Vacuum Polarisation on the Brane for a Higher Dimensional Black Hole Spacetime

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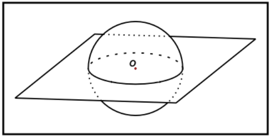
Overview

1. Objectives and Setup
2. Overview of Method [1] [2]
3. Results
4. Conclusions and Further Work

[1] General method: P.R. Anderson Phys. Rev. D 41, 1152 (1990)
[2] Specific reference: E. Winstanley & P. Young Phys. Rev. D 77 024008 (2008)

Setup and Objectives

- Our aim is to calculate a measure of the vacuum polarisation in our universe, the auto-correlation function $\langle \phi^2 \rangle$, with the following setup
- ADD braneworld with our universe as a $(3+1)$d flat, tensionless brane in a flat bulk for a total of $d = 4 \to 11$ dimensions



- Euclideanised Schwarzschild-Tangherlini metric on the brane:

$$ds^2 = f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad f(r) = 1 - \left(\frac{r_h}{r}\right)^{d-3}$$

- Hartle-Hawking vacuum: time symmetry, temperature T and periodic states (due to Wick rotation) of period T^{-1}

Method Part 1: Covariant Point Splitting

We get an unrenormalised expression for $\langle\phi^2\rangle$ from the Euclidean Green's function,

$$\langle\phi^2\rangle_{\text{unren}} = Re[\lim_{x\to x'} G_E(x;x')],$$

which is defined by the point split Klein-Gordon equation

$$(\nabla_\mu\nabla^\mu - m^2 - \xi R(r))\, G_E(x;x') \;=\; -g^{-1/2}\delta^4(x,x')$$

The solution is (with $\omega = 2\pi nT$)

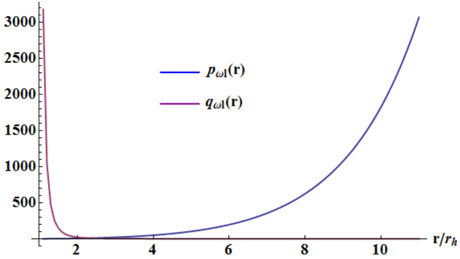
$$G_E(x;x') = \frac{T}{4\pi}\sum_{n=-\infty}^{\infty}\exp[i\omega(t-t')]\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\Upsilon_{\omega l}(r,r')$$

where $\Upsilon_{\omega l}$ is given by the radial ODE:

$$f\frac{d^2\Upsilon_{\omega l}}{dr^2} + \left[\frac{2f}{r} + f'\right]\frac{d\Upsilon_{\omega l}}{dr} - \left[\frac{\omega^2}{f} + m^2 + \xi R(r) + \frac{l(l+1)}{r^2}\right]\Upsilon_{\omega l} = -\frac{1}{r^2}\delta(r-r')$$

Method Part 2: Radial ODE Solutions

The homogeneous form of the radial ODE has two solutions: $p_{\omega l}(r)$ and $q_{\omega l}(r)$



With $\lim_{r \to r'} [\Upsilon_{\omega l}(r, r')] = C_{\omega l} p_{\omega l}(r) q_{\omega l}(r)$ & $C_{\omega l} \left[p_{\omega l} \frac{dq_{\omega l}}{dr} - q_{\omega l} \frac{dp_{\omega l}}{dr} \right] = -\frac{1}{r^2 f}$

Final Green's function expression ($\epsilon = t - t'$)

$$G_E(t, \mathbf{x}; t', \mathbf{x}) = \frac{T}{4\pi} \sum_{n=-\infty}^{\infty} \exp(i\omega\epsilon) \sum_{l=0}^{\infty} (2l+1) C_{\omega l} p_{\omega l}(r) q_{\omega l}(r)$$

Method Part 3: Divergences

We remove an unphysical divergence gained from point splitting

$$G_E(t,\mathbf{x};t',\mathbf{x}) = \frac{T}{4\pi}\sum_{n=-\infty}^{\infty}\exp(i\omega\epsilon)\sum_{l=0}^{\infty}\left[(2l+1)C_{\omega l}p_{\omega l}(r)q_{\omega l}(r) - \frac{1}{rf^{1/2}}\right]$$

