The vacuum density fluctuations in extended chaotic inflation

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Draft 1.3.92

1. The background solution

Consider a scalar field σ with potential $V_i(\sigma) = C\sigma^{2n}$ which is coupled to the Jordan-Brans-Dicke theory of gravity, where G is Newton's constant and ω is the dimensionless JBD coupling constant:

$$S = \int d^4x\sqrt{-\tilde{g}}\Big[\phi\frac{1}{16\pi G}\tilde{R} - \frac{\omega}{16\pi G}\phi^{-1}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\tilde{g}^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma - V_i(\sigma)\Big]. \qquad (1)$$

The dynamics are more simply described in what is called the Einstein frame. We shall find the notations $8\pi G \equiv \kappa$ and $\omega \equiv \kappa\varphi_0^2 - 3/2 \equiv \beta^{-2} - 3/2$ useful. The Einstein frame is given by a redefinition of the metric, $g_{ab} \equiv \phi\tilde{g}_{ab}$, and of the Brans-Dicke scalar, $\phi \equiv \exp(\varphi/\varphi_0)$. The action becomes, up to boundary terms,

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2\kappa}R - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}e^{-\varphi/\varphi_0}g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma - e^{-2\varphi/\varphi_0}V_i(\sigma)\Big]. \qquad (2)$$

For the cosmological application we assume a flat FRW universe with homogeneous σ and φ. Let the scale factor in the Einstein frame be $a(t)$, where t is the Einstein frame cosmic time and $H = \dot{a}/a$. Then the equations of motion are

$$H^2 = \frac{\kappa}{3}\left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}e^{-\varphi/\varphi_0}\dot{\sigma}^2 + e^{-2\varphi/\varphi_0}C\sigma^{2n}\right), \qquad (3)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{1}{2\varphi_0}e^{-\varphi/\varphi_0}\dot{\sigma}^2 - \frac{2}{\varphi_0}e^{-2\varphi/\varphi_0}C\sigma^{2n} = 0, \qquad (4)$$

$$\ddot{\sigma} + \left(3H - \frac{\dot{\varphi}}{\varphi_0}\right)\dot{\sigma} + e^{-\varphi/\varphi_0}2nC\sigma^{2n-1} = 0. \qquad (5)$$

The effective phase space is 4-dimensional, given by φ, $\dot{\varphi}$, σ and $\dot{\sigma}$.

In chaotic inflation one typically considers a class of solutions which locally resemble de Sitter space, because the scale factor changes much more rapidly than the scalar field and hence the Hubble constant. These are called slow rolling solutions. Here we shall look for solutions in which only the scalar field is slowly varying compared to the scale factor, while the dilaton φ and the Hubble constant may be rapidly varying. We define a dimensionless parameter f by

$$f^2 \equiv e^{\varphi/\varphi_0}\frac{1}{\kappa\sigma^2} \qquad (6)$$

and will speak of slow rolling if $f^2 \ll 1$. We find that if we neglect terms of order f^2 compared to terms of order 1 in the equations of motion, and if $\beta^2 < 1$, which will turn out to be necessary for inflation anyway and if n is only a few, then the equations of motion simplify to the following set:

$$H = \frac{\kappa}{2}\,\tilde{C} e^{-\varphi/\varphi_0} \sigma^n, \tag{7}$$

$$\dot{\varphi} = \frac{1}{\varphi_0}\,\tilde{C} e^{-\varphi/\varphi_0} \sigma^n, \tag{8}$$

$$\dot{\sigma} = -n\,\tilde{C}\sigma^{n-1}, \tag{9}$$

where

$$\tilde{C} = \left(\frac{C}{6\pi G}\right)^{\frac{1}{2}} \left(1 - \frac{2}{3}\beta^2\right)^{-\frac{1}{2}}.$$

The slow rolling approximation results in neglecting $\dot{\sigma}^2$ and $\ddot{\sigma}$, but not $\ddot{\varphi}$ and $\dot{\varphi}^2$ in equations (3-5). Furthermore $\ddot{\varphi}$ in (4) is approximated by $-\dot{\varphi}^2/\varphi_0$. A different slow rolling approximation was made in [1], where $\ddot{\varphi}$ and $\dot{\varphi}^2$ where also neglected. The present approximation is more exact for large values of β and differs from [1] by the second factor in $\tilde{C}$.

