DAMTP R-90/9
May 1990

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It is well known that static straight local cosmic strings coupled to gravity produce a spacetime with conical sections transverse to the axis of the string. We examine the effect of coupling local cosmic strings, not to standard Einstein gravity (GR), but to a Jordan-Brans-Dicke (JBD) theory of gravity. The new JBD metric is shown to be approximately that of the cone obtained in GR times a conformal factor of a small power of the radius. We examine the geodesics for this metric. All null geodesics are unchanged, but massive particles will have bound trajectories. The upper bound for these trajectories will be inside the cosmological horizon for particles moving considerably slower than the velocity of light with respect to the string.

1. Introduction

A local cosmic string is usually taken to be a static, cylindrically symmetric solution of a field theory containing a complex scalar field with a "Mexican hat" potential which is minimally coupled to a U(1) gauge field. This model for a straight cosmic string may be coupled to gravity, and it has been shown that the corresponding spacetime is asymptotically conical in the spatial sections transverse to the string [1]. The deficit angle is proportional to the mass per unit length of the string, which is approximately 10^{-6} in Planck units for GUT scales, and the full solution coupled to gravity is stable in the sense that it is energy minimising [2].

The Jordan-Brans-Dicke theory of gravity [3] is a natural extension of general relativity which introduces an additional scalar field $\tilde{\phi}$ besides the metric tensor, and a dimensionless coupling constant ω. In the limit $\omega \to \infty$, general relativity is recovered. The theory is metric, which means that matter couples minimally to the metric, and not directly to $\tilde{\phi}$. Lately there has been renewed interest in the JBD theory of gravity, since it has revived the old inflationary scenario by transforming exponential Hubble expansion into power law expansion, thus making bubble percolation more natural [4].

The action for JBD theory is as follows

$$S = \int \left(\tilde{\phi}\tilde{R} - \frac{\omega}{\tilde{\phi}} \left(\nabla_a \tilde{\phi} \right)^2 \right) \sqrt{-\tilde{g}}\, d^4x + S_{Matter} \left[\tilde{g}_{ab} \right]. \tag{1}$$

It is, however, easier to examine the theory if one expresses the action in terms of a Weyl-transformed metric, and in terms of a redefined JBD field and coupling constant

[5]. With

$$\omega = \frac{1}{4\beta^2} - \frac{3}{2}, \tag{2a}$$

$$\tilde{\phi} = \frac{1}{16\pi G} e^{-2\beta\sigma}, \tag{2b}$$

$$\tilde{g}_{ab} = e^{2\beta\sigma} g_{ab}, \tag{2c}$$

the action in the new metric becomes

$$S = \frac{1}{16\pi G} \int \left(R - (\nabla_a \sigma)^2 \right) \sqrt{-g}\, d^4x + S_M \left[e^{2\beta\sigma} g_{ab} \right], \tag{3}$$

where all quantities without a tilde are with respect to g_{ab}. We shall henceforth refer to $\tilde{g}_{ab}$ as the physical or Jordan metric and to g_{ab} as the unphysical or Einstein metric. In this latter Einstein frame, the JBD action for gravity has been reduced to the standard Einstein action for general relativity coupled to a scalar field. However, matter couples to gravity not through the unphysical metric of our new action, but through the original metric $\tilde{g}_{ab}$. The corresponding field equations obtained by varying g_{ab} and σ are

$$R_{ab} = \nabla_a \sigma \nabla_b \sigma + 8\pi G \left(T_{ab} - \frac{1}{2} g_{ab} T \right) \tag{4a}$$

$$T_{ab} = \frac{1}{2\sqrt{-g}} \frac{\delta S_M \left[e^{2\beta\sigma} g_{ab} \right]}{\delta g_{ab}} \tag{4b}$$

$$\nabla^a \nabla_a \sigma = -8\pi G \beta T. \tag{4c}$$

In addition, the Bianchi identity gives

$$\nabla_a T^{ab} = \beta T \nabla^b \sigma, \tag{4d}$$

showing that the matter energy momentum tensor is not conserved in the Einstein frame. We emphasize that we regard σ not as matter, but as part of the gravitational action. β has been constrained to be < 0.032 (2σ limit) for ordinary matter by radar time delay experiments [6], but if one is willing to sacrifice the weak equivalence principle, it is possible that string matter couples to σ with a β of order unity [7].

