Radiation fluid spacetimes and non-linear stability

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Presenting results of joint work
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Cosmology and stability

Central question

Any model is an approximation of particular features of our universe.
How sensitive are predictions using these models to perturbations?

Stability results using functional analysis (examples):

- **vacuum:** Christodoulou-Klainermann [1993], Anderson [2005], Lindblad-Rodnianski [2010]
- **EM:** Zipser [2000], Loizelet [2008]
- **scalar field:** Ringström [2008], Holzegel, Smulevici [2011]
- **perfect fluid:** Rodnianski-Speck [2010], Speck [2011, 12]

Stability results using the conformal methods:

- **vacuum:** Friedrich [1981, 85, 86] LV [2010, 11]
- **EMYM:** Friedrich 1991, LV 2012 (EM)

Main theorem

This talk will analyse stability of perfect fluid FLRW space-times with an equation of state $\tilde{p} = \frac{1}{3}\tilde{\rho}$, $(\gamma = \frac{4}{3})$. The energy-momentum tensor is of the form:

$$\tilde{T}_{ij} = \frac{4}{3}\tilde{\rho}\tilde{u}_i\tilde{u}_j - \frac{1}{3}\tilde{\rho}\tilde{g}_{ij}$$

These space-times describe incoherent radiation.

In particular we prove:

Theorem

Suppose we are given Cauchy initial data for the Einstein-Euler system with a de Sitter-like cosmological constant λ and equation of state $\tilde{p} = \frac{1}{3}\tilde{\rho}$.
If the initial data is sufficiently close to data for a FLRW cosmological model with $\tilde{p} = \frac{1}{3}\tilde{\rho}$, cosmological constant λ and spatial curvature $k = 1$, then

- *the development exists globally towards the future,*
- *is future geodesically complete,*
- *remains close to the FLRW solution.*

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Conformal approach of Friedrich

General idea

- Conformally embed $(\tilde{M}, \tilde{g})$ into (M, g) where $g = \theta^2 \tilde{g}$.
- Re-formulate the Einstein field equation in terms of the geometry of (M, g).
- Treat global problems in $(\tilde{M}, \tilde{g})$ via local analysis in (M, g)
- Show regularity of PDE and formulate evolution problem.
- Prove existence and uniqueness.
- Find reference space-time and prove stability.

The conformal variables: Geometry

Geometry:

- **coordinates:** $(\tau, x^{\mathcal{A}})$
- **Frame:** g-orthonormal ($g_{\mu\nu}$ indirectly defined via frame metric)
  $\rightsquigarrow$ 1+3 split and space-spinors
- **Connection:** torsion-free, here Levi-Civita connection for g
- **Curvature:** Weyl and Schouten tensor - well suited for conformal approach
- **conformal factor:** and its derivatives

Torsion, curvature conditions and Bianchi identies used to derive evolution and constraints equations

Conformal factor θ must satisfy $\nabla^k \nabla_k \theta = \theta P$ and is *fixed* by setting $P = -1$

Additional gauge source functions chosen to *fix gauge freedoms* of coordinates and frame and give symmetric hyperbolic system (Friedrich 1985).

The conformal variables: Matter

Matter (radiation fluid $\gamma = \frac{4}{3}$): $\quad \tilde{T}_{ij} = \frac{4}{3}\tilde{\rho}\tilde{u}_i\tilde{u}_j - \frac{1}{3}\tilde{\rho}\tilde{g}_{ij}$

Define $T_{ij} = \theta^{-2}\tilde{T}_{ij}$ and new variables:

- **density:** $\rho = \frac{\tilde{\rho}}{\theta^4}$
- **fluid flow:** $u_i = \theta\tilde{u}_i$ with $g(u,u) = 1$
- **derivatives of the above:** $\nabla_i\rho \to \rho_i \quad \nabla_i u_j \to u_{ij}$

$$\Rightarrow \quad T_{ij} = \frac{4}{3}\rho u_i u_j - \frac{1}{3}\rho g_{ij}$$

Evolution equations and constraints derived from $\nabla^i T_{ij} = 0 \; (\Leftrightarrow \tilde{\nabla}^i\tilde{T}_{ij} = 0)$.

Geometry and matter variables are connected by Einstein equation $\tilde{P}_{ij} = \frac{1}{2}\theta^2 T_{ij}$

Existence and uniqueness

The equations satisfied by the geometric and matter variables are known as the *conformal Einstein field equations* (CEFE) - here for a radiation fluid.

