ON THE QUANTISATION OF PERTURBATIONS IN INFLATION

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Abstract

The theory of gauge-invariant scalar perturbations in chaotic inflation is briefly reviewed. We clarify a problem related to their quantisation, pertinent to any inflationary model.

1. Introduction

Inflationary models (see e.g. [1]) predict a phase of exponential expansion, or more generally an expansion in t^p, $p > 1$, of the scale factor of the early universe. In this way a number of outstanding problems faced by the standard Big-Bang model, among them the horizon and flatness problems, are elegantly solved. The prediction of a scale invariant, Harrison-Zel'dovich, spectrum for the energy density fluctuations is also a major achievement of the theory, as it could explain the formation of galaxies. These fluctuations are produced by the vacuum quantum fluctuations of a scalar field during the inflationary stage [2-3]. The resulting power spectrum provides the only genuine test for these theories as it must satisfy stringent observational constraints.

We consider here a massive scalar field minimally coupled to gravitation with Planckian initial conditions, as postulated in chaotic inflation. In section 2 we review the classical description of the perturbations. In section 3 we describe their canonical quantisation and explain, in a hopefully transparent way, the pitfalls of a naive quantisation based on equations of motion instead of Lagrangians.

2. Classical description of the perturbations

The behaviour of a scalar field coupled to the gravitational field in the context of inflation has been extensively studied. We shall therefore give only a brief summary of the dynamics of such a system. We consider a real scalar field φ of mass m, minimally coupled to the gravitational field, and look for solutions which are of the Friedmann-Robertson-Walker type with flat spatial sections :

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 = a^2(\eta)\left[d\eta^2 - d\vec{x}^2\right] \tag{2.1}$$

where t is the cosmic time, η is the conformal time, a is the scale factor and $d\vec{x}^2 \equiv \delta_{ij}dx^i dx^j, i,j = 1,2,3$. The evolution of the system can be found from the (00) component of Einstein's equations :

$$3H^2 = \tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}m^2\varphi^2 \tag{2.2}$$

and the Klein-Gordon equation :

$$\ddot{\varphi} + 3H\dot{\varphi} + m^2\varphi = 0. \tag{2.3}$$

A dot denotes differentiation with respect to t (and a prime will denote differentiation with respect to η), $H \equiv \dot{a}/a$ is the Hubble "constant", and we use Planck units such that $c = \hbar = 8\pi G = 1$. In writing (2.2-3) we assumed that the (background) field φ had no spatial dependence. A useful relation is obtained by combining (2.2) and (2.3) :

$$\dot{H} = -\tfrac{1}{2}\dot{\varphi}^2. \tag{2.4}$$

Given "slow rollover" initial conditions which insure that the scalar field varies slowly ($\ddot{\varphi} \simeq 0$) and that its energy-momentum tensor is dominated by the mass term (which acts therefore as an effective cosmological constant), it follows from (2.2-3) that the universe starts in a quasi-de Sitter phase characterised by :

$$H \simeq \sqrt{\tfrac{1}{6}}\, m\varphi \quad \text{and} \quad \dot{\varphi} \simeq -\sqrt{\tfrac{2}{3}}\, m \tag{2.5}$$

which holds as long as $\varphi \gg 1$ or equivalently $H^2 \gg |\dot{H}|$. This inflationary stage is eventually followed by a phase with scale factor $a(t) \propto t^{\frac{2}{3}}$, as in a matter dominated era. For more details see [4].

So far we have been considering a universe driven by a spatially homogeneous scalar field. Let us now add to φ a small perturbation $\delta\varphi$ which can have spatial dependence. This perturbation induces perturbations of the metric (2.1). Consider then a perturbed metric of the form :

$$ds^2 = a^2(\eta)\left\{(1+2A)d\eta^2 - 2\partial_i B\, dx^i d\eta - [(1-2C)\delta_{ij} + 2\partial_{ij}D]\, dx^i dx^j\right\}. \tag{2.6}$$

