Self-consistent orbital evolution of a particle around a Schwarzschild black hole

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EMRIs and eLISA/NGO

- Extreme Mass Ratio Inspirals have long been a promising source of gravitational waves for the LISA, the space based gravitational wave detector.
- Accurate models are a critical component of any observation.
- Even more true now that LISA is no more and there are proposals for eLISA/NGO which will have less sensitivity.

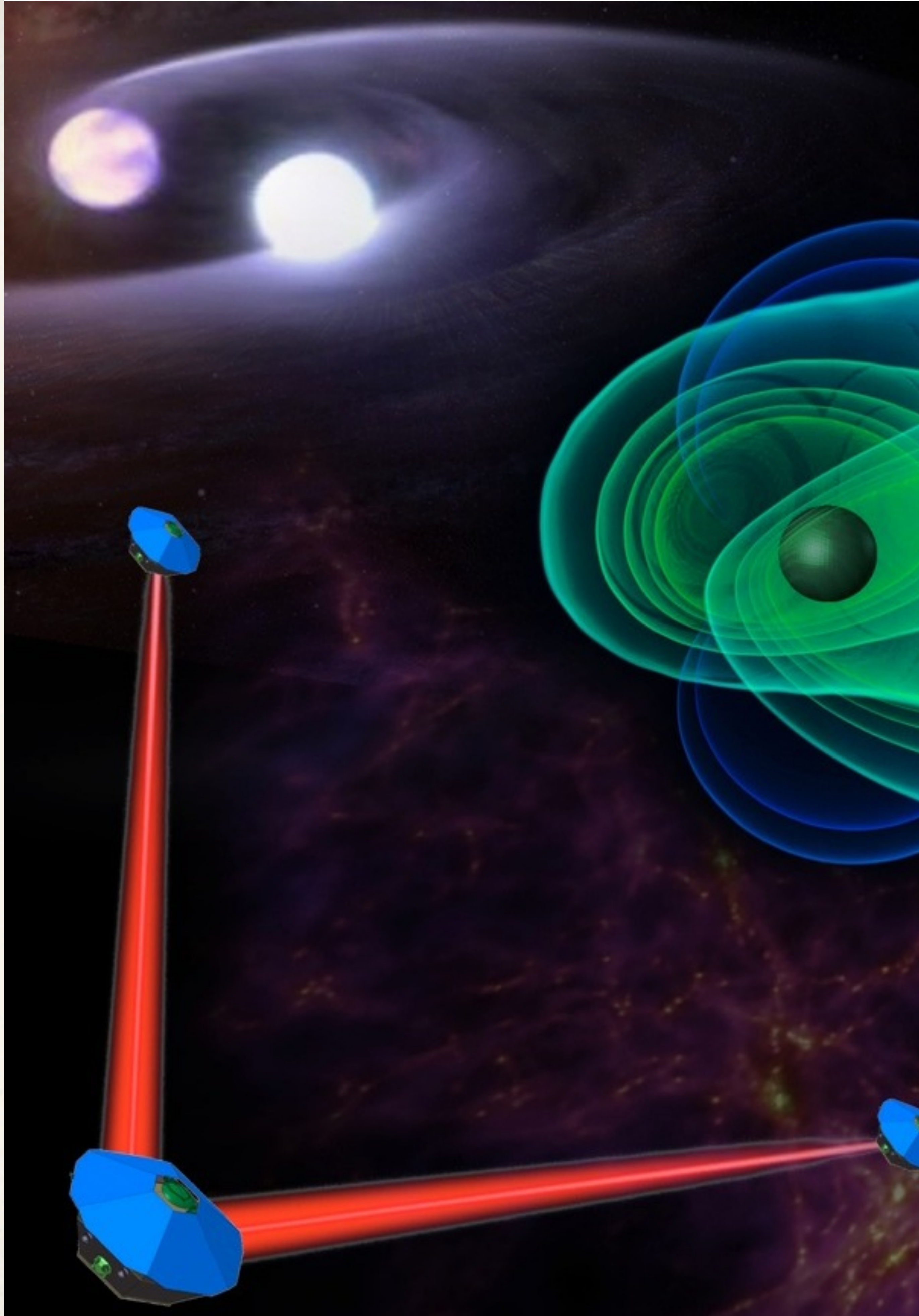


Image credit: eLISA/NGO Yellow book (ftp://ftp.rssd.esa.int/pub/ojennric/NGO_YB/NGO_YB.pdf)

