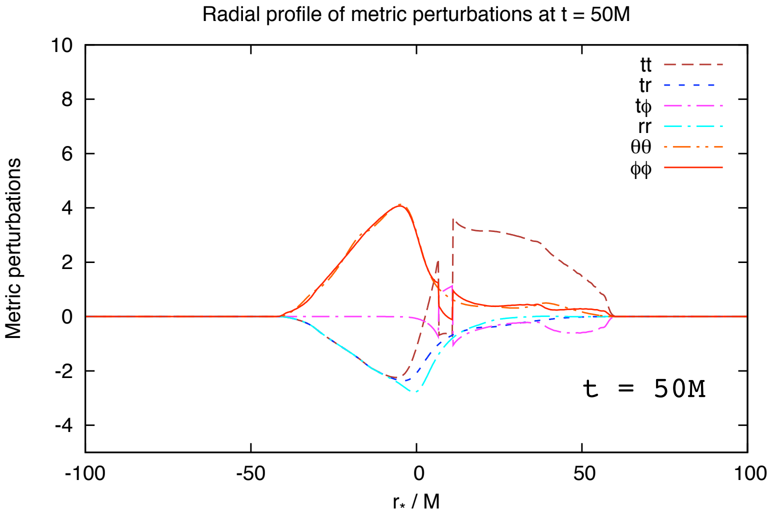
- Locally conserved quantity in monopole ($l=m=0$) equations:

$$-\tfrac{1}{4}\left( r^2 \left( \bar{h}_{tt,r} - \bar{h}_{tr,t} \right) - 2f^{-1}\bar{h}_{tt} + 2f\bar{h}_{rr} \right) = \begin{cases} \mathcal{E}, & r > r_0, \\ 0, & r < r_0. \end{cases}$$

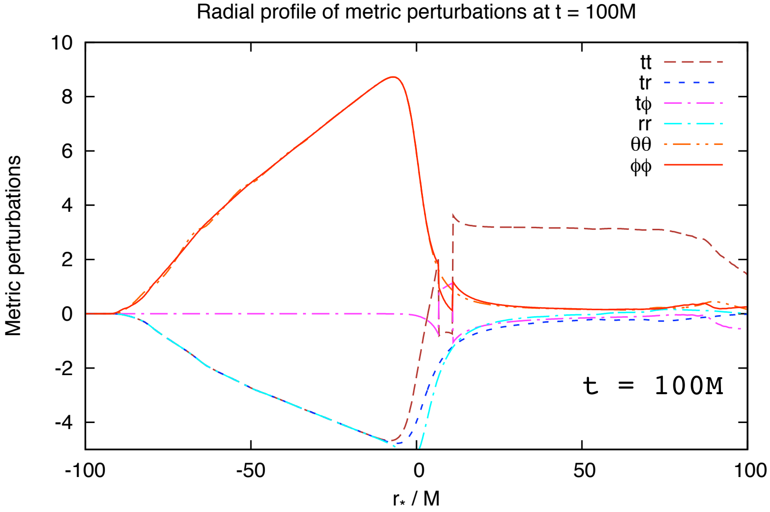
## **Problem:** Gauge mode instabilities

- Modes $m = 0$ and $m = 1$ suffer from linear-in-$t$ instabilities
- The growing solutions are (locally) Lorenz-gauge $\bar{h}_{ab}^{;b} = 0$
- are regular on the future horizon
- are homogeneous and pure-gauge: $h_{ab} = \xi_{a;b} + \xi_{b;a}$
- are generated by 'scalar' gauge modes: $\xi_a = \Phi_{;a}$.
- which are traceless $h = -\bar{h}_a^a = 0$.
- The problem is entirely in $l = m = 0$ and $l = m = 1$ modes on Schw.
- Q. Why has no-one evolved Lorenz-gauge Schw. $l = 0$ and $l = 1$ modes in time-domain?
- A. Negative potentials $(r < 3M)$, unstable evolutions.

