Second-order equation of motion of a small compact body

Adam Pound

University of Southampton

April 4, 2012

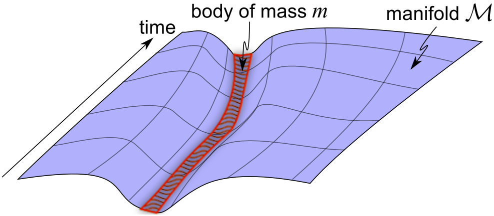
A small extended body moving through spacetime

Fundamental question

- how does a body's gravitational field affect its own motion?

Regime: asymptotically small body

- examine spacetime $(\mathcal{M}, g_{\mu\nu})$ containing body of mass m and external lengthscales $\mathcal{R}$
- seek representation of motion in limit $\epsilon = m/\mathcal{R} \ll 1$

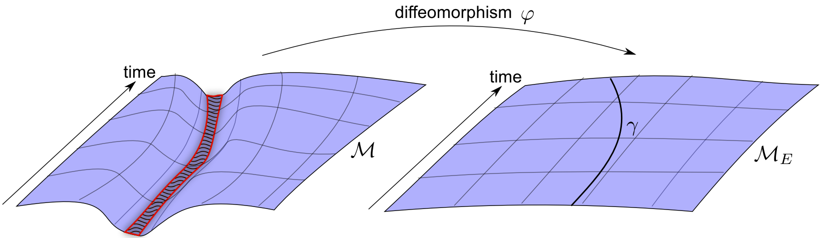


Gravitational self-force

- treat body as source of perturbation of external background spacetime $(\mathcal{M}_E, \mathsf{g}_{\mu\nu})$

$$\mathsf{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \ldots$$

- $h^{(n)}_{\mu\nu}$ exerts *self-force* on body
- self-force at linear order in ϵ first calculated in 1996 [Mino, Sasaki, and Tanaka], now on firm basis [Gralla & Wald; Pound; Harte]

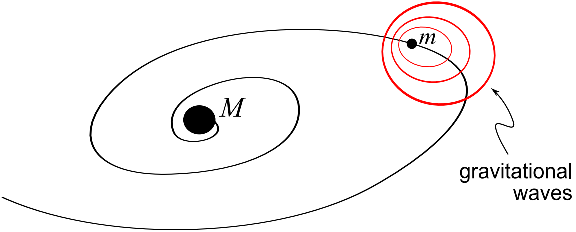


Body in exact spacetime

Approximation of motion in external spacetime

Canonical example: extreme-mass-ratio inspiral

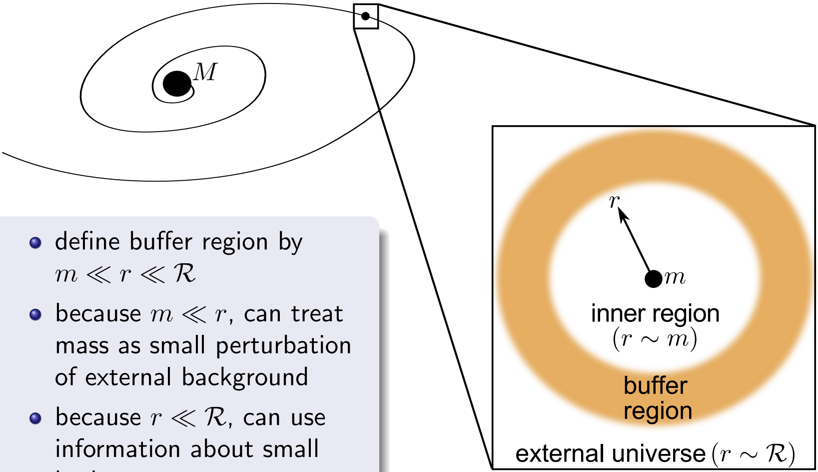
- solar-mass neutron star or black hole orbits supermassive black hole
- m = mass of smaller body, $\mathcal{R} \sim M$ = mass of large black hole
- $(\mathcal{M}, g_{\mu\nu})$ = Kerr spacetime of large black hole



Why second order?

