Static, stationary and inertial Unruh-DeWitt detectors on the BTZ black hole

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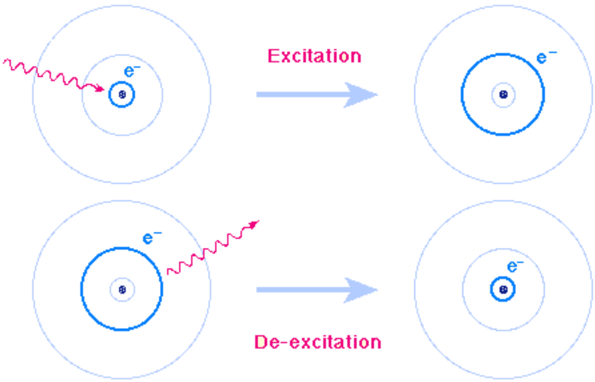
Particle detectors

- Ambiguity in particle notion in QFT
- Operational definition of "particles": Couple a simple QM system (detector) to the quantum field.

Detectors-Excitation and De-excitation



Detector response

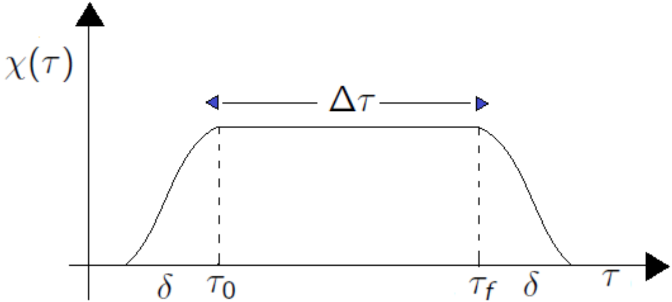
- What is probability of this transition? Use first-order perturbation theory; the interesting part of the probability proportional to
-
$$\mathcal{F}(E) = \int_{-\infty}^{\infty} \mathrm{d}\tau' \int_{-\infty}^{\infty} \mathrm{d}\tau'' \, \mathrm{e}^{-iE(\tau'-\tau'')} \chi(\tau')\chi(\tau'')W(\tau',\tau'')$$
- Where $W(\tau',\tau'')$ is known as the Wightman function,

$$W(\tau',\tau'') = \langle \Psi | \phi(\mathsf{x}(\tau'))\phi(\mathsf{x}(\tau'')) | \Psi \rangle$$

The instantaneous transition rate

- $\mathcal{F}(E)$ depends on $\chi(\tau)$, no room for time dependence $\rightarrow \dot{\mathcal{F}}(E)$.
- Extreme care must be taken when obtaining the transition rate: Schlicht(2004), Satz(2007)
- Must switch smoothly, obtain response function in regulator free form, then take sharp switching limit.

Sharp switching limit $\frac{\delta}{\Delta\tau} \to 0$



Transition rate in three dimensions arbitrary Hadamard state

- $$W_\epsilon(\mathrm{x},\mathrm{x}') = \frac{1}{(4\pi)}\left[\frac{U(\mathrm{x},x')}{\sqrt{\sigma_\epsilon(\mathrm{x},\mathrm{x}')}} + \frac{H(\mathrm{x},\mathrm{x}')}{\sqrt{2}}\right]$$
- Hadamard characterises singularity in coincidence limit $\mathrm{x} \to \mathrm{x}'$
- Wightman function for BTZ spacetime has singularaties even when $\sigma(\mathrm{x},\mathrm{x}') \neq 0$.
- $$\dot{\mathcal{F}}_\tau(E) = \frac{1}{4} + 2\int_0^{\Delta\tau} \mathrm{d}s \ \mathrm{Re}\left[\mathrm{e}^{-iEs} W_0(\tau, \tau - s)\right] .$$

The BTZ black hole

- $(2+1)$-dimensional black hole.
- Periodically identify 3D adS.
- $$ds^2 = -(N^{\perp})^2 dt^2 + f^{-2} dr^2 + r^2 \left(d\phi + N^{\phi} dt \right)^2$$

  with lapse and shift functions

  $$N^{\perp} = f = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \right)^{1/2},$$
  $$N^{\phi} = -\frac{J}{2r^2} \qquad (|J| \leq M\ell).$$