Motion of a point particle

- Solve the coupled system of equations for the motion of the particle and its retarded field.
- Self-interaction of the particle with its retarded field, $\mathbf{\Phi}^{\text{ret}}$.
- $\mathbf{\Phi}^{\text{ret}}$ diverges like 1/r on the world-line.
- "Unphysical" divergence removed by appropriate regularization.

$$\Box\Phi^{\text{ret}} = -4\pi q \int \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} d\tau$$

$$\frac{Du^\alpha}{d\tau} = a^\alpha = \frac{\bar{q}}{m(\tau)}(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi^{\text{ret}}$$

$$\frac{dm}{d\tau} = -\bar{q} u^\beta \nabla_\beta \Phi^{\text{ret}}$$

Effective source regularization

- Split retarded field into locally constructed field and "regularized" remainder.

$$\Phi^{\rm ret} = \Phi^{\rm S} + \Phi^{\rm R}$$

- Derive an equation for $\mathbf{\Phi}^{\rm R}$.

$$\Box\Phi^{\rm R} = \Box\Phi^{\rm ret} - \Box\Phi^{\rm S}$$

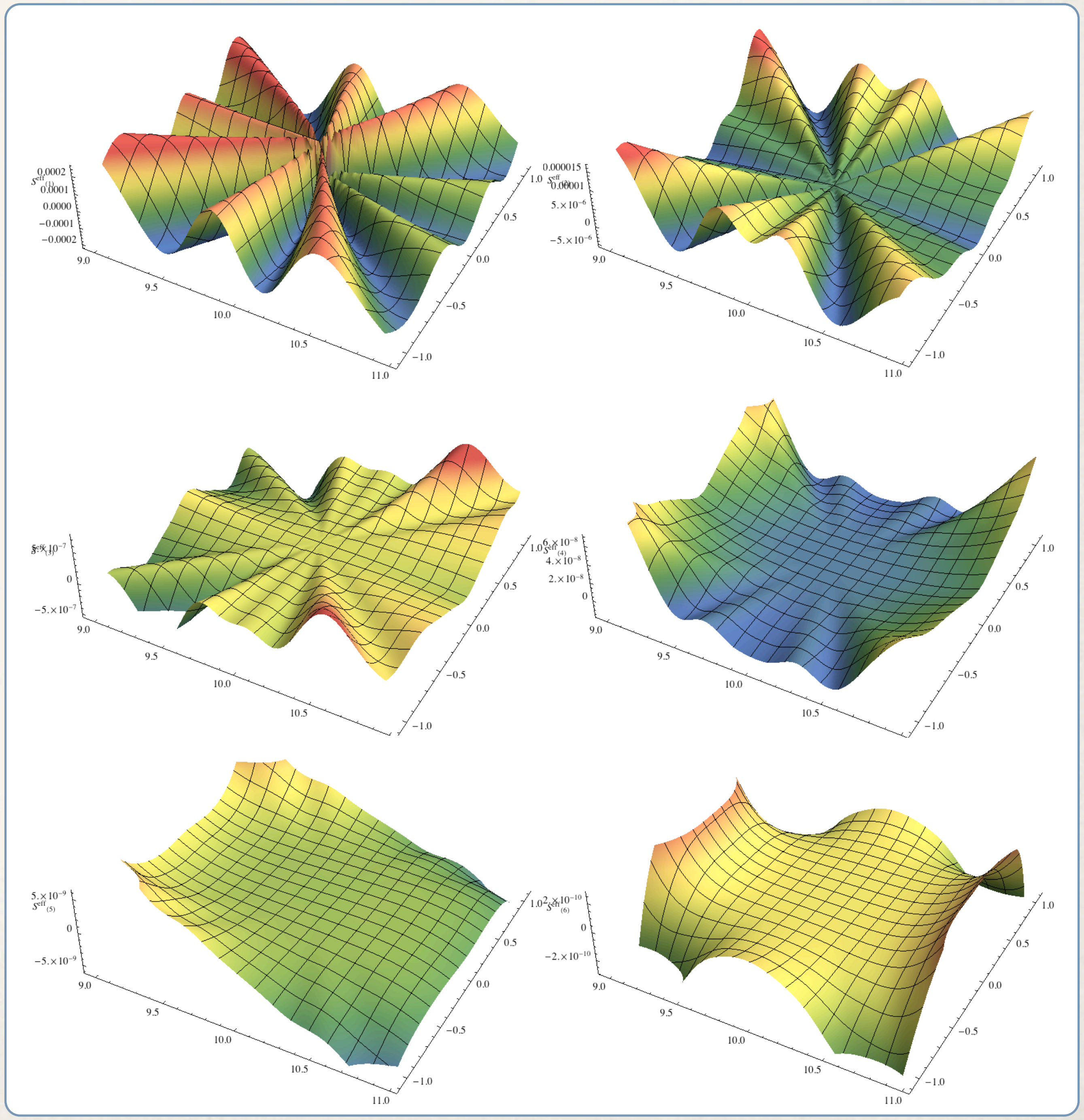
- Always work with $\mathbf{\Phi}^{\rm R}$ instead of $\mathbf{\Phi}^{\rm ret}$.
- If $\mathbf{\Phi}^{\rm S}$ is chosen appropriately, then we can just replace $\mathbf{\Phi}^{\rm ret}$ with $\mathbf{\Phi}^{\rm R}$ in the equations of motion.