- inspiral occurs very slowly, on timescale $1/\epsilon$
  $\Rightarrow$ need $O(\epsilon^2)$ terms in acceleration to get trajectory correct at $O(1)$
- also useful to complement PN and NR

How to determine motion: buffer region

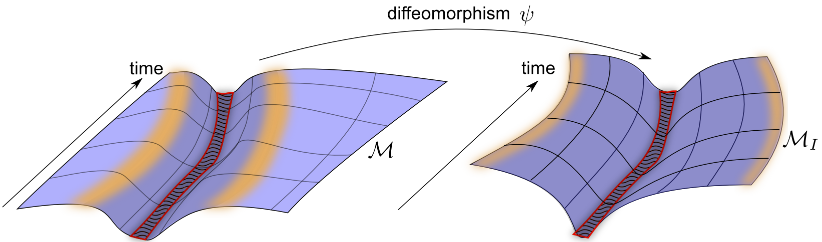


- define buffer region by $m \ll r \ll \mathcal{R}$
- because $m \ll r$, can treat mass as small perturbation of external background
- because $r \ll \mathcal{R}$, can use information about small body

Matched asymptotic expansions: *inner expansion*

Zoom in on body

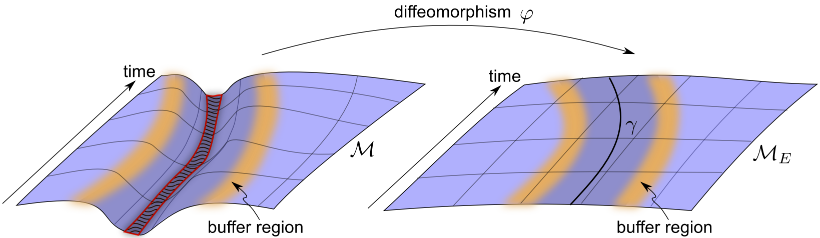
- map ψ keeps size of body fixed, sends other distances to infinity (e.g., using coords $\sim r/\epsilon$)
- unperturbed body defines background spacetime $g_{I\mu\nu}$ in inner expansion
- buffer region at asymptotic infinity
  $\Rightarrow$ can define multipole moments



Matched asymptotic expansions: *outer expansion*

Send body to zero size around a worldline

- map φ shrinks body to zero size, holding other distances fixed
- build metric $g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \ldots$ in external universe (outside buffer region) subject to *matching condition*: in coords centered on γ, metric in buffer region must agree with inner expansion



Metric in buffer region

Expansion for small r

- presence of *any* compact body in inner region leads to

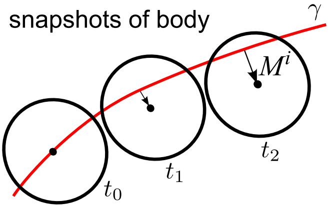
$$h^{(1)}_{\mu\nu} = \frac{1}{r}h^{(1,-1)}_{\mu\nu} + h^{(1,0)}_{\mu\nu} + rh^{(1,1)}_{\mu\nu} + O(r^2)$$
$$h^{(2)}_{\mu\nu} = \frac{1}{r^2}h^{(2,-2)}_{\mu\nu} + \frac{1}{r}h^{(2,-1)}_{\mu\nu} + h^{(2,0)}_{\mu\nu} + O(r)$$

where r is distance from γ

- most divergent terms are background spacetime in inner expansion: $g_{I\mu\nu} = \eta_{\mu\nu} + \frac{1}{r}h^{(1,-1)}_{\mu\nu} + \frac{1}{r^2}h^{(2,-2)}_{\mu\nu} + O(1/r^3)$

Relating worldline to body

- define γ to be worldline of body iff mass dipole terms vanish in coords centered on γ