We shall look for qualitatively new features of local cosmic strings coupled to JBD gravity, studying the system in the Einstein frame. Such solutions must exist, at least for large ω (small JBD coupling). We show that for physical values of the coupling constants in our theory, the problem can be divided into two regions. We match an interior solution in which the effects of σ in the gravitational and string equations of motion are neglected, and in which σ only plays a passive role, to an exterior solution where the string matter is neglected. In other words, we will neglect the σ dependence on the right hand side of equations (4a) and (4d) in the interior solution, and the T_{ab} dependence in (4a) and (4c) in the exterior solution. This approximation scheme will be justified if the changeover from T_{ab} to $\nabla_a\sigma\nabla_b\sigma$ dominance in (4a) is rapid, and if $2\beta\sigma \ll 1$ everywhere in the interior solution.

We see that in the outer region, the JBD field obeys the equation $\nabla^a\nabla_a\sigma = 0$, which implies that the contribution $\nabla^a\sigma\nabla_a\sigma$ to the energy density falls off as $\frac{1}{r^2}$. This is similar to the behaviour of the energy density of a global string, and indeed we shall see that the metric corresponding to a JBD cosmic string (in the Einstein frame) is similar in character to the metric for a global string. It turns out, however, that this effect is secondary, and that to first order in the GUT mass in Planck units, the physical JBD cosmic string metric corresponds to a conformal transformation of the standard conical metric.

Finally, we calculate the geodesics in the sections transverse to the string, and show that every massive particle has bound trajectories. The null geodesics are unchanged.

2. Cosmic string and Brans-Dicke Solutions

The standard model for a U(1) gauge cosmic string arises from the following action

$$S_M = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\left(\nabla_a + ieA_a\right)\Phi\left(\nabla^a + ieA^a\right)\Phi^* + \lambda\left(\Phi^*\Phi - \eta^2\right)^2 + \frac{F_{ab}F^{ab}}{16\pi}\right). \tag{5}$$

Let us deal with an isolated static straight string. If we couple such a model to general relativity, it is natural to assume that the metric is of the form

$$ds^2 = e^A\left(-dt^2 + dz^2\right) + d\rho^2 + e^C d\theta^2 \tag{6}$$

(where boost symmetry for the source implies boost symmetry of the metric [8]), and to make the simplifying ansatz

$$\Phi = \eta X e^{i\theta}, \quad A_a = \frac{1}{e}\left[P - 1\right]\nabla_a\theta,$$

where A, C, X and P are all functions of ρ only. They must in addition satisfy the following boundary conditions

$$X \to 1, \quad P \to 0 \quad \text{as} \quad \rho \to \infty$$

$$X \to 0, \quad P \to 1, \quad e^A \to 1, \quad e^C \to \rho^2 \quad \text{as} \quad \rho \to 0.$$

Note that we have taken the winding number of the string to be 1. Here and in what follows we follow the notation of reference [1], with $\hbar = c = 1$.

The resulting field equations for a cosmic string consist of two equations for X and P, and three Einstein equations, although it may be shown that only two of the Einstein equations are dynamical, and that the third is purely a constraint equation. These field equations depend on two dimensionless parameters, when we make the transformation

$$r = \sqrt{\lambda}\eta\rho,$$

these being the ratio of the Higgs mass to the mass of the ggauge field,

$$8\left(\frac{m_A}{m_\Phi}\right)^2 = \alpha = \frac{4\pi e^2}{\lambda}$$

and the strength of gravitational coupling $G\eta^2 = {l_P}^2\eta^2$. It is most straightforward to solve the equations asymptotically in the case where $\alpha = 8$ [9], so that the exponential fall-off of both X and P is the same. In this case, it is possible to show that asymptotically,

$$\begin{aligned} &A \equiv 0, \quad \lambda\eta^2 e^C \to r^2\left(1 - 4G\eta^2\right)^2 \\ &P \to 2\sqrt{2}\left(1 - 4G\eta^2\right)\gamma r^{\frac{1}{2}} e^{-2\sqrt{2}r} \\ &R \to 1 - \gamma r^{-\frac{1}{2}} e^{-2\sqrt{2}r}. \end{aligned} \tag{8}$$