$$\frac{Du^\alpha}{d\tau} = a^\alpha = \frac{\bar{q}}{m(\tau)}(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi^{\rm R}$$

$$\frac{dm}{d\tau} = -\bar{q}u^\beta\nabla_\beta\Phi^{\rm R}$$

Effective source regularization

- If $\mathbf{\Phi}^S$ is exactly the Detweiler-Whiting singular field, $\mathbf{\Phi}^R$ is a solution of the homogeneous wave equation.
- If $\mathbf{\Phi}^S$ is only approximately the Detweiler-Whiting singular field, then the equation for $\mathbf{\Phi}^R$. has an effective source, S.
- S is typically finite, but of limited differentiability on the world line.



Self-consistent Evolution

- Solve the coupled system of equations for the motion of the particle and its regularized field.
- $\mathbf{\Phi}^{\mathrm{R}} = \mathbf{\Phi}^{\mathrm{ret}}$ in the wave zone
- $\mathbf{\Phi}^{\mathrm{R}}$ finite and (typically) twice differentiable on the world-line

$$\Box \Phi^{\mathrm{R}} = S(x|z(\tau), u(\tau))$$

$$\frac{Du^{\alpha}}{d\tau} = a^{\alpha} = \frac{\bar{q}}{m(\tau)}(g^{\alpha\beta} + u^{\alpha}u^{\beta})\nabla_{\beta}\Phi^{\mathrm{R}}$$