And apply Hadamard renormalisation ($\sigma = \sigma[\epsilon]$):

$$\langle\phi^2\rangle_{\mathrm{div}} = \frac{1}{8\pi^2\sigma} + \frac{1}{8\pi^2}\left(m^2 + (\xi - 1/6)R\right)\left(C + \frac{1}{2}\ln\left[\frac{\mu^2|\sigma|}{2}\right]\right) + \frac{1}{96\pi^2}R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta}{\sigma}$$

Removing the above from $\langle\phi^2\rangle_{\mathrm{unren}}$ and letting $\epsilon \to 0$ we get:

$$\langle\phi^2\rangle_{\mathrm{ren}} = \langle\phi^2\rangle_{\mathrm{analytic}} + \langle\phi^2\rangle_{\mathrm{numeric}}$$

Numeric Component 1

$$\langle\phi^2\rangle_{\text{numeric}} = \frac{T}{2\pi}\sum_{n=1}^{\infty}\left\{\sum_{l=0}^{\infty}\left[(2l+1)C_{\omega l}p_{\omega l}q_{\omega l} - \frac{1}{rf^{\frac{1}{2}}}\right] + \frac{\omega}{f} + \frac{1}{2\omega}\left(m^2 + \left(\xi - \frac{1}{6}\right)R\right)\right\} + \frac{T}{4\pi}\sum_{l=0}^{\infty}\left[(2l+1)C_{0l}p_{0l}q_{0l} - \frac{1}{rf^{\frac{1}{2}}}\right]$$

WKB approximation: Let $\beta_{\omega l} = C_{\omega l}p_{\omega l}q_{\omega l}$ and rewrite radial ODE

$$\beta_{\omega l} = \frac{1}{2\chi_{\omega l}}\left[1 - \frac{\delta^2}{\chi_{\omega l}^2}\left(\frac{1}{\sqrt{\beta_{\omega l}}}\frac{d^2(\sqrt{\beta_{\omega l}})}{d\zeta^2} - \eta\right)\right]^{-\frac{1}{2}} = \beta_{0\omega l}(r) + \delta^2\beta_{1\omega l}(r) + \delta^4\beta_{2\omega l}(r) + \dots$$

Numeric Component 2

Watson-Somerfeld formula:

$$\sum_{l=0}^{\infty} \mathcal{F}(l) = \int_0^{\infty} \mathcal{F}(\lambda - \frac{1}{2}) d\lambda - Re\left[i \int_0^{\infty} \frac{2}{1+e^{2\pi\lambda}} \mathcal{F}(\lambda - \frac{1}{2}) d\lambda \right]$$

$$= I_\alpha(\omega, r) + J_\alpha(\omega, r) \quad (\text{for } \beta_{\alpha\omega l})$$

For $n = 0$ case $\sum_l (2l+1)\beta_{\alpha 0 l} = \Delta_\alpha$ is known in closed form.

$$\begin{aligned} \langle \phi^2 \rangle_{\text{numeric}} = \frac{T}{2\pi} \sum_{n=1}^{\infty} \Bigg[& \sum_{l=0}^{\infty} (2l+1)[C_{\omega l} p_{\omega l} q_{\omega l} - \beta_{0\omega l} - \beta_{1\omega l} - \beta_{2\omega l} - \beta_{3\omega l}] \\ & + J_0 + J_1 + \beta_{2\omega l} + \beta_{3\omega l}] \\ + \frac{T}{4\pi} \sum_{l=0}^{\infty} & [(2l+1)[C_{0l} p_{0l} q_{0l} - \beta_{00l} - \beta_{10l} - \beta_{20l} - \beta_{30l}]] \\ & + \Delta_1 + \Delta_2 + \Delta_3. \end{aligned}$$

Massless, Conformally Coupled Results 1

$$\langle\phi^2\rangle_{\text{analytic}} = \frac{\kappa^2}{48\pi^2 f} - \frac{1}{192\pi^2 f}\left(\frac{df}{dr}\right)^2 + \frac{1}{96\pi^2}\frac{d^2 f}{dr^2} + \frac{1}{48\pi^2 r}\frac{df}{dr} - \frac{1}{8\pi^2}\left(m^2 + \left(\xi - \frac{1}{6}\right)R\right)\left(C + \frac{1}{2}\ln\left[\frac{\mu^2 f}{4\kappa^2}\right]\right)$$