The constant γ is determined by matching an asymptotic solution onto a solution which satisfies the boundary conditions given above at the origin. Numerical calculations have shown γ to be of order 1. The asymptotic behaviour of the energy momentum tensor in this case is given by

$$T^t{}_t = T^z{}_z \to \lambda\eta^4 \frac{32\gamma^2 e^{-4\sqrt{2}r}}{r}, \qquad T^r{}_r \equiv T^\theta{}_\theta \equiv 0. \tag{9}$$

The generalisation to the case where $\alpha \neq 8$ is straightforward, although in this case A, $T^r{}_r$ and $T^\theta{}_\theta$ are nonvanishing. A will nevertheless fall off exponentially fast, and the metric will tend towards the standard conical metric. The rr and $\theta\theta$ components of the energy momentum tensor will both decay exponentially, and will remain smaller in magnitude than the tt and zz components. We shall show in section 3 that these results remain correct if we couple our cosmic string to JBD gravity instead of general relativity.

Let us now consider the question of finding vacuum solutions to the JBD equations, with cylindrical and boost symmetry. We shall see in section 3 that such solutions give the asymptotic behaviour of both g_{ab} and σ far from the string, since in this region we may take neglect T_{ab} with respect to the σ term on the right hand side of the Einstein equations (4a).

The vacuum JBD equations with an ansatz for the Einstein metric of type (6), and with σ a function of the dimensionless r only, are

$$A'' + A'^2 + \frac{A'C'}{2} = 0$$

$$C'' + \frac{C'^2}{2} + A'C' = 0$$

$$\sigma'' + \left(A' + \frac{C'}{2}\right)\sigma' = 0$$

(where the prime denotes differentiation with respect to r), with the third Einstein equation providing the constraint

$$A'^2 + 2A'C' = 2\sigma'^2.$$

These equations admit an exact solution, which is, in its most general form,

$$ds^2 = e^{A_0}\left(r - r_0\right)^a\left(-dt^2 + dz^2\right) + \frac{1}{\lambda\eta^2}\left[dr^2 + e^{C_0}\left(r - r_0\right)^{2-2a} d\theta^2\right] \tag{10a}$$

$$\sigma = \sigma_1 \ln\left(r - r_0\right) + \sigma_0 \tag{10b}$$

$$\sigma_1^2 = a\left(2 - \frac{3}{2}a\right). \tag{10c}$$

e^{A_0} may be absorbed by a redefinition of t and z. Note that in the limit where $\sigma = \text{const}$, the metric given by (7a) and (7b) is just the Levi-Civita form with $a = \frac{4}{3}$ or $a = 0$. The latter case corresponds to the cone, and e^{C_0} determines the deficit angle.

3. Jordan-Brans-Dicke Cosmic Strings

In this section we shall justify the previous assertion that a JBD cosmic string may be split up into two regions corresponding to neglecting the effects of the JBD field and the string fields respectively. It is straightforward to match these two regions, to obtain a picture of the metric of a JBD cosmic string in the Einstein frame and of the conformal factor $e^{2\beta\sigma}$ connecting it to the physical metric.