$$\frac{dm}{d\tau} = -\bar{q}u^{\beta}\nabla_{\beta}\Phi^{\mathrm{R}}$$

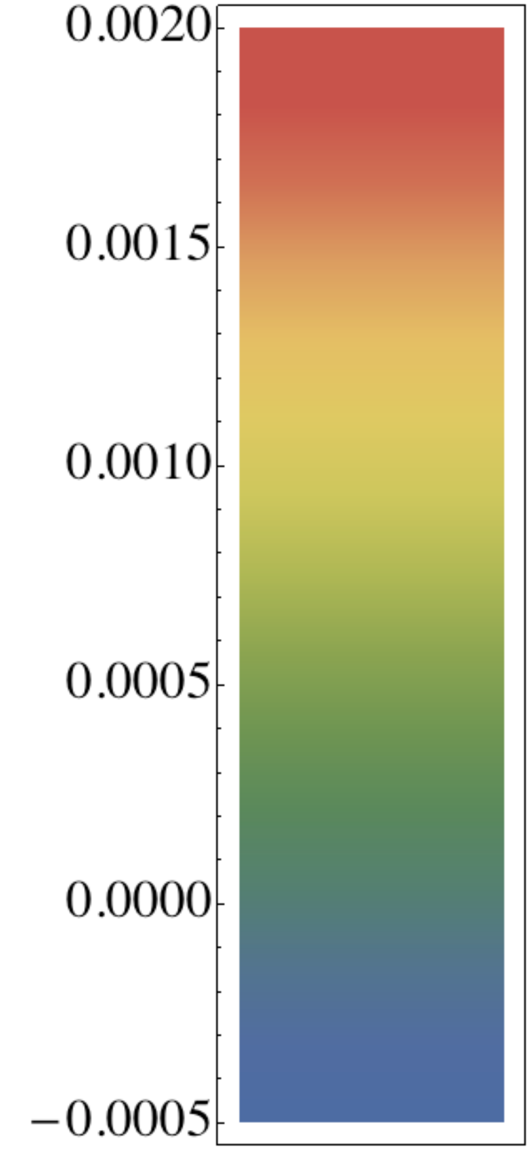
$\Phi^{R}(t)$

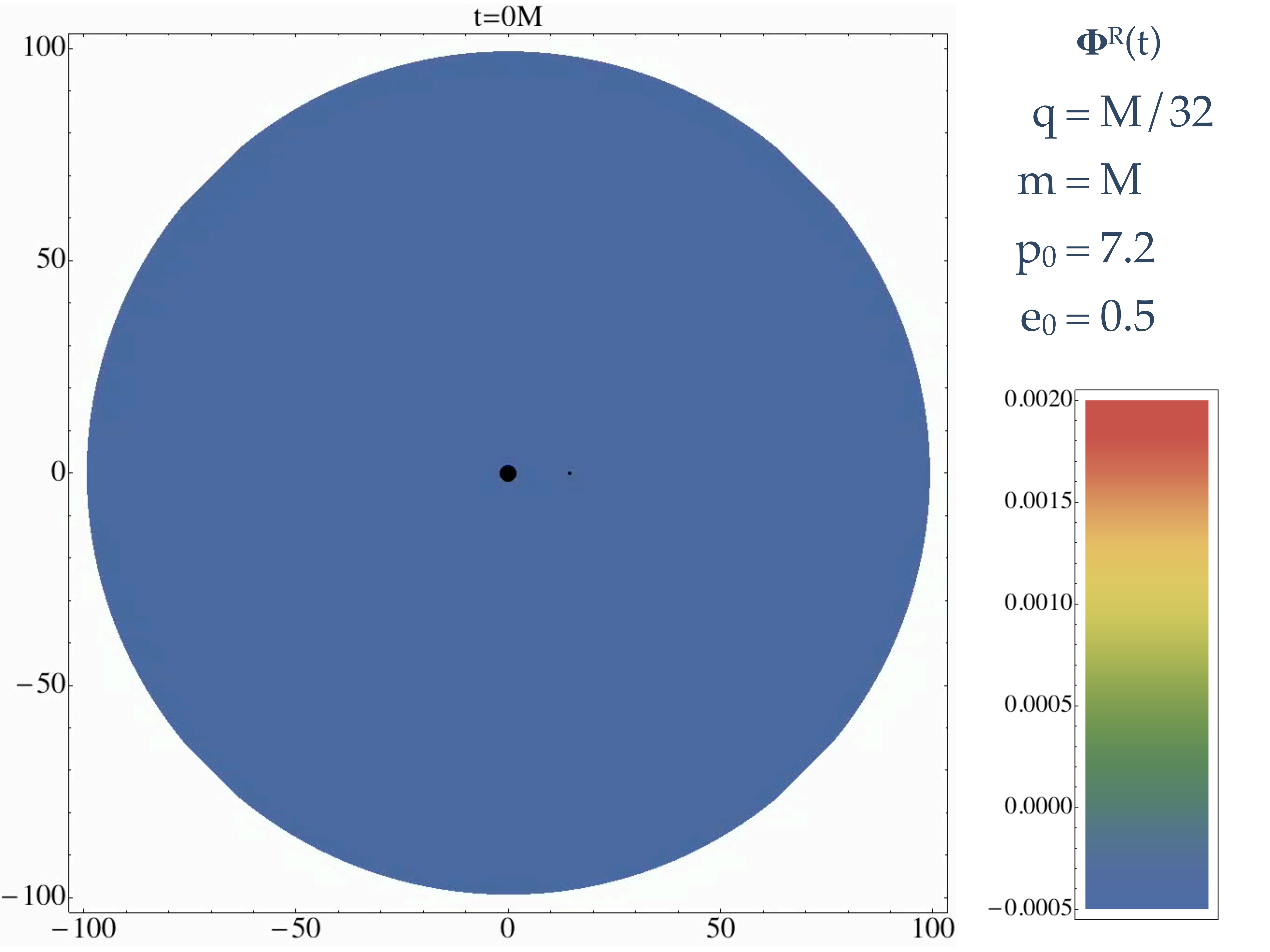
$q = M/32$

$m = M$