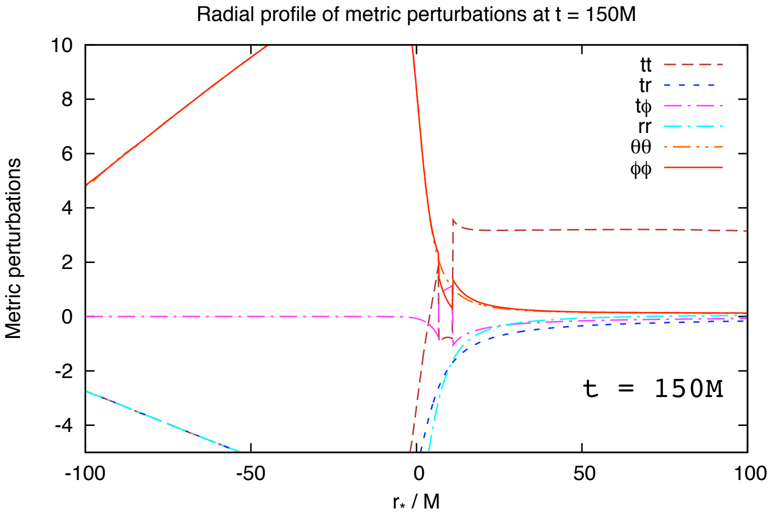
# Radial Profile : $m = 0$ mode



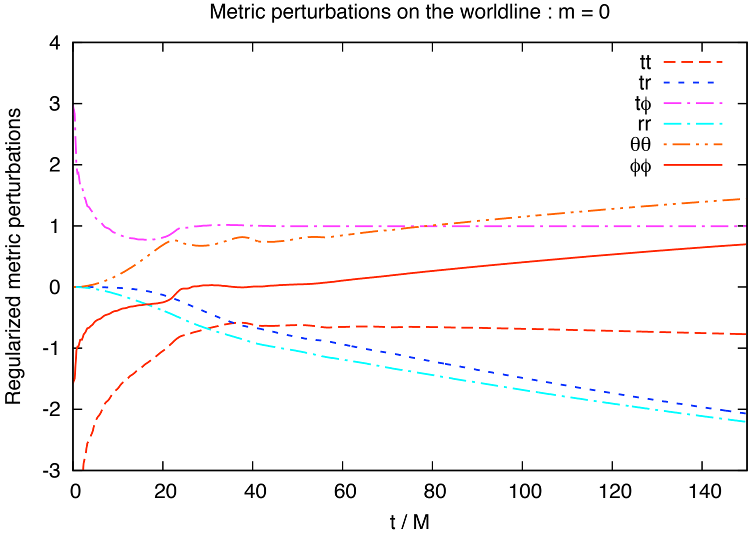
# Radial Profile : $m = 0$ mode



# Radial Profile : $m = 0$ mode



# Problem: Time Evolution of $m = 0$ mode



# **Problem:** Gauge mode instabilities

- Modes $m = 0$ and $m = 1$ suffer from linear-in-$t$ instabilities
- The growing solutions are (locally) pure (Lorenz-)gauge modes

## (Partial) solution:

- Use generalized Lorenz gauge to promote stability,

$$\bar{h}^{;\nu}_{\mu\nu} = H_\mu(\bar{h}_{\alpha\beta}, x^\gamma).$$

  where $H_\mu$ are gauge drivers.

- 'Implicit' gauge-drivers used with success in Num. Rel.
- I've established a proof-of-principle for an explicit gauge driver for $m = 0$ sector on Schw. for circular orbits).

# Summary

- GSF approach has come-of-age: first comparisons of GSF with PN, EOB and NR have been successful.
- Orbital resonance phenomenon provides key motivation for computing GSF on Kerr.
- Second-order-in-$\mu$ formalism has been developed (Pound 2012); numerical work now needed ...
- First 'self-forced' orbits and gravitational waveforms recently produced (Warburton *et al.*; Diener *et al.*)
- First GSF calculations on Kerr are now in progress, via $m$-mode regularization with $2+1D$ time-domain evolution
- Linear-in-$t$ gauge mode instabilities are a challenge for time-domain, Lorenz-gauge $Z4$ schemes
- Stable schemes based on generalized Lorenz-gauge are now needed.

# Literature

## Reviews of self force approach

- L. Barack, Class. Quantum Grav. 26 (2009) 213001
- E. Poisson *et* al. (2011) arXiv:1102.0529

## $m$-mode regularization in time domain

- Scalar field, first-order puncture: Barack & Golbourn [arXiv:0705.3620].
- Second-order GSF formulation: Barack, Golbourn & Sago [arXiv:0709.4588].
- Scalar-field, 4th-order punc, Schw.: Dolan & Barack [arXiv:1010.5255]
- Scalar-field, Kerr, circ. orb.: Dolan, Wardell & Barack [arXiv:1107.0012]
- Scalar-field, Kerr, eccentric orbits: Thornburg (in progress)
- GSF, Schw, circ. orbits, 2nd order: Dolan & Barack (in progress)
- GSF, Kerr, circ. orbits, 4th order: coming soon (I hope!).