It describes the four scalar perturbations of the metric, so called as they derive from scalar functions [5]. As they are the only perturbations to couple to the perturbation of the scalar field we shall restrict our attention to them. Now the perturbations of the metric and of any physical field will depend on the way we identify points on the perturbed space-time with points on the unperturbed background space-time. Different identifications yield different descriptions of the same physical perturbations [5] [6]. One goes over from one description to another by means of a gauge transformation, that is a change of coordinates on the background space-time. For scalar perturbations, infinitesimal gauge transformations are generated by :

$$x^\alpha \to x^\alpha + \zeta^\alpha \quad \text{with} \quad \zeta_\alpha = (T, \partial_i L) \tag{2.7}$$

where T and L are two arbitrary functions. They can be used to fix two of the four scalar metric perturbations, for example $B = D = 0$, which defines the "longitudinal gauge". Hence one can construct altogether three independent "gauge invariant" perturbations Φ, Ψ, χ which are invariant under the transformations (2.7) and which can be chosen so as to reduce respectively to A, C and $\delta\varphi$ in the longitudinal gauge. For more details see [3][5-8].

The classical equations of motion for the perturbations can be found in two different ways. One can start from the Einstein equations and linearise them around the background solution (2.1-5) [3] [9], or one can write the Einstein-Hilbert action of gravity coupled to a scalar field at quadratic order in the perturbations and vary it [7-8]. We begin with the first method.

Let us then write the linearised Einstein equations :

$$\delta G_{\alpha\beta} = \delta T_{\alpha\beta} \tag{2.8}$$

in the longitudinal gauge. ($G_{\alpha\beta}$ is Einstein's tensor and $T_{\alpha\beta}$ the energy-momentum tensor of φ.) Since the gauge invariant quantities Φ, Ψ, χ reduce to A, C and $\delta\varphi$ in the longitudinal gauge, the results obtained by doing the calculations in that gauge are valid in any gauge provided we substitute Φ, Ψ and χ for A, C and $\delta\varphi$. From the (ij) component of (2.8) one obtains :

$$\Phi = \Psi. \tag{2.9}$$

Eliminating Ψ from the $(0i)$ component of (2.8) one derives :

$$\tfrac{1}{2}\dot{\varphi}\chi = \dot{\Phi} + H\Phi. \tag{2.10}$$

Finally, when (2.10) is used to eliminate χ from the (00) component of (2.8), the following evolution equation for the perturbations is obtained (see [3] [7-9]) :

$$\ddot{\Phi} + \dot{\Phi}\left(H - 2\frac{\ddot{\varphi}}{\dot{\varphi}}\right) - \Phi\left(2H\frac{\ddot{\varphi}}{\dot{\varphi}} + \dot{\varphi}^2\right) - \frac{\triangle\Phi}{a^2} = 0 \tag{2.11}$$

where $\triangle \equiv \vec{\nabla}\cdot\vec{\nabla}$ with $\vec{\nabla} = (\partial_1, \partial_2, \partial_3)$. From (2.11) one can deduce the dynamics of all the perturbations. The constraints (2.9-10) are dynamical constraints (see e.g. [10]). They follow from the equations of motion of the system and are not imposed in addition to them, in contrast to the fixing of the gauge. Finally we note that if we define the quantity :

$$Y \equiv 2\frac{\Phi}{\dot{\varphi}} \tag{2.12}$$

then (2.11) is transformed into [7-8] :

$$Y'' - \triangle Y - \frac{w''}{w}Y = 0 \quad \text{where} \quad w \equiv H/\varphi'. \tag{2.13}$$

During the inflationary stage H and $\dot{\varphi}$ are approximated by (2.5) so that $w \simeq -\varphi/2a$, $w''/w \simeq \frac{2}{3}m^2a^2$, and (2.11-13) reduce to :

$$\Phi_k \simeq -\sqrt{\tfrac{1}{6}}\,mY_k \quad ; \quad Y_k'' + Y_k\left(k^2 - \tfrac{2}{3}m^2a^2\right) \simeq 0 \tag{2.14}$$

where we have written Φ (and Y) as a sum over $\vec{k}$ of modes $(2\pi)^{-\frac{3}{2}}\Phi_k(\eta)\exp(\mathrm{i}\vec{k}\cdot\vec{x})$ with $k \equiv \sqrt{\vec{k}^2}$. Equation (2.14) is the equation first obtained in [3], but with an additional mass term. In the adiabatic approximation where the time dependence of H and φ is ignored, so that the scale factor reduces to $a = 1/H|\eta|$ (that is the background geometry is taken to be exactly de Sitter space-time), (2.14) further reduces to :

$$Y_k'' + Y_k\left(k^2 - \frac{4}{\varphi^2\eta^2}\right) \approx 0. \tag{2.15}$$