The slow rolling approximation is of course only relevant if it corresponds to solutions which are stable against small perturbations, i.e. which are attractors at least locally. This is indeed the case. We define $x(\dot{\varphi}, \varphi, \sigma)$ and $y(\dot{\sigma}, \varphi, \sigma)$ to be the values of $\dot{\varphi}$ and $\dot{\sigma}$ minus their slow rolling values, given as functions of σ and φ by (8,9). We can then use x, y, σ and φ as coordinates on phase space, where the slow rolling approximation correspnds to $x = y = 0$. Linearizing in x and y, that is around the slow rolling surface in phase space, we obtain

$$\dot{x} = -(12\pi G + \varphi_0^{-2})\tilde{C} e^{-\varphi/\varphi_0} \sigma^n x \tag{10}$$

$$\dot{y} = -(12\pi G - \varphi_0^{-2})\tilde{C} e^{-\varphi/\varphi_0} \sigma^n y - (n^3/2 - n^2)\tilde{C}\sigma^{2n-3} \tag{11}$$

The slow rolling approximation is not a solution, but approximates a solution which is locally an attractor.

Now we solve the approximate equations of motion. Let us first consider the case $n \neq 0$. Then the solution of (9) is

$$\sigma(t) = \sigma_0 - \tilde{C}t, \quad n = 1, \tag{12}$$

$$\sigma(t) = \sigma_0 \exp -2\tilde{C}t, \quad n = 2, \tag{13}$$

$$\sigma(t) = \sigma_0 \left(1 + n(n-2)\tilde{C}\sigma_0^{n-2} t\right)^{-\frac{1}{n-2}}, \quad n > 2, \tag{14}$$

where $\sigma_0 \equiv \sigma(0)$. Equation (8) can be rewritten as

$$\frac{d}{d\sigma} e^{\varphi/\varphi_0} = -\frac{\sigma}{n\varphi_0^2}$$

and hence

$$\phi(t) \equiv e^{\varphi(t)/\varphi_0} = \phi_0 - \frac{1}{2n\varphi_0^2}(\sigma^2 - \sigma_0^2), \tag{15}$$

where $\phi_0 \equiv \phi(0)$. This can also be formulated as an invariant of the equations of motion [1],

$$2n\varphi_0^2\phi + \sigma^2 = \text{const.}$$

¿From (7,8) we obtain

$$H = \frac{1}{2\beta^2}\frac{d}{dt}\ln\phi$$

and hence

$$a(t) = a_0\left(\frac{\phi}{\phi_0}\right)^{\frac{1}{2\beta^2}}, \tag{16}$$

where $a_0 \equiv a(0)$. Note that a_0, ϕ_0 and σ_0 are the initial values of a, ϕ and σ at $t = 0$ while φ_0 is a coupling constant. For $t = 0$ we choose the end of inflation – a precise definition will not be necessary, and not possible within the slow rolling approximation. For $n \neq 2$ we may choose $\sigma_0 = 0$ in assuming that the slow rolling approximation holds to the end of inflation. a_0 will be chosen so that $a = 1$ today.

Finally we consider the old extended inflation case, $n = 0$. The solution of (8) is $\phi = \phi_0 + \varphi_0^{-2}\tilde{C}t$ and $a(t)$ is again given by (16). Here equations (5-7) hold exactly and on integration give rise to a well-known exact solution [3].

We see that the slow rolling approximation becomes arbitrarily good towards $a = 0$. The "big bang" $a = 0$ takes place in finite proper time and at a finite value of σ given by

$$\sigma_{\mathrm{BB}}^2 = \sigma_0^2 + 2n\varphi_0^2\phi_0. \tag{17}$$

In the general relativity limit $\varphi_0 \to \infty$ of course $\sigma_{\mathrm{BB}} \to \infty$.

In the limit $\varphi_0 \to \infty$ we should recover the slow rolling approximation in standard chaotic inflation. The limit of (12-14) is clearly the desired one. The formal limit of (15,16) is

$$a = a_0\left(1 - 2\beta^2\phi_0^{-1}\frac{2\pi G}{n}(\sigma^2 - \sigma_0^2)\right)^{\frac{1}{2\beta^2}} \tag{18}$$

$$\sim a_0 \exp\left(\phi_0^{-1}\frac{2\pi G}{n}(\sigma^2 - \sigma_0^2)\right) \quad \text{as} \quad \varphi_0 \to \infty. \tag{19}$$

This means that in the GR limit $\phi_0 = 1$ must be chosen, as G in (19) is Newton's constant. In the JBD case ϕ_0 is also not a free physical parameter, but must be adjusted so that $\phi = 1$ today, as G has to agree with the observed gravitational constant. (In an inhomogenous universe, however, the effective gravitational constant G/ϕ can be spatially inhomogeneous [1].) The effective phase space of the slow rolling solutions is 2-dimensional. With the fixing of ϕ_0 the solution is then unique.