$p_0 = 7.2$

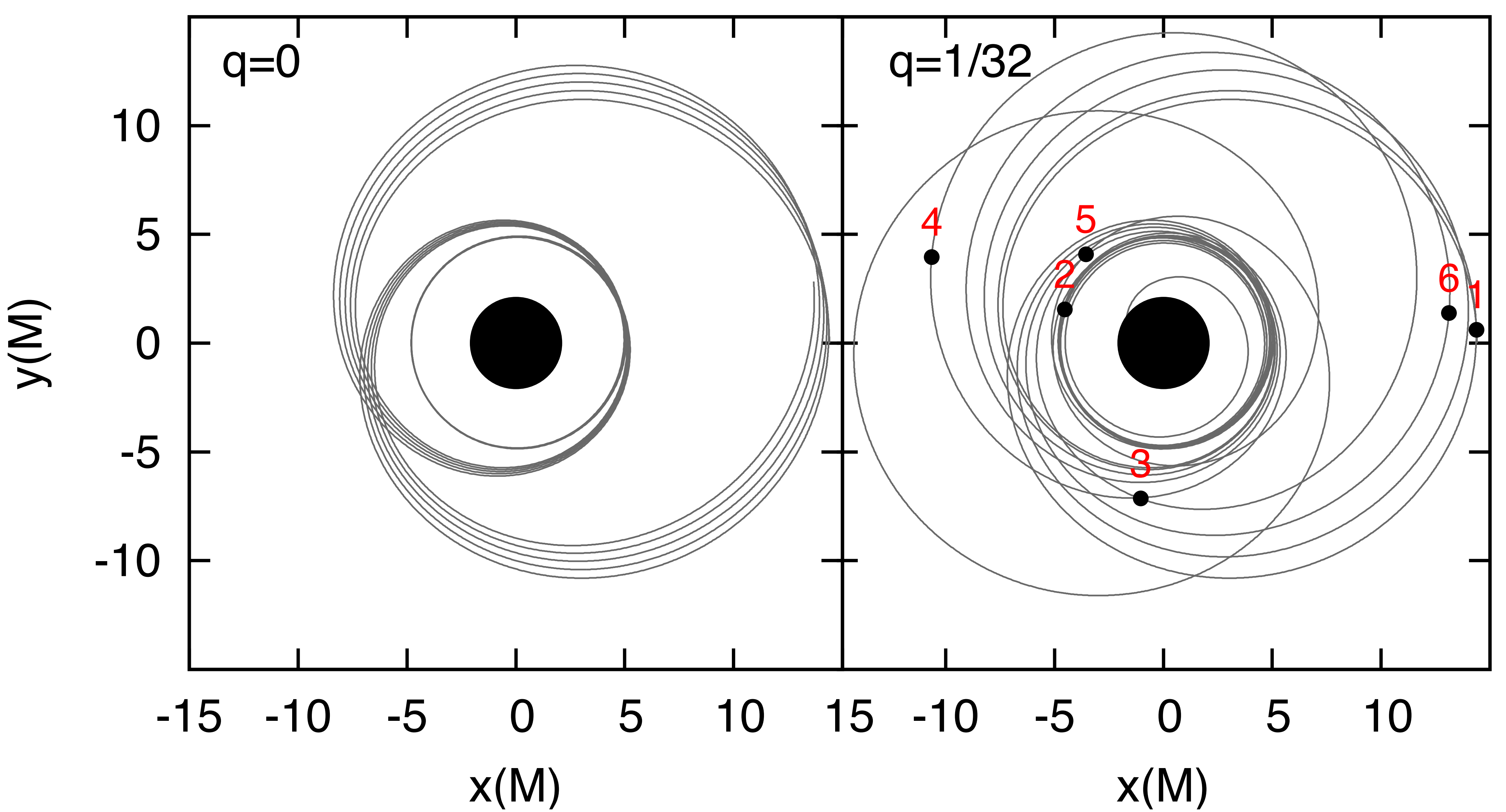
$e_0 = 0.5$





t=0M
100
50
0
−50
−100
−100
−50
0
50
100
Φ^R(t)
q = M/32
m = M
p0 = 7.2
e0 = 0.5
0.0020
0.0015
0.0010
0.0005
0.0000
−0.0005