The full field equations for the JBD cosmic string take the form

$$R^z_z = R^t_t = -\frac{\lambda\eta^2}{2}\left(A'' + A'^2 + \frac{A'C'}{2}\right) = 8\pi G\left(T^t_t - \frac{1}{2}T\right) \tag{11a}$$

$$R^\theta_\theta = -\frac{\lambda\eta^2}{2}\left(C'' + \frac{C'^2}{2} + A'C'\right) = 8\pi G\left(T^\theta_\theta - \frac{1}{2}T\right) \qquad (11b)$$

$$R^r_r = -\frac{\lambda\eta^2}{2}\left(2A'' + C'' + A'^2 + \frac{C'^2}{2}\right) = \lambda\eta^2\sigma'^2 + 8\pi G\left(T^r_r - \frac{1}{2}T\right) \qquad (11c)$$

$$\sigma'' + \left(A' + \frac{C'}{2}\right)\sigma' = -8\pi G\beta T \qquad (11d)$$

$$\nabla^a T_{ab} = \beta T \nabla_b \sigma \qquad (11e)$$

$$X'' + X'\left(2\beta\sigma' + A' + \frac{C'}{2}\right) - X\left(P^2\frac{e^{-C}}{\lambda\eta^2} + 4(X^2-1)e^{2\beta\sigma}\right) = 0$$

$$P'' + P'\left(A' - \frac{C'}{2}\right) - \alpha P X^2 e^{2\beta\sigma} = 0$$

where the primes are with respect to r. The energy momentum tensor T_{ab} is now a function of σ as well as X and P

$$T^t_t = T^z_z = -\frac{\lambda\eta^4}{2}\left[X'^2 e^{2\beta\sigma} + \frac{e^{-C}}{\lambda\eta^2}X^2P^2e^{2\sigma\beta} + 2\left(X^2-1\right)^2 e^{4\beta\sigma} + \frac{1}{\alpha}e^{-C}P'^2\right]$$

$$T^r_r = \frac{\lambda\eta^4}{2}\left[X'^2 e^{2\beta\sigma} - \frac{e^{-C}}{\lambda\eta^2}X^2P^2e^{2\sigma\beta} - 2\left(X^2-1\right)^2 e^{4\beta\sigma} + \frac{1}{\alpha}e^{-C}P'^2\right]$$

$$T^\theta_\theta = \frac{\lambda\eta^4}{2}\left[-X'^2 e^{2\beta\sigma} + \frac{e^{-C}}{\lambda\eta^2}X^2P^2e^{2\sigma\beta} - 2\left(X^2-1\right)^2 e^{4\beta\sigma} + \frac{1}{\alpha}e^{-C}P'^2\right].$$

Let us first of all consider the interior solution to these equations. In this region, we shall assume that the contribution of the string energy-momentum dominates in the Einstein equations and that we may take $e^{\beta\sigma} = 1$ in the energy-momentum tensor and the X and P field equations, and then show that these assumptions are self-consistent. We consider for simplicity the case where $\alpha = 8$; the extension of our results to $\alpha \neq 8$ is straightforward.

If $e^{\beta\sigma} \simeq 1$, the string fields X and P will behave asymptotically as in (8). The energy momentum tensor will therefore behave asymptotically as in (9), except that equation (11e) shows us that the small perturbation of the asymptotic solutions (8) and (9) induced by σ will generate a non-zero T^r_r and T^θ_θ. These will however be at least $o(\frac{1}{r})$ smaller than T^t_t and T^z_z.

To compute $\sigma(r)$ using (11d), it is sufficient to approximate T by

$$T = -\lambda\eta^4 \begin{cases} 1\ ; & 0 < r < 1 \\ \dfrac{e^{-4\sqrt{2}(r-1)}}{r}\ ; & r > 1, \end{cases}$$

where the value of γ of equation (8) is fixed by the chosen value of T in the core; the choice above agrees with numerical estimates [10]. Equations (11a)-(11c), subject to our assumption that the contribution from σ is small, show that A and C tend rapidly to their conical space values. Then,

$$\sigma = 8\pi G\eta^2\beta \begin{cases} \dfrac{r^2}{4}\ ; & 0 < r < 1 \\ \dfrac{1}{4} + \left(\dfrac{1}{2} + \dfrac{1}{4\sqrt{2}}\right)\ln r + \displaystyle\int_1^r \dfrac{e^{-4\sqrt{2}(r'-1)}}{-4\sqrt{2}r'}dr'\ ; & r > 1. \end{cases}$$

We have demanded that σ be bounded at the origin so that the physical metric will be regular there, and that $\sigma(0) = 0$. The latter choice is arbitrary, corresponding to a choice of units.