The action (1) does not describe a realistic cosmological model. In a more complete cosmological scenario one must match the end of the slow rolling phase of inflation to a long radiation-dominated phase, in which ϕ is known to tend to a constant, followed by a matter-dominated phase in which ϕ changes again. We can estimate what the value of ϕ_0 has to be. In a cosmological context it is not obvious that the coupling constant β is the same for all kinds of matter, for example for ordinary baryonic matter, for the non-baryonic dark matter in the universe, and for the inflaton field σ. We denote here by β_i (just β everywhere else in this paper!) the coupling constant of the inflaton field σ. We denote by β_d the coupling of the matter which dominates the late universe, probably nonbaryonic dark matter, and by β_b that of ordinary baryonic matter. Experimental tests of the weak and and Einstein equivalence principles do not constrain the different β to coincide, as they involve only baryonic matter. On the other hand, solar system experiments constrain, via the Parametrized PostNewtonian theoretical framework, only β_b. There are cosmological limits on β_d, and they are typically much weaker [4]. This is interesting because extended old inflation can be made viable without giving the dilaton ϕ a mass if β is larger for the dilaton than for baryonic matter [5]. An approximate expression for ϕ_0 can be derived by adapting equation (31) of [4]. (Note $\beta = 2\beta_{\mathrm{DGG}}$!) It is

$$\phi_0 = Z_{\mathrm{rad}}^{-\frac{\beta_i \beta_d}{1-\frac{1}{2}\beta_b\beta_d}} . \tag{20}$$

Here Z_{rad} is the redshift at which the universe stopped being radiation-dominated, measured in the Jordan frame of ordinary matter. If one wanted to use the value in the Jordan frame of the inflaton or the dominant matter one would have to replace β_b by β_i or β_d. We are not proposing a complete cosmology here, and so we note from equation (20) only that $\phi_0 = 1$ is possible if $\beta_d = 0$.This is as natural an assumption as any other, given that β_b is very small while we assume of course that β_i is of interesting magnitude.

Later we shall need two more background relations, namely the conformal Hubble constant $h \equiv aH \equiv \dot{a} \equiv a'/a$, and the slow rolling parameter f, as functions of the conformal time η defined by $dt \equiv a\, d\eta$. We calculate these in closed form, again in the slow rolling approximation $f^2 \ll 1$. We find that

$$\eta \equiv \int a^{-1}\, dt = \int -\frac{1}{1-2\beta^2}\left(1 - \frac{2n^2 f^2}{1-2\beta^2}\right)^{-1} \frac{d}{dt}\left(h^{-1}\right)\, dt \tag{21}$$

Neglecting the f^2 term we can integrate and obtain, in the slow rolling approximation and with the choice of origin $\eta = 0$ at the end of inflation,

$$h = -\frac{1}{1-2\beta^2}\frac{1}{\eta} + O(f^2) \tag{22}$$

It is not surprising that this result is the same as for old or extended old inflation where σ is exactly constant.

We can use (22) to write f as

$$f = \left(\frac{3}{\kappa^{2-n} C} \frac{1 - \frac{2}{3}\beta^2}{\left(1 - 2\beta^2\right)^2} \right)^{\frac{\beta^2}{2(1-2\beta^2)}} \left(\sqrt{\kappa}\sigma\right)^{-1-\frac{n\beta^2}{1-2\beta^2}} \left(\frac{-\eta}{\sqrt{\kappa}}\right)^{-\frac{\beta^2}{1-2\beta^2}} \tag{23}$$

As our expression for $h(\eta)$ has been derived by neglecting the logarithmic time dependence of σ with respect to that of ϕ, we must in consequence neglect in (23) the logarithmic time dependence of σ with respect to that of η. We must therefore choose an appropriate constant value of σ. Which? We can rewrite the slow rolling criterium as

$$f^2 \equiv \frac{\phi}{\kappa\sigma^2} = \frac{\phi_0}{\kappa\sigma^2} - \frac{\beta^2}{2n}\left(1 - \frac{\sigma_0^2}{\sigma^2}\right) \ll 1 \tag{24}$$