When k^2 dominates the solution oscillates and in that regime Y/a has the same equation of motion as a massless *conformally* coupled scalar field propagating on the space-time (2.1) (see e.g. [11] and below). "Horizon" crossing, or more accurately Hubble radius crossing, occurs when the physical wave-length a/k of the mode equals the Hubble radius H^{-1}, that is when $k|\eta| \approx 1$. Now the end of inflation corresponds to $\eta \approx 0$ so that after horizon crossing there is not enough conformal time left for Y to oscillate : this is the "freeze-out" picture ([2-3]) as seen in conformal time. The perturbation however is not quite frozen as the mass term in (2.14) has the wrong sign which renders it unstable. Indeed when k^2 becomes negligible, which occurs soon after horizon crossing, the solution of (2.15) tends to the sum of a rapidly decaying and a growing exponential. The unstable mode however grows very slowly like $\exp(\frac{2m^2}{3H}t)$ and can be considered for all practical purposes as frozen till nearly the end of inflation. For more details, see [1-3] and [7-9].

We can also arrive at the equations of motion through variation of the action. To obtain the equations linear in the perturbations one must compute the action δS quadratic in the perturbations. This was done in [7] (see also [8]). An economical way to obtain a manifestly gauge invariant action is to perform the calculation in the longitudinal gauge ($B = D = 0$) and replace A, C and $\delta\varphi$ by Φ, Ψ and χ. In analogy with the electromagnetic case (see Appendix A), the Euler-Lagrange variation with respect to Φ, Ψ and χ yields a set of equations of motion which should be equivalent to the evolution equation (2.11-13) and the constraints (2.9-10) (see [7], where the demonstration however is not made completely explicit ; for an Hamiltonian approach, see [12-13] and Appendix B). Using the constraints (2.9-10) to eliminate two of the three variables, an action for the only true dynamical degree of freedom is obtained. It was found in [7] that it can be cast into the familiar form :

$$\delta S = \tfrac{1}{2} \int d^4x \left[v'^2 - (\vec{\nabla} v)^2 + \frac{z''}{z} v^2 \right], \tag{2.16}$$

where $z \equiv \varphi'/H = 1/w$, and where v is defined as :

$$v \equiv a \left(\chi + \frac{\dot\varphi}{H} C \right). \tag{2.17}$$

The equation of motion for v is readily obtained from (2.16) :

$$v'' - \triangle v - \frac{z''}{z} v = 0. \tag{2.18}$$

To show explicitely the equivalence of (2.18) with (2.13) we note that v as a function of Y reads, using (2.17) (2.9-10) and (2.12) :

$$v = Y' + \frac{z'}{z} Y \tag{2.19}$$

so that the equation (2.18) for v reads, in terms of Y :

$$w \left(\frac{Y'' - \triangle Y - \frac{w''}{w} Y}{w} \right)' = 0, \tag{2.20}$$

or $Y'' - \triangle Y - \frac{w''}{w} Y = cw$ with c a constant. We discard the solutions of the inhomogeneous equation ($c \neq 0$), which do not describe small perturbations. The remaining, homogeneous, equation ($c = 0$) is equivalent to the equation (2.13) for Y. Inversely, differentiating (2.19) and using (2.13) gives :

$$\triangle Y = -z\left(\frac{v}{z}\right)' \tag{2.21}$$

and computing the Laplacian of the equation (2.13) for Y yields :

$$z\left(\frac{v'' - \triangle v - \frac{z''}{z}v}{z}\right)' = 0. \tag{2.22}$$

During the slow rollover stage $z \simeq -2a/\varphi$. In the adiabatic approximation, where the time dependence of φ is ignored, z is therefore proportional to the scale factor. During this stage then, v/a has the same equation of motion as a massless *minimally* coupled scalar field propagating in the space-time (2.1) (see [11] and below). Since $v/a \simeq \chi$ during inflation and $\chi = \delta\varphi$ in the longitudinal gauge, this makes contact with earlier work [2] which considered only the quantisation of $\delta\varphi$, which is also minimally coupled and almost massless during slow roll-over.