Solving the EFE with an accelerated source

Expansion of EFE

- allow γ to depend on ϵ and assume outer expansion of form

$$\begin{aligned} \mathrm{g}_{\mu\nu}(x,\epsilon) &= g_{\mu\nu}(x) + h_{\mu\nu}(x;\gamma) \\ &= g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x;\gamma\ ) + \epsilon^2 h^{(2)}_{\mu\nu}(x;\gamma\ ) + \ldots \end{aligned}$$

- need a method of systematically solving for each $h^{(n)}_{\mu\nu}$
  $\Rightarrow$ impose Lorenz gauge on total perturbation: $\nabla_\mu \bar{h}^{\mu\nu} = 0$
- linearized Einstein tensor $\delta G_{\mu\nu}$ becomes a wave operator and EFE becomes a weakly nonlinear wave equation:

$$\Box \bar{h}_{\mu\nu}[\gamma] + 2R_\mu{}^\rho{}_\nu{}^\sigma \bar{h}_{\rho\sigma}[\gamma] = 2\delta^2 G_{\mu\nu}[h] + \ldots$$

  (no stress-energy tensor because equation written outside body)
- can be split into wave equations for each subsequent $h^{(n)}_{\mu\nu}[\gamma]$ and exactly solved for arbitrary γ
- $\nabla_\mu \bar{h}^{\mu\nu} = 0$ determines γ

Solving the EFE with an accelerated source

Expansion of EFE

- allow γ to depend on ϵ and assume outer expansion of form

$$\mathsf{g}_{\mu\nu}(x,\epsilon) = g_{\mu\nu}(x) + h_{\mu\nu}(x;\gamma)$$
$$= g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x;\gamma_\epsilon) + \epsilon^2 h^{(2)}_{\mu\nu}(x;\gamma_\epsilon) + \ldots$$

- need a method of systematically solving for each $h^{(n)}_{\mu\nu}$
  $\Rightarrow$ impose Lorenz gauge on total perturbation: $\nabla_\mu \bar{h}^{\mu\nu} = 0$
- linearized Einstein tensor $\delta G_{\mu\nu}$ becomes a wave operator and EFE becomes a weakly nonlinear wave equation:

$$\Box \bar{h}_{\mu\nu}[\gamma] + 2R_\mu{}^\rho{}_\nu{}^\sigma \bar{h}_{\rho\sigma}[\gamma] = 2\delta^2 G_{\mu\nu}[h] + \ldots$$

  (no stress-energy tensor because equation written outside body)
- can be split into wave equations for each subsequent $h^{(n)}_{\mu\nu}[\gamma]$ and exactly solved for arbitrary γ
- $\nabla_\mu \bar{h}^{\mu\nu} = 0$ determines γ

General solution in buffer region

First order

- splits into two solutions: $h^{(1)}_{\mu\nu} = h^{S(1)}_{\mu\nu} + h^{R(1)}_{\mu\nu}$
- $h^{S(1)}_{\mu\nu} \sim 1/r + \ldots$ defined by mass monopole m
- $h^{R(1)}_{\mu\nu} \sim r^0 + \ldots$ undetermined homogenous solution regular at $r = 0$
- $\nabla_\mu \bar{h}^{\mu\nu} = 0 \Rightarrow \dot{m} = 0$ and $a^\mu_{(0)} = 0$

Second order

- splits into two solutions: $h^{(2)}_{\mu\nu} = h^{S(2)}_{\mu\nu} + h^{R(2)}_{\mu\nu}$
- $h^{S(2)}_{\mu\nu} \sim 1/r^2 + 1/r + \ldots$ defined by
  1. mass correction δm
  2. mass dipole M^μ (set to zero with appropriate choice of γ)
  3. spin dipole S^μ
- $\nabla_\mu \bar{h}^{\mu\nu} = 0 \Rightarrow \dot{S}^\mu = 0$, $\delta\dot{m} = \ldots$, and $a^\mu_{(1)} = \ldots$