Orbital motion



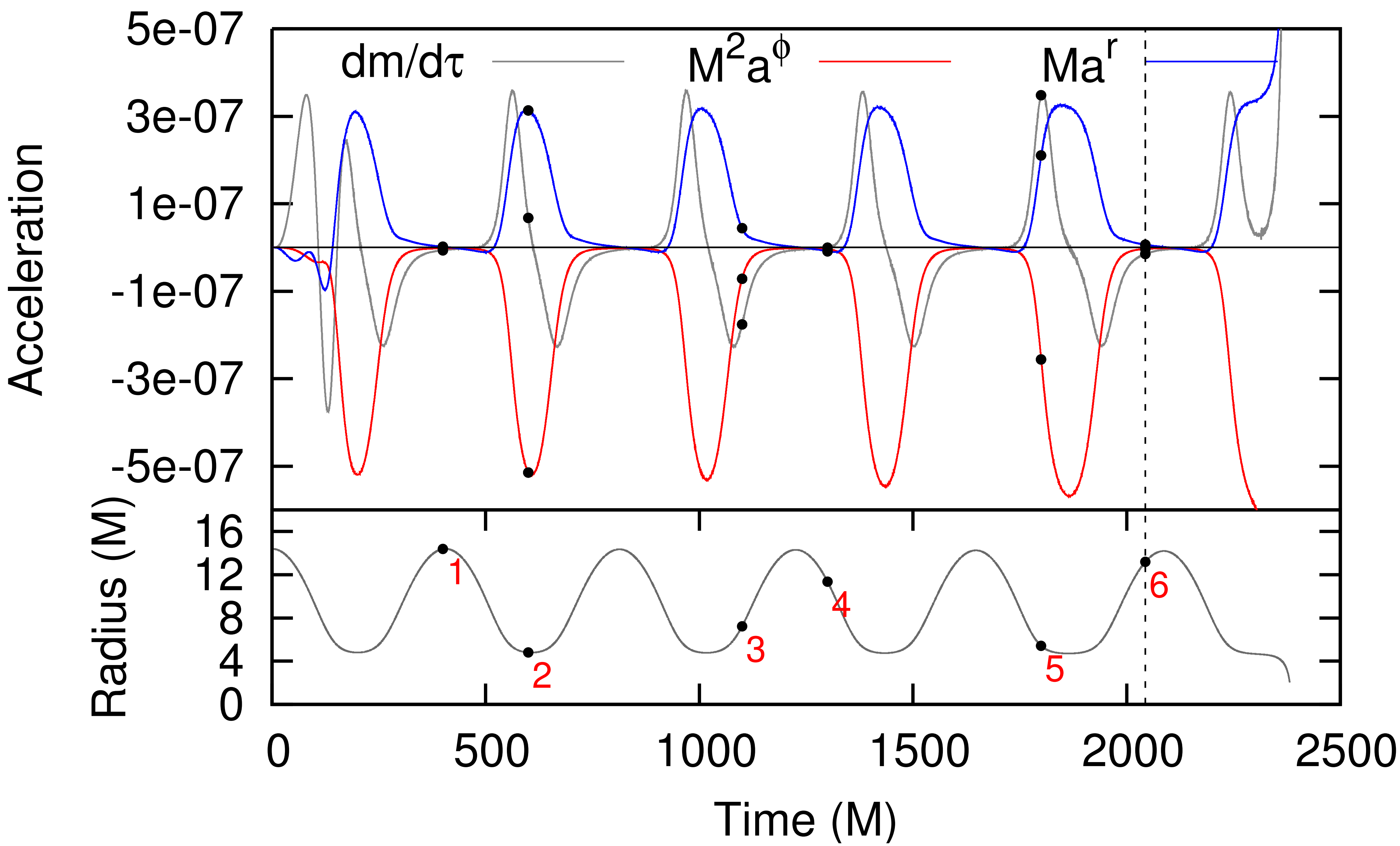
Orbital motion

- Parametrize orbits in terms of a dimensionless semilatus rectum p and eccentricity e, such that $r_{\pm} = Mp/(1 \mp e)$.
- Separatrix, $p = 6 + 2e$, corresponds to unstable circular orbits and represents the boundary in p–e space separating bound from plunging orbits.