Massless, conformally coupled ($\xi = 1/6$) surface gravity:

$$\kappa = \frac{1}{2}\frac{df}{dr}\bigg|_{r=1} = \frac{d-3}{2} \quad \text{letting } r_h = 1 \text{ WLOG}$$

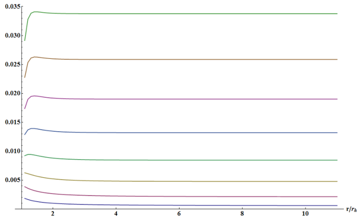
$$\text{Ricci scalar: } R(r) = \frac{(d-4)(d-5)}{r^{d-1}}$$

Massless, Conformally Coupled Results 1

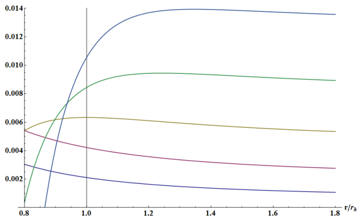
$$\langle\phi^2\rangle_{\text{analytic}} = \frac{(d-3)}{192\pi^2 f}\left((8-2d)\left(\frac{1}{r}\right)^{d-1} + (d-5)\left(\frac{1}{r}\right)^{2d-4} + d - 3\right)$$

$$\langle\phi^2\rangle^{r\to1}_{\text{analytic}} = \frac{d-3}{48\pi^2} \quad \text{and} \quad \langle\phi^2\rangle^{r\to\infty}_{\text{analytic}} = \frac{(d-3)^2}{192\pi^2}$$

Plots of $\langle\phi^2\rangle_{\text{analytic}}$: ($d$ increases up the graphs)



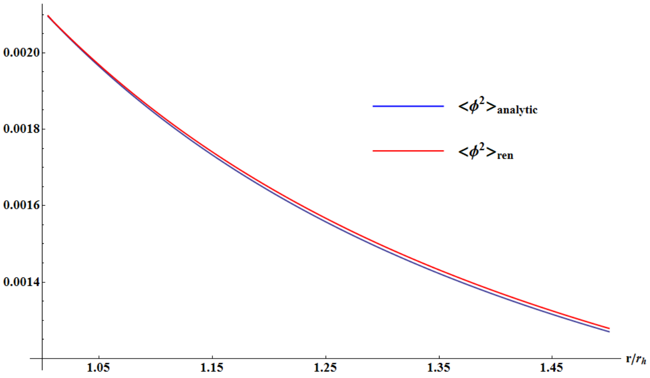
(a) $d = 4 \to 11$ for $r = 1 \to 11$



(b) $d = 4 \to 8$ for $r = 0.8 \to 1.8$

4d Comparison

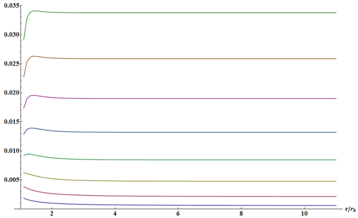
Plots of $\langle\phi^2\rangle_{\text{analytic}}$ and $\langle\phi^2\rangle_{\text{ren}}$ in 4d



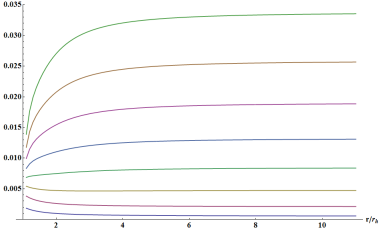
Maximum difference $\simeq 0.69\%$

Massless, Conformally Coupled Results 2

Comparison of $\langle\phi^2\rangle_{\text{analytic}}$ and $\langle\phi^2\rangle_{\text{ren}}$ in $d = 4 \to 11$ for $r = 1.1 \to 11$ (d increases up the graphs)