Conformal Diagram of BTZ

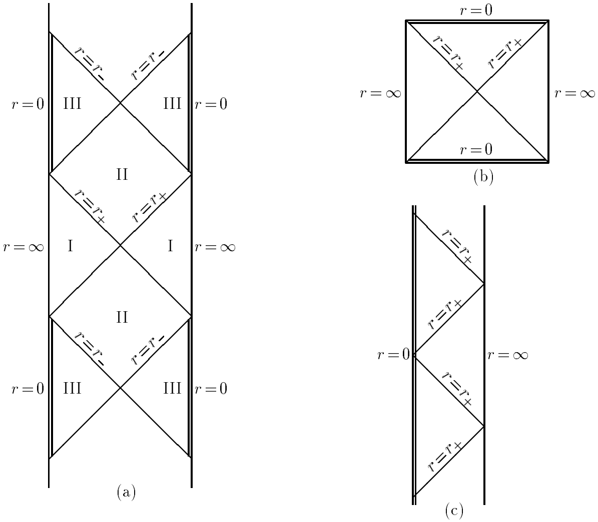


Figure 1

Wightman function for BTZ

- Express in terms of adS Wightman by method of images

$$G_{\mathrm{BTZ}}(\mathrm{x},\mathrm{x}') = \sum_n G_A(\mathrm{x},\Lambda^n \mathrm{x}')$$

- For field in Hartle-Hawking Vacuum, AdS Wightman functions are (Carlip 1995)

$$G_A^{(\zeta)} = \frac{1}{4\pi}\left[\frac{1}{\sqrt{\Delta \mathrm{X}^2}} - \zeta \frac{1}{\sqrt{\Delta \mathrm{X}^2 + 4\ell^2}}\right],$$

-

$$\Delta \mathrm{X}^2 := \left[(X_1 - X_1')^2 - (T_1 - T_1')^2 + (X_2 - X_2')^2 - (T_2 - T_2')^2\right]$$

Transition rate for BTZ

- $$\begin{aligned}\Delta\tilde{X}_n^2 &:= \Delta X^2(X(\tau), \Lambda^n X(\tau - \ell\tilde{s}))/2\ell^2 \\ &= \Bigg[-1 + \sqrt{\alpha(r)\alpha(r')}\cosh\left[(r_+/\ell)\left(\phi - \phi' - n2\pi\right) - (r_-/\ell^2)\left(t - t'\right)\right] \\ &- \sqrt{(\alpha(r)-1)(\alpha(r')-1)}\cosh\left[(r_+/\ell^2)\left(t - t'\right) - (r_-/\ell)\left(\phi - \phi' - n2\pi\right)\right]\Bigg]\end{aligned}$$
- where,
$$\alpha(r) = \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right)$$
- $$\dot{\mathcal{F}}_\tau\left(\tilde{E}/\ell\right) = \frac{1}{4} + \frac{1}{2\pi\sqrt{2}}\sum_n \int_0^{\Delta\tau/\ell} d\tilde{s}\ \mathrm{Re}\left[e^{-i\tilde{E}\tilde{s}}\left(\frac{1}{\sqrt{\Delta\tilde{X}_n^2}} - \zeta\frac{1}{\sqrt{\Delta\tilde{X}_n^2 + 2}}\right)\right]$$

Detector co-rotating with horizon angular velocity

- Spacetime dragged along with hole's rotation
- As approach outer event horizon r_+ can define ang vel of horizon itself, Ω_H: the ang vel of particle at horizon.
- Look at detector co-rotating with BH at this horizon velocity (for a non-spinning hole this reduces to static detector)
- Turns out stationary but NOT co-rotating will not satisfy KMS, therefore didn't analyse explicitly.

Transition rate of co-rotating detector

- $$\dot{\mathcal{F}}_{\tau}\left(E/\ell\right) =$$
  $$\frac{1}{4} + \frac{1}{4\pi\sqrt{\alpha(\tilde{r}\ell) - 1}} \operatorname{Re}\sum_{n} \int_0^{\infty} \mathrm{d}s \left[\mathrm{e}^{-iEs} \left(\frac{1}{\sqrt{K_n - \sinh^2\left(\Xi s + n\pi\left(r_-/\ell\right)\right)}} - \zeta \frac{1}{\sqrt{Q_n - \sinh^2\left(\Xi s + n\pi\left(r_-/\ell\right)\right)}} \right) \right]$$