As we have chosen σ_0 to correspond to the end of inflation, $\sigma_0^2/\sigma^2 \ll 1$ during slow rolling, and hence we neglect the third term. Clearly $f^2 \simeq \frac{\phi_0}{\kappa\sigma^2}$ for $f^2 > \frac{\beta^2}{2n}$. For $f^2 < \frac{\beta^2}{2n}$ however, the first and second term must approximately cancel, so that $\sigma \simeq \sigma_{\rm BB}$. This is also the regime where the σ factor in (23) is slowly varying with respect to the η factor. Then

$$f = \left(\frac{3}{\kappa^{2-n} C} \frac{1 - \frac{2}{3}\beta^2}{\left(1 - 2\beta^2\right)^2} \right)^{\frac{\beta^2}{2(1-2\beta^2)}} \left(\frac{\beta^2}{2n\phi_0}\right)^{\frac{1}{2}+\frac{n\beta^2}{2(1-2\beta^2)}} \left(\frac{-\eta}{\sqrt{\kappa}}\right)^{-\frac{\beta^2}{1-2\beta^2}} \quad \text{for} \quad f^2 < \frac{\beta^2}{2n}. \tag{25}$$

is it consistent for $f^2 > \frac{\beta^2}{2n}$ to use a time-dependent σ adiabatically?

The Planck limit, beyond which quantum gravity effects must play an important role according to some simple thought experiments, corresponds to $H^2 \approx \kappa^{-1}$, in units in which $\hbar = c = 1$. Hence

$$\kappa^2 C \sigma_{\rm Pl}^{2n} \phi_{\rm Pl}^{-2} \approx 1 \tag{26}$$

If $f_{\rm Pl}^2 < \frac{\beta^2}{2n}$ we can again assume that $\sigma_{\rm Pl} \simeq \sigma_{\rm BB}$, which gives us

$$\phi_{\rm Pl}^2 \approx \left(\frac{2n\phi_0}{\beta^2}\right)^n \kappa^{2-n} C \quad \text{for} \quad \kappa^{2-n} C < \left(\frac{\beta^2}{2n}\right)^n \phi_0^{2-n}. \tag{27}$$

2. Perturbation equations of motion

A general expression for the Lagrangian of the two true degrees of freedom among the scalar type perturbations has been given, for arbitrary inflaton potential $V_i(\sigma)$, and also for an arbitraty dilaton self-interaction $V_d(\varphi)$, in a previous paper [2]. Here we limit ourselves to the case of extended chaotic inflation, or $V_d(\varphi) = 0$ and $V_i(\sigma) = C\sigma^{2n}$. The Lagrangian of the perturbations of extended inflation is quadratic in two true degrees of freedom v and w [2]. It is useful to Fourier decompose with respect to the comoving spatial coordinates,

$$v(x, \eta) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} v_k(\eta) e^{ikx} \tag{28}$$

and so for $w(x, \eta)$. Note that the dimension of v_k is not the same as that of the scalar field v. By reparametrizing time and the two fields and discarding total divergences, the 10 free time-dependent coefficients of the Lagrangian for each Fourier component can be reduced to 4. The time coordinate needed to bring the Lagrangian into this form happens to be conformal time η. The action for the scalar perturbation can be written as

$$S = \int d^3k \int d\eta \left(\frac{1}{2} v_k'^2 + \frac{1}{2} w_k'^2 - D v_k w_k' - \frac{1}{2} A v_k^2 - B v_k w_k - \frac{1}{2} C w_k^2 \right). \tag{29}$$

In the following we always consider the Fourier components, but will mostly suppress the index k.

A Fourier component of the action (29) leads to the canonical momenta

$$\pi_v = v', \qquad \pi_w = w' - Dv, \tag{30}$$

and to the equations of motion

$$v'' + Dw' + Av + Bw = 0 \tag{31}$$
$$w'' - Dv' - D'v + Bv + Cw = 0. \tag{32}$$

¿From the background solution for these potentials derived in the last section we find for the coefficients of the perturbation Lagrangian the following approximate values:

$$A = k^2 - 2h^2(1 - \beta^2), \tag{33}$$
$$C = k^2 - 2h^2(1 - \beta^2)(1 - \tfrac{3}{2}\beta^2), \tag{34}$$
$$B = 2n\beta(1 - \beta^2) f h^2, \tag{35}$$
$$D = -2nfh, \tag{36}$$
$$D' = 2n\beta(1 - \beta^2) f h^2 = -B \tag{37}$$

As in the background equations of motion terms of order f^2 have been neglected compared to terms of order 1 in (33-37).

