Scalar field Hadamard renormalisation in AdS_n

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Outline

1. Scalar field theory in AdS_n
2. Overview of research
3. Calculation of the 'vacuum polarisation' $\langle \Phi^2 \rangle_{\rm ren}$

Scalar field theory in AdS_n

Geometry

Anti-de Sitter space

AdS_n is the *maximally symmetric* vacuum solution to Einstein's classical field equations with constant *negative* curvature.

Hyperspherical coordinates

Timelike	Radial	Polar	Azimuthal
$-\pi \leq \tau \leq \pi$	$0 \leq \rho < \frac{\pi}{2}$	$0 \leq \theta_j < \pi$ $(j = 1, 2, \ldots n-3)$	$0 \leq \theta_{n-2} < 2\pi$

Metric

$$ds^2 = -a^2 \sec^2 \rho \left[d\tau^2 - d\rho^2 - \sin^2 \rho \left(d\theta_1^2 + \sum_{i=2}^{n-2} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2 \right) \right].$$

Scalar field theory in AdS_n

Propagation

Homogenous scalar field wave equation

$$\left(\Box - m^2 - \xi\mathcal{R}\right)\phi(x) = 0$$

Inhomogenous scalar field wave equation

$$\boxed{\left(\Box_x - m^2 - \xi\mathcal{R}\right)G_F\left(x, x'\right) = g^{-\frac{1}{2}}\delta\left(x, x'\right), \quad g := |\det g_{\mu\nu}|}$$

Short-distance behaviour of $G_F\left(x, x'\right)$

$$\boxed{-iG_F\left(x, x'\right) \to \text{`}\langle\Phi^2(x)\rangle\text{'} \quad \text{as } x' \to x}$$

Scalar field theory in AdS_n

Maximal Symmetry

General effect of maximal symmetry

$$G_F\left(x, x'\right) \mapsto G_F(s), \quad s := s\left(x, x'\right).$$

Synge's 'world function'

$$\sigma := \frac{1}{2} s^2$$

Inhomogenous scalar field wave equation

$$\boxed{\left(\Box - m^2 - \xi \mathcal{R}\right) G_F(\sigma) = \delta(\sigma) \quad \forall\, \sigma < 0}$$

Short-distance behaviour of $G_F\left(\sigma\right)$

$$\boxed{-iG_F\left(\sigma\right) \rightarrow \text{`}\langle \Phi^2 \rangle\text{'} \quad \text{as } \sigma \rightarrow 0}$$

Overview of research

$\Phi^{(+)}_{n,\ell}(x) \longrightarrow G_F(\sigma) \longrightarrow \langle \Phi^2 \rangle_{\text{ren}} \longrightarrow \langle T_{\mu\nu} \rangle_{\text{ren}}$

$\Phi^{(+)}_{n,\ell}(x)$	$G_F(\sigma)$	$\langle \Phi^2 \rangle_{\text{ren}}$	$\langle T_{\mu\nu} \rangle_{\text{ren}}$
Field modes	Feynman propagator	Renormalised v.e.v. of the quadratic field fluctuations	Renormalised v.e.v. of the stress-energy tensor

Analytic

$\downarrow$ $\qquad$ $\downarrow$

$\langle \Phi^2(x) \rangle^{\beta}_{\text{ren}}$	$\langle T_{\mu\nu}(x) \rangle^{\beta}_{\text{ren}}$
Renormalised t.e.v. of the quadratic field fluctuations	Renormalised t.e.v. of the stress-energy tensor

Numeric

Calculation of $\langle \Phi^2 \rangle_{\mathrm{ren}}$

Hadamard theorem

Hadamard form of the Feynman Green's function

$$\boxed{G_F^H(\sigma) = i\nu(n)\left[U(\sigma)\sigma^{1-\frac{n}{2}} + V(\sigma)\ln\bar{\sigma} + W(\sigma)\right] \quad \forall\, n > 2}$$

(where $\nu(n)$ is a constant).

Hadamard functions

- $U(\sigma)$, $V(\sigma)$ and $W(\sigma)$ are *regular* as $\sigma \to 0$.
- Expansion coefficients are obtained from recursion relations [1].
- $U(\sigma)$, $V(\sigma)$ are *uniquely defined* but $W(\sigma)$ is *not*.