The energy-momentum tensor of the string decays rapidly, and the metric co-efficients A and C will indeed tend rapidly to a vacuum solution of the type given by (10a,10b), with $a = 0$ (the standard cone), provided that the contribution from the JBD field σ in the Einstein equations (11a)-(11c) remains smaller than the string matter term (which is dominated by $8\pi GT$). This is certainly true near the core if we assume that the dimensionless quantities $8\pi G\eta^2$ and β are within the bounds discussed above, since the σ'^2 term is a factor of $8\pi G\eta^2\beta^2$ smaller than $8\pi GT$. It is also true near the core that $\beta\sigma$ will be much less than 1, which shows our approximations to be self-consistent.

Since T decays exponentially fast, the contribution from σ'^2 to the right-hand side of (11c) will eventually become comparable with $8\pi GT$, and will then rapidly dominate. Once this contribution dominates, the solution to equations (11a-e) will become of the JBD vacuum type (10a-d). Let us calculate at approximately what distance from the core this transition will occur.

We compare the magnitudes of the two terms on the right hand side of (11c). Equality of the magnitudes of the two terms will occur when

$$e^{4\sqrt{2}r_{crit}} \sim \frac{e^{4\sqrt{2}}r_{crit}}{2\pi G\eta^2\beta^2}.$$

If we assume that $G\eta^2 \sim 10^{-6}$, and $\beta < 0.032$, then r_{crit} is greater than 4 string widths. This tells us that the transition from the inner to the outer region occurs far enough outside the core for the asymptotic solution (8) to be used self-consistently, but close enough that our assumptions about $e^{\beta\sigma}$ remain valid. For much larger values of $G\eta^2\beta^2$ (which has to be $\ll 1$ to keep away from quantum gravity) our approximation is no longer valid quantitatively, but the following conclusions about the asymptotic metric should still hold qualitatively.

As stated above, once $r > r_{crit}$, the JBD cosmic string solution quickly resembles the vacuum JBD solution (10). We may estimate the arbitrary constants in solution (10) by matching the solution to the solution at r_{crit}. We deduce that the exterior solution takes the following form

$$e^A \sim (r/r_{crit})^{\left(4\pi G\eta^2\beta\right)^2/2} \tag{12a}$$

$$e^C \sim \frac{1}{\lambda\eta^2}\left(1-4G\eta^2\right)^2 r^2 \left(r/r_{crit}\right)^{-\left(4\pi G\eta^2\beta\right)^2} \tag{12b}$$

$$e^{2\beta\sigma} \sim r^{8\pi G\eta^2\beta^2} \tag{12c}$$

All three exponents above are small in the standard model, and so we may approximate r^a by $1 + a\,\mathrm{ln}r$, for all r such that $a\,\mathrm{ln}r \ll 1$. If we cast the Einstein metric in this form,

$$\begin{aligned} ds^2 &= \left(1+\frac{1}{2}\left(8\pi G\eta^2\beta\right)^2 \ln\left(r/r_{crit}\right)\right)\left(-dt^2+dz^2\right) \\ &+ \frac{1}{\lambda\eta^2}\left[dr^2+\left(1-4G\eta^2\right)^2\left(1-\left(8\pi G\eta^2\beta\right)^2\ln\left(r/r_{crit}\right)\right)r^2 d\theta^2\right], \end{aligned} \tag{13}$$

we may view the (r,θ) section of this metric as describing a two surface which is conical in shape, but with its deficit angle increasing slowly with distance from the core, and hence non-flat. This description is valid out to $r \sim e^{\frac{1}{(8\pi G\eta^2\beta)^2}}$ string widths. The properties of the metric described above are reminiscent of those of a global U(1) cosmic string. A global cosmic string has been shown, in the linear approximation, to have metric [11]

$$\begin{aligned} ds^2 &= \left(1-4\pi G\eta^2 \mathrm{ln}\left(r/r_{core}\right)\right)\left(-dt^2+dz^2\right) \\ &+ \frac{1}{\lambda\eta^2}\left[dr^2+\left(1-4G\eta^2\right)^2\left(1-8\pi G\eta^2\mathrm{ln}\left(r/r_{core}\right)\right)r^2 d\theta^2\right]. \end{aligned} \tag{14}$$