In [2] we have also given the expression for the gauge-invariant density perturbation δ in terms of v and w. δ is the relative density perturbation on comoving hypersurfaces, i.e. on those with vanishing energy flux. It is given by

$$\delta = \frac{2\Delta\Phi}{3a^2H^2}, \tag{38}$$

where Δ denotes the Laplacian with respect to the comoving spatial coordinates. The gauge-invariant quantity Φ can be interpreted as the gravitational potential on conformally Newtonian hypersurfaces, obeying a Poisson equation with δ as its source.

Specializing the expression for Φ in [2], we find in the slow rolling approximation for the Fourier component Φ_k

$$-k^2\Phi_k = \frac{\sqrt{\kappa}h}{a}\Big[\beta(v_k' - hv_k) - nf(w_k' - (1-\beta^2)hw_k)\Big]. \tag{39}$$

3. Vacuum fluctuations

We can construct the quantum theory of the perturbations in the Heisenberg picture from the solutions of the classical theory by writing the quantum operators $\hat{v}_k(\eta)$ and $\hat{w}_k(\eta)$ as products of time-independent annihilation and creation operators and of time-dependent c-numbers, the "mode functions". Let us define, for each k separately,

$$\begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = Q(\eta)\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} + Q^*(\eta)\begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix} \tag{40}$$

where the matrix of mode functions is

$$Q(\eta) = \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}. \tag{41}$$

$\dagger$ denotes the hermitian conjugate. Each column of Q is a linearly independent complex solution of the coupled equations of motion. By equation (30) we assign to Q a matrix P giving the canonical momenta of the mode functions. We conventionally assign commutation relations

$$[\hat{a}_i(k), \hat{a}_j^\dagger(k')] = i\delta_{ij}\delta^3(k-k'), \tag{42}$$

while all other pairs commute. The canonical commutation relations are then equivalent to the following constraints on the mode functions:

$$PP^+ - P^*P^t = 0, \quad QQ^+ - Q^*Q^t = 0, \quad QP^+ - Q^*P^t = i, \tag{43}$$

where t denotes transposition and $*$ complex conjugation of a matrix, and $+$ their combination.

A quantum state of particular interest is the state $|Q\rangle$ defined by

$$\hat{a}_i(k)|Q\rangle = 0 \quad \text{for all}\, k, i. \tag{44}$$

It is the Fock vacuum associated with the mode functions Q. There is a 6-real-parameter family of such states (for two degrees of freedom). They are identical with the semiclassical states and those whose wave functional in the Schroedinger picture is Gaussian. Furthermore, the ground state at any given time is among them. The quadratic expectation values in the Fock vacua are

$$\begin{pmatrix} \langle Q|\hat{v}\hat{v}|Q\rangle & \langle Q|\hat{v}\hat{w}|Q\rangle \\ \langle Q|\hat{w}\hat{v}|Q\rangle & \langle Q|\hat{w}\hat{w}|Q\rangle \end{pmatrix} = Q^+Q, \tag{45}$$

while quadratic expectation values involving π_v and π_w are read off accordingly from Q^+P and P^+P.

It is convential in calculations of cosmological quantum fluctuations to assume that all perturbations around a given background solution are in their ground state in the asymptotic past. A general criterium for the ground state can be given, but here we can make use of the fact that the coupling terms B and D vanish with f as $\eta \to -\infty$. The ground state in the asymptotic past, which we denote by $|0\rangle$, is then clearly given by the mode functions

$$Q_{|0\rangle} \sim \begin{pmatrix} \frac{e^{-ik\eta}}{\sqrt{2k}} & 0 \\ 0 & \frac{e^{-ik\eta}}{\sqrt{2k}} \end{pmatrix} \quad \text{as}\, \eta \to -\infty. \tag{46}$$

(Similar initial conditions for the mode functions of several scalar fields in inflation have been used, for example, in [7].) At this stage a constant relative phase between v_1 and w_2 does not have physical significance. As $\eta \to 0$, Q develops diagonal terms. It is possible to treat f formally as a small parameter in evolving the equations of motion (31,32) from the initial conditions (46), developing the solution in powers of f. It would be inconsistent to go further than $O(f)$, as the error in the coefficients themselves is of $O(f^2)$. To $O(f^0)$ we have to solve two decoupled equations for $v_1^{(0)}$ and $w_2^{(0)}$:

$$v_1^{(0)''} + \left(k^2 - \frac{2(1-\beta^2)}{(1-2\beta^2)^2}\frac{1}{\eta^2}\right)v_1^{(0)} = 0, \tag{47}$$

$$w_2^{(0)''} + \left(k^2 - \frac{2(1-\beta^2)(1-\frac{3}{2}\beta^2)}{(1-2\beta^2)^2}\frac{1}{\eta^2}\right)w_2^{(0)} = 0, \tag{48}$$