- $$K_n := \left(1 - \alpha(\tilde{r}\ell)^{-1}\right)^{-1} \sinh^2\left[\left(r_+/\ell\right) n\pi\right]$$
  $$\Xi := \left(2\sqrt{\alpha(\tilde{r}\ell) - 1}\right)^{-1}$$
  $$Q_n := \left(\alpha(\tilde{r}\ell) - 1\right)^{-1} + K_n$$

KMS co-rotating detector

- Zeroth term is

$$\dot{\mathcal{F}}_{\tau}^{\mathrm{n}=0}(E/\ell) = \frac{1}{2\left(\exp\left[\beta E\right]+1\right)} - \frac{\zeta}{(2\pi)}\exp\left[-\beta E/2\right]\int_0^\infty \mathrm{d}y\,\frac{\cos\left(yE\beta/\pi\right)}{\sqrt{Q_0+\cosh^2(y)}}.$$

- KMS condition of the form

$$\dot{\mathcal{F}}_{\tau}^{\mathrm{n}=0}(E/\ell) = \mathrm{e}^{-\beta E}\dot{\mathcal{F}}_{\tau}^{\mathrm{n}=0}(-E/\ell)$$

- where $\beta := 2\pi\sqrt{\alpha(\tilde{r}\ell)-1}$. Identify our β with the *dimensionless* local inverse temperature: $\beta^{-1} = (g_{00})^{-1/2}T_0$ with $T_0 = \kappa/(2\pi)$ (Carlip:1995).
- Similarly with aid of contour manipulations we can observe the non-zero piece obeys same KMS condition.

Asymptotics co-rotating detector

- Large r_+/ℓ asymptotics (large black hole mass M in static case) for $E \neq 0$ (physically interesting case)

$$
\begin{aligned}
\dot{\mathcal{F}}_\tau\left(E/\ell\right) &= \frac{1}{2}\frac{1}{\mathrm{e}^{\beta E}+1} - \frac{\zeta}{2\pi}\mathrm{e}^{-\beta E/2}\int_0^\infty \mathrm{d}y\,\frac{\cos\left(yE\beta/\pi\right)}{\sqrt{Q_0+\cosh^2 y}} \\
&+ \mathrm{e}^{-\beta E/2}\frac{\cos\left(Er_-\beta/\ell\right)}{\sqrt{\pi}\beta E}\times \\
&\left[\mathrm{Im}\left\{\left(\frac{(4K_1)^{i(\beta E/2\pi)}}{\sqrt{K_1}} - \frac{\zeta\,(4Q_1)^{i(\beta E/2\pi)}}{\sqrt{Q_1}}\right)\Gamma\left(1+i\left(\beta E/2\pi\right)\right)\Gamma\left(1/2-i\left(\beta E/2\pi\right)\right)\right\}\right. \\
&\left.+ O\left(\mathrm{e}^{-2\pi(r_+/\ell)}\right)\right]
\end{aligned}
$$

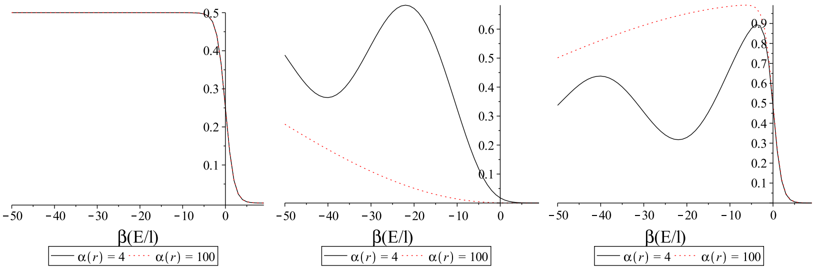
Asymptotics co-rotating detector

- small r_+/ℓ asymptotics (small black hole mass M in static case)
- to get qualitatively different behaviour to the static case in this limit (i.e. in order that the hole retains some angular momentum as we take $(r_+/\ell) \to 0$) we impose the condition $r_- = \kappa r_+$, where κ is fixed $0 < \kappa < 1$:

$$\dot{\mathcal{F}}_\tau (E/\ell) = \frac{1}{\pi^2 (r_+/\ell)} \mathrm{e}^{-\beta E/2} \int_0^\infty \mathrm{d}r \int_0^\infty \mathrm{d}y \, \cos\left[r\left(E\beta\kappa/\pi\right)\right] \cos\left(yE\beta/\pi\right) \times$$

$$\left[\frac{1}{\sqrt{\frac{\sinh^2 r}{(1-1/\alpha(\tilde{r}\ell))} + \cosh^2 y}} - \frac{\zeta}{\sqrt{\frac{1}{\alpha(\tilde{r}\ell)-1} + \frac{\sinh^2 r}{(1-1/\alpha(\tilde{r}\ell))} + \cosh^2 y}} \right] + O(1)$$