There are clear similarities between metrics (13) and (14), and one might be tempted to say that the effect of coupling a local cosmic string to JBD theory is to change it, in terms of its gravitational effects, into a global string. There are, however, certain fundamental qualitative differences between our JBD local string and a global string.

The energy momentum tensor of a global cosmic string behaves as $\frac{8\pi G\eta^2}{r^2}$, whilst in the Einstein frame, the effective contribution from σ to the right hand side of the Einstein tensor behaves as $\frac{(8\pi G\eta^2\beta)^2}{r^2}$. However, the global string has an energy-momentum tensor with negative pressure in all but the θ-direction, whereas the σ-induced energy-momentum in the JBD string has negative pressure in all but the r-direction. The result of this difference is the change in sign of the logarithmic term in the e^A component of the metric. A second difference is that our asymptotic metric given by (12) is valid infinitely far from the core, whereas the metric (14) is valid only whilst linearised gravity remains a good approximation. Beyond this, the metric rapidly becomes singular, with the source term in the Einstein equations, given by $8\pi G\eta^2X^2e^{-C}$ in our notation, no longer behaving like $\frac{1}{r^2}$, but blowing up [12].

Most importantly, the *physical* metric of the JBD string carries a conformal factor $e^{2\beta\sigma}$. If $8\pi G\eta^2$ is small, this factor will always dominate the effects described above. Note that all deviations from the conical metric will be small when $G\eta^2\beta^2$ is small. The typical length scale on which the Einstein metric (13) will differ appreciably from the cone is given by $r_1 \sim e^{\frac{1}{(8\pi G\eta^2\beta)^2}}$, which is considerably larger than the horizon scale for all values of $g\eta^2$ and β for which our classical analysis is appropriate. The length scale on which the conformal factor $e^{2\beta\sigma}$ becomes important is given by $r_1 \sim e^{\frac{1}{8\pi G\eta^2\beta^2}}$ string widths, which is still larger than the horizon scale for typical GUT scales, but is nevertheless 10^5 orders of magnitude closer to the core than r_1. If $e^{\frac{1}{8\pi G\eta^2\beta^2}} \sim \sqrt{\lambda}\eta\rho_H = r_H$, which corresponds to $\eta \geq 2\times 10^{17} GeV$ or (2×10^{-2} in Planck units), for current bounds on β, the effects will come within the current horizon scale, and may become of cosmological importance.

The above remarks justify taking the physical metric for the JBD cosmic string as a conformally rescaled cone,

$$ds^2 = r^{8\pi G\eta^2\beta^2}\left(-dt^2+dz^2+\frac{1}{\lambda\eta^2}\left[dr^2+\left(1-4G\eta^2\right)^2 r^2 d\theta^2\right]\right), \tag{15}$$

for all cosmological applications, since this will restrict r to be less than r_H.

4. Geodesics

In this section, we shall discuss some of the qualitative differences in the geodesic motion in the usual conical metric for a cosmic string, and in the conformally rescaled metric

$$ds^2 = \Omega^2(\rho)\left(-dt^2+dz^2+d\rho^2+(1-\delta)^2\rho^2 d\theta^2\right), \tag{16}$$

where for the JBD cosmic string, $\Omega^2(\rho) = (\sqrt{\lambda}\eta\rho)^{8\pi G\eta^2\beta^2}$. As is well known, in the flat conical metric, trajectories are straight lines, which are deflected by the constant deficit angle $\Delta\theta = 2\pi\delta$, due to the spacetime's global properties. It is clear that in the conformally rescaled metric (16), light rays will also be straight lines, only deflected by the constant deficit angle $\Delta\theta = 2\pi\delta$, as their geodesic equation is conformally invariant.