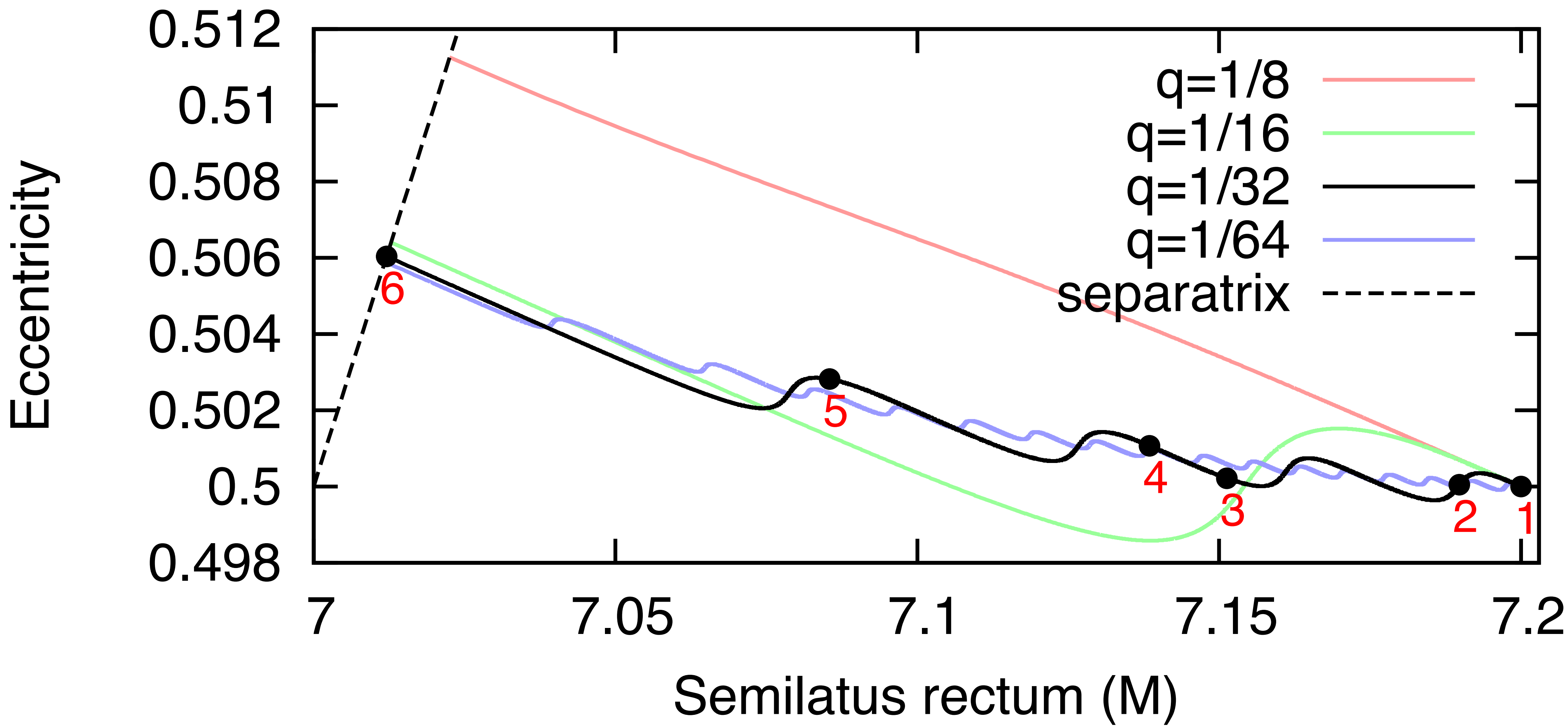
$$r(t) = \frac{Mp}{1 + e\cos(\chi - w)}$$

$$\frac{d\phi}{dt} = \left[1 - \frac{2Mr'}{r - 2M}\right] \times \frac{[p - 2 - 2e\cos(\chi - w)][1 + e\cos(\chi - w)]^2}{M\sqrt{p^3[(p-2)^2 - 4e^2]}}$$