Regularity of the CEFE

If $\rho > 0, u_0 \neq 0$ then the CEFE form a regular symmetric hyperbolic system. In particular, this system is regular at conformal infinity $\mathscr{I}$, i.e. when $\theta = 0$.

Existence and uniqueness

Given sufficiently smooth initial data $\mathbf{w_0}$ for the (radiation fluid) CEFE on $\mathcal{U} \subset \mathbb{S}^3$ there exists a unique solution $\mathbf{w}$ in a neighbourhood of $\mathcal{U}$.

Radiation fluid space-time

A solution (M, g) to the CEFE (for a radiation fluid) implies a solution $(\tilde{M}, \tilde{g})$ to the Einstein field equations for a radiation fluid, where $\tilde{M} = M|_{\{\theta>0\}}, \tilde{g} = \theta^{-2} g$.

FLRW and the conformal reference space-time

Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2_{FLRW} = dt^2 - \frac{a(t)^2}{(1+\frac{1}{4}kr^2)^2}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2))$$

$$k = 1 \Rightarrow \qquad ds^2_{FLRW} = a(t)^2\left[d\tau^2 - d\sigma^2_{\mathbb{S}^3}\right] = a(t)^2 ds^2_{EC}$$

where $\tau = \int_{t_0}^{t} \frac{dt'}{a(t')}$ and $d\sigma^2_{\mathbb{S}^3}$ is the standard metric on $\mathbb{S}^3$.

- Work with $(M, g) = (\mathbb{R} \times \mathbb{S}^3, g_{EC})$ and $(\tilde{M}, \tilde{g}) = (I \times \mathbb{S}^3, g_{FLRW})$.
- For $\lambda < 0$ (deSitter-like case), $\mathscr{I}^+$ is a space-like hypersurface $\tau = \tau_\infty(x^{\mathcal{A}})$.
- Use FLRW with $\gamma = \frac{4}{3}$, $k = 1$, $\lambda < 0$ as reference space-time and read off initial data $\mathring{\mathbf{w}}_0$ for the CEFE (note $P_{EC} = -1$).

Stability theorem for a tracefree perfect fluid

We work in $H^m(\mathbb{S}^3, \mathbb{R}^N)$, where $m > 4$

Theorem

Let $\mathbf{w}_0$ be initial data on $\mathbb{S}^3$ for CEFE for radiation fluids with $\lambda < 0$ such that $\mathbf{w}_0$ is sufficiently close to $\mathring{\mathbf{w}}_0$ (FLRW data with $\lambda < 0$ and $k = 1$). Then

- *a solution $\mathbf{w}$ to the CEFE exists on $[0, T] \times \mathbb{S}^3$ with $T > \tau_\infty$,*
- *$\mathbf{w}$ implies a C^{m-2} solution of the Einstein equations for a radiation fluid on $\tilde{M} = \{p \in [0, T] \times \mathbb{S}^3 : \theta(p) > 0\}$,*
- *the development exists globally towards the future,*
- *$\tilde{M}$ is future geodesically complete and $\mathscr{I}^+$ a space-like hypersurface.*
- *$\mathbf{w}$ remains close to the FLRW solution, which is hence non-linearly stable.*

Remarks:

- Similar results for $k = 0, 1$ and $\lambda \leq 0$ can be obtained
- The results complement Speck[2011], where $\gamma \neq \frac{4}{3}$.
- Speck [2012] covers $\gamma = \frac{4}{3}$ using conformal invariance of Einstein-Euler equations.

Future work and open questions

1. *Can a similar result be obtained for null dust?*
2. *Can the conformal analysis be extend to perfect fluids with* $\gamma \neq \frac{4}{3}$*?*
   Anguige, Tod [1999] analysed isotropic singularities in perfect fluid spacetimes with $1 < \gamma \leq 2$
3. *Can one use congruences to fix the gauge choice?* $\rightsquigarrow$ Weyl connections
   - conformal geodesics for vacuum (Friedrich 1995, 2003, LV 2010, 2011)
   - conformal curves for Einstein-Maxwell (LV 2012)
4. *Can one use congruences to locate conformal infinity?*
   The above examples allow to prescribe / predict the location of the conformal infinity due to explicit knowledge of the conformal factor in terms of the time parameter of the curves.

Thank you for listening

Reference: arXiv:1111.4691