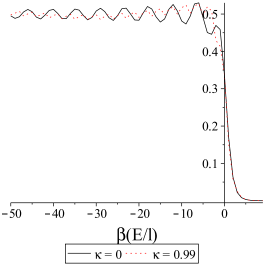
Co-rotating (static) plots



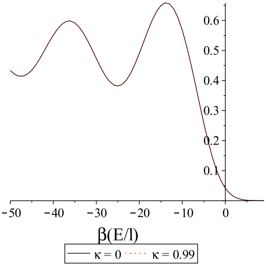
(a) $\zeta = 0$ (b) $\zeta = 1$ (c) $\zeta = -1$

Figure: Transition rate against $\beta\,(E/l)$ for $(r_+/l) = 10$, $(r_-/l) = 0$.

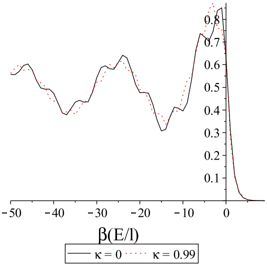
Co-rotating (static) sensitivity to BH rotation



(a) $\zeta = 0$



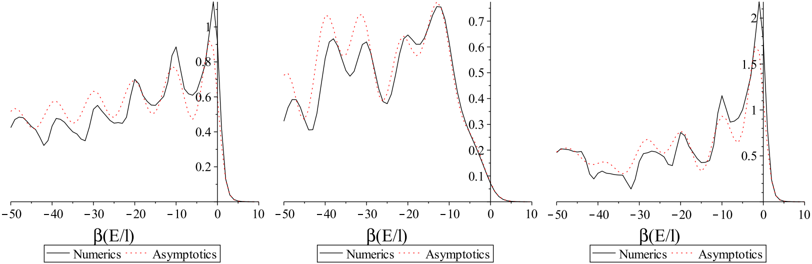
(b) $\zeta = 1$



(c) $\zeta = -1$

Figure: Transition rate against $\beta\,(E/l)$ for $(r_+/l) = 1$, $\alpha(\tilde{r}l) = 2$. Plotted from numerics.

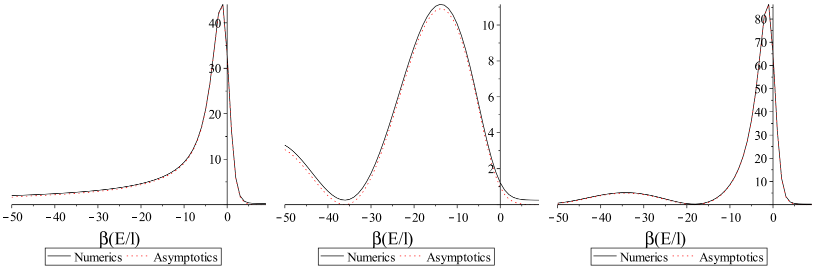
How are our asymptotics performing?



(a) $\zeta = 0$ (b) $\zeta = 1$ (c) $\zeta = -1$

Figure: Transition rate against $\beta\,(E/l)$ for $(r_+/l) = 0.3$, $(r_-/l) = 0.299$ and $\alpha(r) = 2$. Numerics plotted

Domain of small (r_+/l) asymptotics



(a) $\zeta = 0$ (b) $\zeta = 1$ (c) $\zeta = -1$

Figure: Transition rate against $\beta\,(E/l)$ for $(r_+/l) = 0.01$, $(r_-/l) = 0$ $(\kappa = 0)$ and $\alpha(\tilde{r}l) = 4$. The plots show the numerics and the small (r_+/l) asymptotics