Orbital evolution

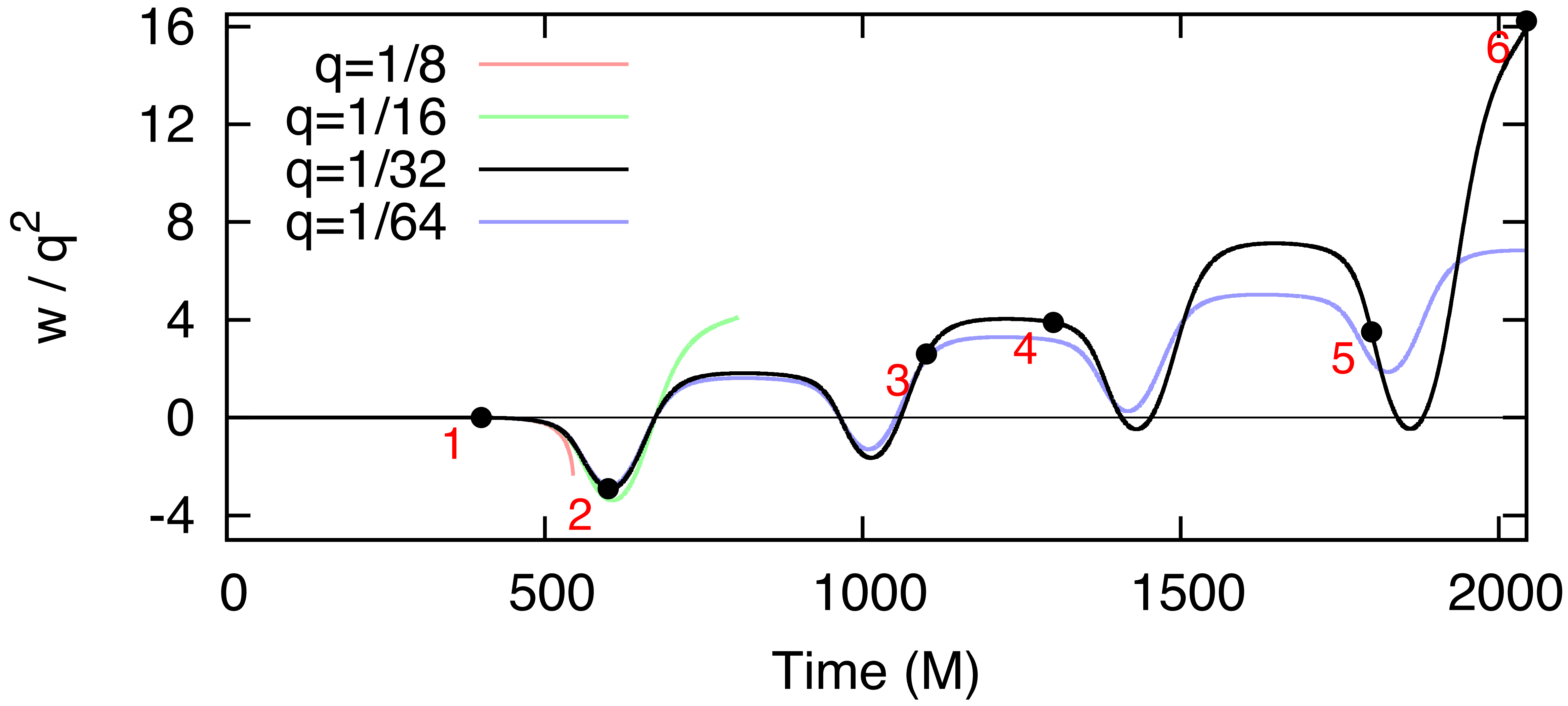


Orbital evolution - "dissipative"



$$r(t) = \frac{Mp}{1 + e\cos(\chi - w)}$$

Orbital evolution - “conservative”



$$r(t) = \frac{Mp}{1 + e\cos(\chi - w)}$$

Waveforms at $\mathscr{I}^+$