The solution for the above initial conditions can be given in terms of Hankel functions of the second kind [8]:

$$v_1^{(0)} = \frac{1}{\sqrt{2k}}\sqrt{\frac{\pi}{2}}e^{-i\frac{\pi}{2}\nu - i\frac{\pi}{4}}\,\sqrt{k\eta}\,H_\nu^{(2)}(k\eta) \quad \text{where} \quad \nu = \frac{3-2\beta^2}{2-4\beta^2}, \tag{49}$$

$$w_2^{(0)} = \ldots \quad \text{where} \quad \nu = \frac{3-4\beta^2}{2-4\beta^2}. \tag{50}$$

For the vacuum expectation value of Φ_k^2 we find in the leading order in f

$$\langle 0|\hat{\Phi}_k^2|0\rangle = k^{-4}\left(\frac{\sqrt{\kappa}h}{a}\right)^2 \left[\beta^2 \left|v_1^{(0)\prime} - hv_1^{(0)}\right|^2 + n^2 f^2 \left|w_2^{(0)\prime} - (1-\beta^2)hw_2^{(0)}\right|^2\right]. \tag{51}$$

In calculating $v' - hv$ and a similar term for w, we have found the following relation useful, which holds for any Bessel function of index ν:

$$\frac{d}{dz}H_\nu(z) + \frac{\nu}{z}H_\nu(z) = H_{\nu-1}(z). \tag{52}$$

Using this and the series expansion for Bessel functions, we obtain

$$v_1^{(0)\prime} - hv_1^{(0)} \sim \frac{1}{\sqrt{2k}}\sqrt{\frac{\pi}{2}}e^{-i\frac{\pi}{2}\nu - i\frac{\pi}{4}}\frac{i2^{\nu-1}\Gamma(2-\nu)}{\sin(\nu-1)\pi}(k\eta)^{\frac{3}{2}-\nu}$$
$$\text{as} \quad k\eta \to 0 \quad \text{where} \quad \nu = \frac{3-2\beta^2}{2-4\beta^2}, \tag{53}$$

$$w_2^{(0)\prime} - (1-\beta^2)hw_2^{(0)} \sim \ldots \quad \text{as} \quad k\eta \to 0 \quad \text{where} \quad \nu = \frac{3-4\beta^2}{2-4\beta^2}, \tag{54}$$

The last ingredient in our expression for $\langle 0|\hat{\Phi}_k^2|0\rangle$ is

$$\frac{\sqrt{\kappa}h}{a} = \frac{1}{a_1}\left(\frac{-\eta}{\sqrt{\kappa}}\right)^{\frac{2\beta^2}{1-2\beta^2}} \tag{55}$$

where a_1 is the value of the scale factor when $-\eta = \sqrt{\kappa}$. The leading powers of the η-dependence cancel! We obtain the following expression:

$$\langle 0|\hat{\Phi}_k^2|0\rangle = \frac{\pi^2}{2}k^{-3}a_1^{-2}\Big[N_v\beta^2(k\sqrt{\kappa})^{-\frac{2\beta^2}{1-2\beta^2}}$$
$$+ N_w n^2\left(\frac{3}{\kappa^{2-n}C}\frac{1-\frac{2}{3}\beta^2}{(1-2\beta^2)^2}\right)^{\frac{\beta^2}{1-2\beta^2}}\left\{\frac{\beta^2}{2n\phi_0}\right\}^{1+\frac{n\beta^2}{1-2\beta^2}}(k\sqrt{\kappa})^{-\frac{\beta^2}{1-2\beta^2}}\Big]$$
$$\text{for} \quad f^2 \ll 1, \quad k\eta \ll 1, \quad f^2 < \frac{\beta^2}{2n}, \tag{56}$$

where $N \equiv (2^\nu\Gamma(2-\nu)/4\pi\sin(\nu-1)\pi)^2 = 1 + O(\beta^2)$ contains the irrelevant bits. *The last condition, $f^2 < \frac{\beta^2}{2n}$, can be relaxed by replacing the term in curly brackets by $1/(\kappa\sigma^2)$. ??*

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