Singularity structure

$$G^H_{F,\,\mathrm{sing}}(\sigma) = i\nu(n)\left[U(\sigma)\sigma^{1-\frac{n}{2}} + V(\sigma)\ln\bar{\sigma}\right], \qquad \text{purely geometric}$$

$$G^H_{F,\,\mathrm{reg}}(\sigma) = i\nu(n)W(\sigma), \qquad \text{state-dependent}$$

Calculation of $\langle\Phi^2\rangle_{\text{ren}}$

Hadamard renormalisation

Given

$$G_F(\sigma) = G^H_{F,\,\text{reg}}(\sigma) + G^H_{F,\,\text{sing}}(\sigma) \qquad \text{and} \qquad G_F(\sigma) \to \text{`}i\langle\Phi^2\rangle\text{'} \text{ as } \sigma \to 0,$$

$$\text{`}\langle\Phi^2\rangle\text{'} = \langle\Phi^2\rangle_{\text{phys}} + \text{`}\langle\Phi^2\rangle_{\text{unphys}}\text{'} \quad \text{when } \sigma = 0,$$

where

$$\langle\Phi^2\rangle_{\text{phys}} = -i \lim_{\sigma\to 0} G^H_{F,\,\text{reg}}(\sigma) = \nu(n) \lim_{\sigma\to 0} W(\sigma).$$

Therefore

$$\langle\Phi^2\rangle_{\text{phys}} = \langle\Phi^2\rangle_{\text{ren}}$$

because

$$\begin{aligned}\langle\Phi^2\rangle_{\text{ren}} &= -i \lim_{\sigma\to 0} \left[G_F(\sigma) - G^H_{F,\,\text{sing}}(\sigma) \right] \\ &= -i \lim_{\sigma\to 0} \left[G_F(\sigma) - i\nu(n) \left\{ U(\sigma)\sigma^{1-\frac{n}{2}} + V(\sigma) \ln \bar{\sigma} \right\} \right]\end{aligned}$$

Calculation of $\langle \Phi^2 \rangle_{\text{ren}}$

Scalar field propagator on AdS_n

The ISFWE is a hypergeometric differential equation [2] in $z(\sigma)$ with solution

$$\boxed{G_F(\sigma) = CF\left[\frac{n-1}{2}+\mu, \frac{n-1}{2}-\mu; \frac{n}{2}; z\right] + DF\left[\frac{n-1}{2}+\mu, \frac{n-1}{2}-\mu; \frac{n}{2}; 1-z\right]}$$

where

- $z(\sigma) := -\left(\sinh\left(\frac{1}{a}\sqrt{\frac{\sigma}{2}}\right)\right)^2, \qquad \mu := \sqrt{\frac{(n-1)^2}{4} + m^2a^2 + \xi \mathcal{R} a^2}\ ,$
- C, D are constants,
- F is the hypergeometric function,

and

$$\boxed{G_F(\sigma) = G_{F,\ \text{reg}}(\sigma) + G_{F,\ \text{sing}}(\sigma)}$$

Calculation of $\langle \Phi^2 \rangle_{\mathrm{ren}}$

Code structure

Code was first written in Maple to generate expressions for

1. $G_{F,\,\mathrm{sing}}(\sigma)$ as $\sigma \to 0$
2. $G^{H}_{F,\,\mathrm{sing}}(\sigma)$ as $\sigma \to 0$

Example: $n = 5$ **as** $\sigma \to 0$

$$G_{F,\,\mathrm{sing}}(\sigma) = \frac{i\sqrt{2}}{32\pi^2}\left\{\frac{1}{\sigma^{\frac{3}{2}}} + \left(-m^2 - \frac{4}{a^2} + 20\xi\frac{1}{a^2}\right)\frac{1}{\sigma^{\frac{1}{2}}}\right\},$$

$$G^{H}_{F,\,\mathrm{sing}}(\sigma) = \frac{i\sqrt{2}}{32\pi^2}\left\{\frac{1}{\sigma^{\frac{3}{2}}} + \left(-m^2 - \frac{4}{a^2} + 20\xi\frac{1}{a^2}\right)\frac{1}{\sigma^{\frac{1}{2}}}\right\}.$$

Calculation of $\langle\Phi^2\rangle_{\rm ren}$

Code structure

Example: $n = 6$ **as** $\sigma \to 0$

$$
\begin{aligned}
&G_{F,\ \mathrm{sing}}(\sigma) \\
&= \frac{i}{16\pi^3}\left\{\frac{1}{\sigma^2} + \left(-\frac{1}{2}m^2 - \frac{10}{3}\frac{1}{a^2} + 15\xi\frac{1}{a^2}\right)\frac{1}{\sigma}\right. \\
&\qquad + \left[-\frac{1}{8}m^4 + \left(-\frac{5}{4} + \frac{15}{2}\xi\right)\frac{m^2}{a^2} + \left(-3 + \frac{75}{2}\xi - \frac{225}{2}\xi^2\right)\frac{1}{a^4}\right]\ln\bar{\sigma} \\
&\qquad \left. + \frac{1}{48}\frac{m^2}{a^2} + \left(\frac{101}{720} - \frac{5}{8}\xi\right)\frac{1}{a^4}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&G^H_{F,\ \mathrm{sing}}(\sigma) \\
&= \frac{i}{16\pi^3}\left\{\frac{1}{\sigma^2} + \left(-\frac{1}{2}m^2 - \frac{10}{3}\frac{1}{a^2} + 15\xi\frac{1}{a^2}\right)\frac{1}{\sigma}\right. \\
&\qquad + \left[-\frac{1}{8}m^4 + \left(-\frac{5}{4} + \frac{15}{2}\xi\right)\frac{m^2}{a^2} + \left(-3 + \frac{75}{2}\xi - \frac{225}{2}\xi^2\right)\frac{1}{a^4}\right]\ln\bar{\sigma} \\
&\qquad \left. + \frac{5}{48}\frac{m^2}{a^2} + \left(\frac{173}{288} - \frac{25}{8}\xi\right)\frac{1}{a^4}\right\}.
\end{aligned}
$$