The 4-velocity of a massive test particle obeys

$$-1 = g_{ab}u^a u^b = \Omega^2 \left(-\dot{t}^2 + \dot{\rho}^2 + (1-\delta)^2 \rho^2 \dot{\theta}^2\right)$$

where a dot signifies a derivative with respect to the particle's proper time. Substituting the two constants of the motion

$$E/M = -g_{ab}\, u^a \left(\frac{\partial}{\partial t}\right)^b = \Omega^2 \dot{t}$$

$$L/M = g_{ab}\, u^a \left(\frac{\partial}{\partial \theta}\right)^b = \Omega^2 (1-\delta)^2 \rho^2 \dot{\theta},$$

we obtain

$$\dot{\rho}^2 = \Omega^{-4} \left((E/M)^2 - \frac{(L/M)^2}{(1-\delta)^2 \rho^2} - \Omega^2 \right).$$

Note that the effect of the conformal factor is to give the particle a variable mass. Substituting the conformal factor (12c) obtained for the string, the turning points are at

$$(E/M)^2 = \frac{(L/M)^2}{(1-\delta)^2 \rho^2} + \left(\sqrt{\lambda}\eta\rho\right)^{2S},$$

where $S = 4\pi G\eta^2\beta^2$ and $\delta = 4G\eta^2$. It is clear that there are always two solutions for ρ of this equation for any given L/M and (not too small) E/M. Thus the orbit of a massive test particle is always bound. However, it should be noted that the outer bound of a highly eccentric orbit will occur at a distance of approximately

$$r_{max} \sim (E/M)^{1/S} = r_{min} \left(1 - v^2\left(r_{min}\right)\right)^{-1/2S}.$$

In a realistic cosmological setting we can only speak of bound orbits if the point of maximum distance is still well within the horizon distance, or $r_{max} \ll r_H = \sqrt{\lambda}\eta H_0^{-1} \sim 10^{57}$. The other interesting case is a circular orbit, with $\ddot{\rho} = \dot{\rho} = 0$, which minimizes E for given L. The orbital velocity of a circular orbit seen by a local observer at rest with respect to the metric is

$$v = \sqrt{S/1+S} \sim (8\pi)^{1/2}\beta\eta l_P,$$

independent of the radius of the orbit or the deficit angle, and the orbital period T is proportional to the proper orbital radius R,

$$T = 2\pi(1-\delta)\sqrt{S(S+1)}\,R \sim 2\pi\beta\eta l_P\,R.$$

To keep the sun in its galactic orbit at $v \sim 10^{-3}$ by the attraction of a JBD string in the center of the galaxy alone, one would need $\beta\eta l_P \sim 10^{-4}$, which is larger than a typical value by one order of magnitude.

5. Conclusions

The coupling of a local cosmic string to the JBD theory of gravity, rather than standard Einstein gravity, produces a change in the corresponding spacetime metric. The usual conical metric, to a first approximation, is rescaled by a conformal factor behaving as $(\rho/\rho_{core})^{8\pi G\eta^2\beta^2}$. This conformal factor is extremely close to 1 for all radii within the current horizon scale, if we take $G\eta^2 \sim 10^{-6}$, the typical GUT scale, and $\beta \sim 0.03$. However, the length scale at which the effects of the conformal factor become important are sensitive to $G\eta^2\beta^2$, and this occurs within the current horizon if $G\eta^2\beta^2 \sim 4 \times 10^{-4}$.

We have shown that the motion of massive particles in this spacetime are bound. However, for velocities above 10^{-3} times the velocity of light the size of the orbits is larger than the present day Hubble radius, so that fast particles are effectively unbound in a cosmological setting. Trajectories of particles with velocities close to the speed of light are virtually the same as for a conical spacetime; as a result, the dragging of matter by cosmic strings would be largely unaffected, the motion of strings is thought to be at a velocity approaching that of light with respect to surrounding matter.