(a) $\langle\phi^2\rangle_{\text{analytic}}$

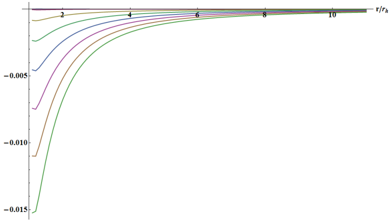


(b) $\langle\phi^2\rangle_{\text{ren}}$

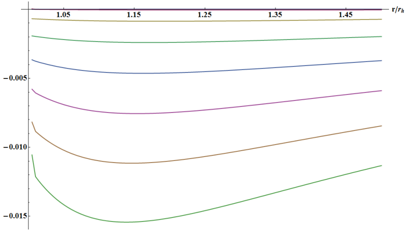
$d = 4$ diff $\simeq 0.69\%$	$d = 5$ diff $\simeq 1.5\%$
$d = 6$ diff $\simeq 14\%$	$d = 7$ diff $\simeq 25\%$
$d = 8$ diff $\simeq 35\%$	$d = 9$ diff $\simeq 42\%$
$d = 10$ diff $\simeq 48\%$	$d = 11$ diff $\simeq 52\%$

Numerical Results on Two Scales

$\langle\phi^2\rangle_{\text{numeric}}$ in $d = 4 \to 11$ for $r = 1.1 \to 11$ and $r = 1.005 \to 1.5$ (d increases down the graphs)



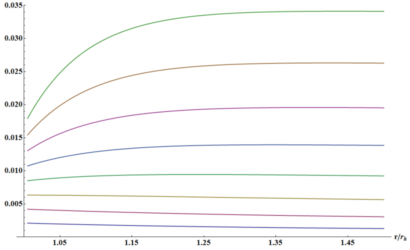
(a) $r = 1.1 \to 11$



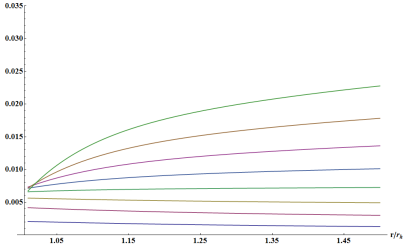
(b) $r = 1.005 \to 1.5$

Massless, Conformally Coupled Results in New Scale

Comparison of $\langle\phi^2\rangle_{\text{analytic}}$ and $\langle\phi^2\rangle_{\text{ren}}$ in $d = 4 \to 11$ for $r = 1.01 \to 1.5$ (d increases up the graphs)



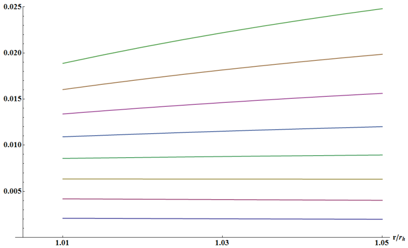
(a) $\langle\phi^2\rangle_{\text{analytic}}$



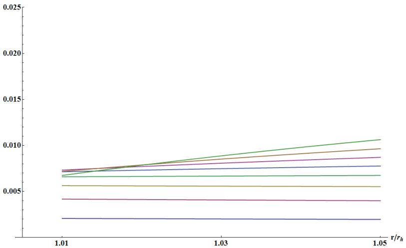
(b) $\langle\phi^2\rangle_{\text{ren}}$

Behaviour Close to the Horizon

Comparison of the first 10 points of $\langle\phi^2\rangle_{\text{analytic}}$ and $\langle\phi^2\rangle_{\text{ren}}$ in $d = 4 \to 11$ (d increases up the graphs)



(a) $\langle\phi^2\rangle_{\text{analytic}}$



(b) $\langle\phi^2\rangle_{\text{ren}}$

Conclusions

- We have developed a method to calculate all components of $\langle\phi^2\rangle_{\text{ren}}$ for our system
- Individual calculations can be accurately done in Mathematica to 30 significant figures within reasonable time
- We have shown that the use of $\langle\phi^2\rangle_{\text{analytic}}$ for an approximation to $\langle\phi^2\rangle_{\text{ren}}$ quickly becomes inappropriate for $d > 4$
- We can deduce from our results $\langle\phi^2\rangle_{\text{ren}}$ has a maximum value on the horizon for $d = 8$ or 9
- Further work will be to extend our method to gaining results in the bulk where the key aspect will be higher dimensional renormalisation