Calculation of $\langle \Phi^2 \rangle_{\rm ren}$

Code structure

3. Subtraction of singular parts as $\sigma \to 0$

   Defining

$$\boxed{f_n^0 := \lim_{\sigma \to 0} \left[G_{F,\,\rm sing}(\sigma) - G^H_{F,\,\rm sing}(\sigma) \right]},$$

$$f_n^0 \begin{cases} = 0 & n \text{ odd}, \\ \neq 0 & n \text{ even}. \end{cases}$$

4. Calculation of $\langle \Phi^2 \rangle_{\rm ren}$

   Recalling

$$\langle \Phi^2 \rangle_{\rm ren} = -i \lim_{\sigma \to 0} \left[G_F(\sigma) - G^H_{F,\,\rm sing}(\sigma) \right]$$

   and

$$G_F(\sigma) = G_{F,\,\rm reg}(\sigma) + G_{F,\,\rm sing}(\sigma),$$

   then

$$\boxed{\langle \Phi^2 \rangle_{\rm ren} = -i \lim_{\sigma \to 0} \left[G_{F,\,\rm reg}(\sigma) + f_n^0 \right]}.$$

Calculation of $\langle\Phi^2\rangle_{\mathrm{ren}}$

Results

Example: $n = 10$

$$\langle\Phi^2\rangle_{\mathrm{ren}} = -\frac{1}{12288\pi^5 a^8} \times \left\{ \left(\mu^8 - 21\mu^6 + \frac{987}{8}\mu^4 - \frac{3229}{16}\mu^2 + \frac{11025}{256}\right)\left[\psi\left(\frac{1}{2}+\mu\right) + \gamma - \ln\bar{a}\right] - \frac{25}{24}\mu^8 + \frac{461}{24}\mu^6 - \frac{87983}{960}\mu^4 + \frac{3854941}{40320}\mu^2 + \frac{288563}{30720} \right\};$$

Example: $n = 11$

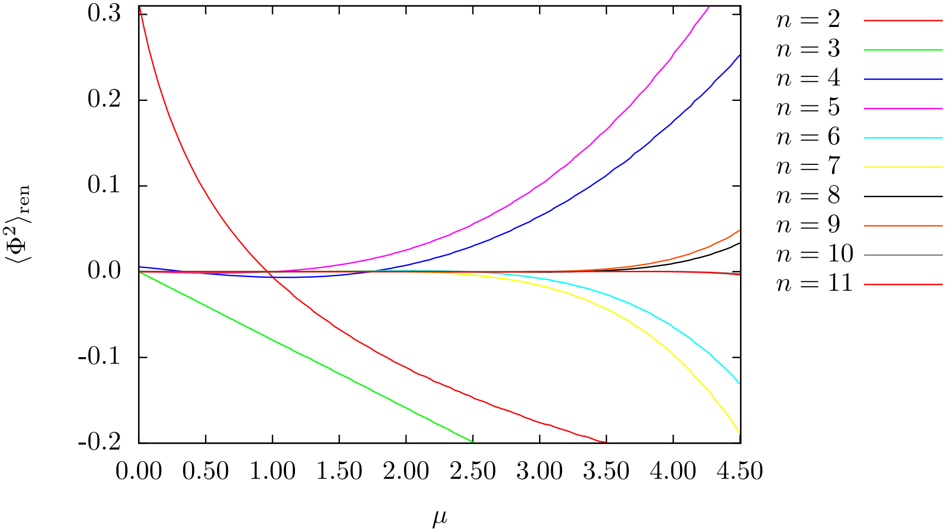
$$\langle\Phi^2\rangle_{\mathrm{ren}} = -\frac{1}{60480\pi^5 a^9}\left\{\mu^9 - 30\mu^7 + 273\mu^5 - 820\mu^3 + 576\mu\right\};$$

where

$$\mu := \sqrt{\frac{(n-1)^2}{4} + m^2 a^2 + \xi\mathcal{R}a^2}.$$

Calculation of $\langle \Phi^2 \rangle_{\text{ren}}$

Results (for $\bar{a} = 1$)



Conclusions

- Using the Hadamard theorem, we have obtained expressions of $\langle \Phi^2 \rangle_{\rm ren}$ analytically.

- Expressions have been obtained for a massive neutral scalar field in $n = 2$ to $n = 11$ inclusive involving an arbitrary coupling ξ.

- The method used is easily extended to higher n with sufficient processing power.

Key references

1. Décanini, Y., Folacci, A., *Phys. Rev. D.* **78**, 044025 (2008).
2. Allen, B., Jacobson, T., *Commun. Math. Phys.* **103**, 669 (1986).