Generalized hyperbolicity in the context of nonlinear distributional geometry

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Outline

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1 Introduction

2 An existence result for wave equations

3 Outlook and Bibliography

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What?

Local existence & uniqueness results for the Cauchy problem of wave equations on low regularity spacetimes.

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- Generalized hyperbolicity [Clarke 98]: alternative approach to singularities of spacetime
  - Standard approach: obstruction to the extension of geodesics
  - Generalized hyperbolicity: obstruction to the local well-posedness of the Cauchy problem for the D'Alembertian
  - Allows for non-singular spacetimes of low regularity, provided a good solution concept for singular wave equations

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  - Generalized hyperbolicity: obstruction to the local well-posedness of the Cauchy problem for the D'Alembertian
  - Allows for non-singular spacetimes of low regularity, provided a good solution concept for singular wave equations
- Paving the way for solving Einstein's equations
  - Cauchy problem formulated in terms of quasilinear wave equations
  - Solutions via an iterative scheme

Colombeau algebras

Algebras of generalized functions in the sense of Colombeau:

[Colombeau 1984, 1985]

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- Differential algebras
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  - maximal consistency with classical analysis (Schwartz' impossibility result), preserve
    - product of smooth functions
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- Differential algebras
  - contain vector space of distributions
  - maximal consistency with classical analysis (Schwartz' impossibility result), preserve
    - product of smooth functions
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- Main ideas of construction:
  - Regularization of distributions by nets of smooth functions
  - Asymptotic estimates in terms of a regularization parameter (quotient construction)

[Colombeau 1984, 1985]

Special Colombeau algebra

Definition

- Moderate families $\mathcal{E}_{\mathrm{M}}(M) \subseteq (C^\infty(M))^{(0,1]}$

$$(u_\varepsilon)_\varepsilon : \forall K\ \forall P \in \mathcal{P}\ \exists N : \sup_{p \in K} |P u_\varepsilon(p)| = O(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

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- For the tensor bundle $\mathrm{T}^r_s(M)$, similar quotient construction

$$\mathcal{G}^r_s(M) \cong \mathcal{G}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M)$$

Generalized Lorentzian metrics

Definition

$\mathbf{g} \in \mathcal{G}^0_2(M)$ a Lorentzian metric for each ε, such that any representative of $\det \mathbf{g}$ is invertible, i. e. for all compact sets $K \subseteq M$

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We have

$$\mathcal{G}^0_2(M) \cong \mathrm{L}_{\mathcal{G}(M)}(\mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M), \mathcal{G}(M)).$$

Compare with the distributional case

$$\mathcal{D}'^0_2(M) \cong \mathrm{L}_{\mathcal{C}^\infty(M)}(\mathfrak{X}(M) \times \mathfrak{X}(M), \mathcal{D}'(M)).$$

[Grosser, Kunzinger, Oberguggenberger, Steinbauer 2001]

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- [Grant, Mayerhofer, Steinbauer 09]: modelled in $\mathcal{G}$ from the start, in a way locally bounded, i. e. $\sup|\partial^k \mathbf{g}| = O(\varepsilon^{-k})$
- [Hörmann, Kunzinger, Steinbauer 11]: global result, asymptotics as in [GMS09], classical global theory [Bär, Ginoux, Pfäffle 07]

Proofs use geometrical approach and rely on parametrized *higher order energy estimates* with *energy tensors* for generalized metrics.

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- $\hat{\mathbf{g}}$ a smooth Lorentzian metric with associated connection $\widehat{\nabla}$ and volume element $\hat{\boldsymbol{\mu}}$,
- $\mathbf{m}$ a smooth Riemannian metric to define pointwise norms of tensor fields.

Initial value problem

The initial value problem for a normally hyperbolic operator in the Special Colombeau Algebra:

$$P\phi = g^{ab}\widehat{\nabla}_a\widehat{\nabla}_b\phi + B^a\widehat{\nabla}_a\phi + C\phi = F$$

$$\phi\big|_{\Sigma_0} = \phi_0 \quad \widehat{\nabla}_{\sigma^\sharp}\phi\big|_{\Sigma_0} = \phi_1 \qquad \text{(IVP)}$$

- $\mathbf{g}\ldots$ generalized Lorentzian metric
- $\boldsymbol{B}$, $C\ldots$ lower order coefficients in $\mathcal{G}$
- $\widehat{\nabla}\ldots$ smooth Levi-Cività connection associated to $\hat{\mathbf{g}}$
- $F\ldots$ source term in $\mathcal{G}$
- $\Sigma_0\ldots$ initial surface
- ϕ_0, $\phi_1\ldots$ initial conditions in $\mathcal{G}$

Two essential conditions

(R) **Regularity:** Let $U \subseteq M$ be open and relatively compact and let $\mathbf{g}$, $\boldsymbol{B}$, $C \in \mathcal{G}$. For all compact $K \subseteq U$ as $\varepsilon \to 0$

$$\sup_K |\mathbf{g}_\varepsilon|_{\mathbf{m}},\ \sup_K |\mathbf{g}_\varepsilon^{-1}|_{\mathbf{m}} = O(1),$$
$$\sup_K |\widehat{\nabla}\mathbf{g}_\varepsilon^{-1}|_{\mathbf{m}},\ \sup_K |\boldsymbol{B}_\varepsilon|_{\mathbf{m}} = O(1),$$
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(C) **Existence of classical solutions on a common domain:** For any representative $(\mathbf{g}_\varepsilon)_\varepsilon$ on U the level set Σ_0 is a past compact spacelike hypersurface such that $\partial J_\varepsilon^+(\Sigma_0) = \Sigma_0$, where J_ε^+ is the closure of the future emission $I_\varepsilon^+(\Sigma_0) \subseteq U$. Moreover, there exists a nonempty open set $A \subseteq M$ and some $\varepsilon_0 > 0$ such that $A \bigcap_{\varepsilon \leq \varepsilon_0} J_\varepsilon^+(\Sigma_0)$.

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Then the initial value problem (IVP) has locally a unique generalized solution in the sense of Colombeau.

That is: For each $p \in \Sigma_0$, there exists an open neighbourhood V with a unique solution $\phi \in \mathcal{G}(V)$.

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- Existence: Show that the solution candidate is a moderate net.

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- Solve the differential equation for fixed ε in the smooth case on some common domain to obtain a net $\rightsquigarrow$ a solution candidate.
- Existence: Show that the solution candidate is a moderate net.
- Uniqueness: Show that varying the data by negligible elements only changes the solution by a negligible element.

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- Generalization to tensorial equations.
- Normally hyperbolic operators with a generalized connection, i. e. $\hat{\mathbf{g}} \in \mathcal{G}$.
- Refined regularity:
  - Improvement on [GMS09]; see theorem.
  - Conditions as in [GMS09] $\rightsquigarrow$ additive regularity of the solution.
- Connection to existence and uniqueness results for hyperbolic first order systems, cf. [Hörmann, Spreitzer 11].

Outlook & future research

- Explore more deeply the connection with first order systems. Regularity issues are nontrivial when rewriting.

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- Extension to non-linear problems wanted: Cauchy problem for Einstein equations can be formulated with quasilinear, normally hyperbolic tensorial differential equations.

References

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