It should be remarked that the effects described in this work are for an isolated static straight cosmic string, in an otherwise vacuum spacetime. One might expect some qualitative changes in the long range gravitational field of the string if one took into account other strings and the presence of other matter in the universe.

Let us consider the effect of the presence of a cosmic string in a Friedmann-Robertson-Walker spacetime, taking into account the coupled matter and Einstein equations. This question has not been treated in the literature. However, if one considers a local string coupled to Einstein gravity and assumes, for simplicity, a $K = 0$ FRW background, one sees that the conical singularity of the string is compatible with a FRW metric with a wedge taken out, where the slices of constant cosmological time have a deficit angle generated by the string.

The new feature for a JBD local string in an otherwise vacuum spacetime is the conformal factor in the physical metric. The equation of motion for the JBD field σ is linear, and so in a cosmological setting, σ can be split into two parts

$$\sigma = \sigma_{string}\,(a(t)\rho) + \sigma_{cosmol.}(t)$$

where it is appropriate to view the dependence of σ_{string} on $a(t)\rho$ in an adiabatic approximation. σ_{string} can be treated as in section 3, and so will still contain a logarithmic term, $\sigma_{string} \sim \sigma_1 \ln r$, whose magnitude is fixed by the strong string fields at the core, as in equations (8) and (9). We would therefore expect the metric for a cosmic string for a JBD theory of gravity, in an FRW spacetime to be of the form

$$d\tilde{s}^2 \sim \tilde{a}^2(t) \left(\sqrt{\lambda}\eta\tilde{a}(t)\rho\right)^{8\pi G\eta^2\beta^2} \left[-dt^2 + dz^2 + d\rho^2 + \rho^2 \left(1 - 4G\eta^2\right)^2 d\theta^2\right]$$

well outside the string core, but well inside the horizon. Our results for massive particle trajectories would therefore remain unaffected for orbits whose period is much less than the Hubble time.

Acknowledgements

We would like to thank Gary Gibbons for suggesting this problem and for helpful discussions, and the SERC (UK) for financial support. One of us (M.E.O.) would like to thank the I.H.E.S., Paris, for their hospitality whilst part of this work was carried out.

References

[1] D. Garfinkle, Phys. Rev. **D 32** (1985) 1323.

[2] R. Gregory, Phys. Rev. Lett. **59** (1987) 740.

[3] P. Jordan, Nature **164** (1949) 637; P. Jordan, Schwerkraft und Weltall, Fried. Vieweg

& Sohn, Braunschweig (1955); P. Jordan, Z. Phys. **157** (1959) 112; C. Brans and R. H. Dicke, Phys. Rev. **124** (1961) 925.

[4] D. La and P. Steinhardt, Phys. Rev. Lett. **62** (1989) 376.

[5] W. Pauli, quoted in §28 of Jordan's book, see ref. [1]; M. Fierz, Helv. Phys. Acta **29** (1956) 128; R. H. Dicke, Phys. Rev. **125** (1962) 128.

[6] R.D. Reasenberg et al., Astrophys. J. **234** (1979) L219.

[7] T. Damour, G.W. Gibbons and C. Gundlach, Phys. Rev. Lett. **64** (1990) 123.

[8] G.W. Gibbons, F.R. Ruiz and M.E. Ortiz, Phys. Lett. **B 240** (1990) 50.

[9] B. Linet, Phys. Lett. **A** (198) .

[10] P. Laguna-Castillo and R. Matzner, Phys. Rev. **D 36** (1987) 3663.

[11] D. Harari and P. Sikivie, Phys. Rev. **D 37** (1988) 3448.

[12] R. Gregory, Phys. Lett. **B 215** (1988) 663; A. Cohen and D. Kaplan, Phys. Lett **B 215** (1988) 67; G.W. Gibbons, M.E. Ortiz and F.R. Ruiz, Phys. Rev. **D 39** (1989) 1546.