Matching to an inner expansion

Inner expansion

- could continue with same method to find $a^{\mu}_{(2)}$ from $h^{(3)}_{\mu\nu}$
- instead, get more information from inner expansion
- assume metric in inner expansion is Schwarzschild as tidally perturbed by external universe
- write tidally perturbed Schwarzschild metric in mass-centered coordinates

Matching

- expand inner metric in buffer region (i.e., for $r \gg m$)
- demand inner and outer expansions in buffer region are related by unique gauge transformation $x^{\mu} \to x^{\mu} + \epsilon\xi^{\mu} + \dots$
- restrict gauge transformation to include no translations at $r = 0$ to ensure worldline correctly associated with center of mass

Equation of motion

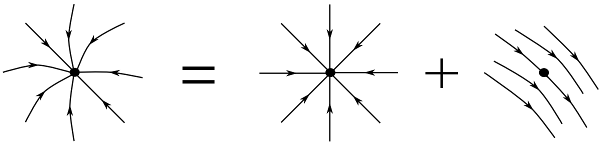
Self-force

- matching procedure yields acceleration

$$a^\mu = \frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(g_\nu{}^\rho - h^{\mathrm{R}\rho}_\nu\right)\left(h^{\mathrm{R}}_{\sigma\lambda;\rho} - 2h^{\mathrm{R}}_{\rho\sigma;\lambda}\right)u^\sigma u^\lambda + O(\epsilon^3)$$

where $a^\mu = a^\mu_{(0)} + \epsilon a^\mu_{(1)} + \epsilon^2 a^\mu_{(2)} + \dots$
and $h^R_{\mu\nu} = \epsilon h^{R(1)}_{\mu\nu} + \epsilon^2 h^{R(2)}_{\mu\nu} + \dots$

- this is geodesic equation in metric $g_{\mu\nu} + h^R_{\mu\nu}$
- equation for more generic body will be the same, modified only by body's multipole moments



body's field $h_{\alpha\beta}$ = singular field $h^S_{\alpha\beta}$ + regular field $h^R_{\alpha\beta}$

Summary

Determining the motion of a small body

- define a worldline of an asymptotically small body, even a black hole, by comparing metric in a buffer region around body in full spacetime and in background spacetime
- determine equation of motion from consistency of Einstein's equation

Future work

- find equation for spinning, non-spherical body
- with collaborators, numerically solve wave equation and determine trajectory of body in an EMRI

Puncture scheme

Effective stress energy tensor

- can prove that $h^{S(1)}_{\mu\nu}$ and δm terms in $h^{S(2)}_{\mu\nu}$ are identical to fields sourced by point-particles of mass m and δm on γ

Obtaining global solution

- outside a hollow worldtube Γ around body, solve

$$E_{\mu\nu}[\bar{h}^{(1)}] = 0, \quad E_{\mu\nu}[\bar{h}^{(2)}_{\mu\nu}] = 2\delta^2 G_{\mu\nu}[h^{(1)}]$$

  where, e.g., $E_{\mu\nu}[\bar{h}^{(2)}] = \Box\bar{h}^{(2)}_{\mu\nu} + 2R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}\bar{h}^{(2)}_{\rho\sigma}$

- inside Γ define $h^{P(n)}_{\mu\nu}$ as small-r expansion of $h^{S(n)}_{\mu\nu}$ truncated at highest order available. Solve

$$E_{\mu\nu}[\bar{h}^{R(1)}] = T^{(1)}_{\mu\nu} - E_{\mu\nu}[\bar{h}^{P(1)}]$$
$$E_{\mu\nu}[\bar{h}^{R(2)}] = 2\delta^2 G_{\mu\nu}[h^{(1)}] + T^{(2)}_{\mu\nu} - E_{\mu\nu}[\bar{h}^{P(2)}]$$

- match $h^{(n)}_{\mu\nu}$ to $h^{S(n)}_{\mu\nu} + h^{R(n)}_{\mu\nu}$ at Γ