Inertial detector transition rate

- Consider detector radially infalling on geodesic to $J = 0$ BTZ.
- $$\dot{\mathcal{F}}_\tau\left(E/\ell\right) = 1/4 + \frac{1}{2\pi\sqrt{2}}\,\mathrm{Re}\sum_n \int_0^{\Delta\tilde{\tau}} \mathrm{d}s \left[\frac{\mathrm{e}^{-iEs}}{\sqrt{-1 + K_n\cos\left(\tilde{\tau}\right)\cos\left(\tilde{\tau}-s\right) + \sin\left(\tilde{\tau}\right)\sin\left(\tilde{\tau}-s\right)}} - \zeta\frac{\mathrm{e}^{-iEs}}{\sqrt{1 + K_n\cos\left(\tilde{\tau}\right)\cos\left(\tilde{\tau}-s\right) + \sin\left(\tilde{\tau}\right)\sin\left(\tilde{\tau}-s\right)}}\right]$$
- $q \equiv \left(\tilde{r}_{\max}/\tilde{r}_+\right)^2 = \left(\varepsilon^2/M + 1\right)$ and $K_n \equiv 1 + 2q^2\sinh^2\left(n\pi\sqrt{M}\right)$

Zeroth term and KMS

- Zeroth term:

$$\dot{\mathcal{F}}_{\tau}^{n=0}(E/\ell) = 1/4 - (1/4\pi)\int_0^{\Delta\tilde{\tau}} \mathrm{d}s \left[\frac{\sin(Es)}{\sin(s/2)} + \zeta\frac{\cos(Es)}{\cos(s/2)}\right].$$

- Can observe numerically that KMS is not satisfied.
- Not surprising (Deser-Levin 1998): lack of real horizon in AdS means detectors with $a < 1/\ell$, no well defined temp.

Large M asymptotics

- $$\dot{\mathcal{F}}_{\tau}^{n\neq 0}(E/\ell) \approx \frac{1}{\pi q}\frac{1}{\sqrt{\cos(\tilde{\tau})}}\int_0^{\Delta\tilde{\tau}} \frac{\cos(Es)\mathrm{d}s}{\sqrt{\cos(\tilde{\tau}-s)}}\times$$
$$\left[(1-\zeta)\left(\mathrm{e}^{-\pi\sqrt{M}}+\mathrm{e}^{-2\pi\sqrt{M}}\right)+\left(\frac{\left(2q^2-1+f_-\right)-\zeta\left(2q^2-1-f_+\right)}{q^2}\right)\mathrm{e}^{-3\pi\sqrt{M}}\right].$$

- where
$$f_\pm(\tau,\tilde{\tau}-s)=\left(\frac{1}{\cos(\tilde{\tau})\cos(\tilde{\tau}-s)}\pm\tan(\tilde{\tau})\tan(\tilde{\tau}-s)\right)$$

- This formula assumes the detector is switched on(off) before hitting singularities $\tilde{\tau},\tilde{\tau_0}\neq\{-\pi/2,+\pi/2\}$

3D Plot for zeroth term

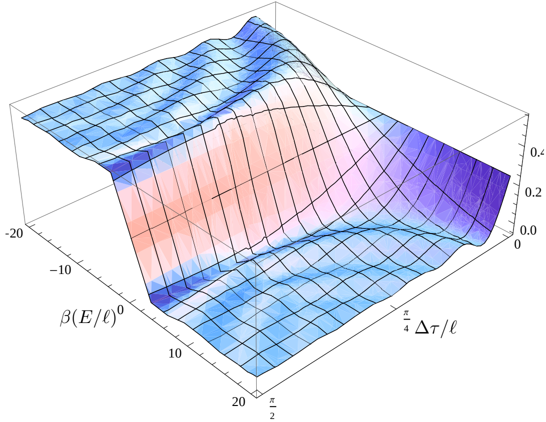


Figure: 3D plot of the zeroth term of the transition rate for boundary condition $\zeta = 0$

Conclusions and Summary

- Regulator-free transition probability and rate in arbitrary 3D Hadamard state-both well-defined and finite under sharp switching.
- Analysed response in the Hartle Hawking vacua on BTZ: found boundary conditions at infinity play a significant role.
- KMS exhibited for the co-rotating detector but not for radially in-falling (although nothing singular happens crossing horizon as expected for Hartle-Hawking vacuum).
- Analytic results obtained agree well with numerics.
- Future directions: 4D Schwarzschild.