Let us finally give the "Poisson" equation which relates the quantities Φ and δ, where δ (see [5] [3]) is the gauge invariant quantity which, in the "comoving" gauge, reduces to $\delta\rho/\rho$, with $\rho = 3H^2$ being the background energy density :

$$\frac{3}{2}H^2\delta = \frac{\triangle\Phi}{a^2} \qquad \text{or} \qquad \delta = \frac{1}{3}\frac{\dot{\varphi}}{(Ha)^2}\triangle Y. \tag{2.23}$$

3. Quantisation of the perturbations

Let us first briefly recall how a real scalar field f of mass μ, propagating on the background (2.1), is quantised (see e.g. [11]). The action is :

$$S = \tfrac{1}{2}\int d^4x\sqrt{-g}\left[g^{\alpha\beta}\partial_\alpha f\partial_\beta f - \frac{\xi}{6}Rf^2 - \mu^2 f^2\right], \tag{3.1}$$

where $R = 6a''/a$ is the scalar curvature and where $\xi = 0$ and $\xi = 1$ correspond to minimal and conformal couplings respectively. It can be rewritten as an action for a scalar field $F \equiv af$ with (time-dependent) mass M, propagating in Minkowski space-time :

$$S = \tfrac{1}{2}\int d^4x\left[F'^2 - \left(\vec{\nabla}F\right)^2 - M^2F^2\right] \quad \text{with} \quad M^2 \equiv \mu^2a^2 - \frac{a''}{a}(1-\xi). \tag{3.2}$$

The equation of motion for f is the Klein-Gordon equation in the background (2.1) :

$$\Box_a f + \left(\mu^2 - \frac{\xi}{6}R\right)f = 0 \tag{3.3}$$

where $\Box_a f \equiv (1/\sqrt{-g})\partial_\alpha(g^{\alpha\beta}\partial_\beta f)$. Equivalently the equation of motion for F is :

$$F'' - \triangle F + M^2 F = 0, \tag{3.4}$$

which is a Klein-Gordon equation on Minkowski space-time.

The canonical quantisation of f, or equivalently F, proceeds in the standard way. Construct the field operator :

$$\hat{F} = \int d^3k \left(\tilde{F}_k \hat{a}_k + \tilde{F}_k^* \hat{a}_k^+ \right) \tag{3.5}$$

where the eigenmodes

$$\tilde{F}_k \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} F_k(\eta) \exp(\mathrm{i}\vec{k}\cdot\vec{x}) \tag{3.6}$$

form a complete set of orthonormal solutions of (3.4) with respect to the usual Klein-Gordon norm. This implies the following condition on the Wronskian :

$$F_k \partial_\eta F_k^* - F_k^* \partial_\eta F_k = \mathrm{i}. \tag{3.7}$$

The action being (3.2), the conjugate momentum of the field $\hat{F}$ is $\partial_\eta \hat{F}$, and quantisation is implemented by imposing the canonical commutation relations :

$$\left[\hat{F}(\eta,\vec{x}), \partial_\eta \hat{F}(\eta,\vec{x}')\right] = \mathrm{i}\delta(\vec{x}-\vec{x}') \tag{3.8}$$

or equivalently :

$$\left[\hat{a}_k, \hat{a}_k^+\right] = \delta(\vec{k}-\vec{k}'). \tag{3.9}$$

Finally, the vacuum state $|\,0>$ is defined, for a given choice of eigenmodes, by :

$$\hat{a}_k|\,0>= 0 \quad \forall\vec{k}. \tag{3.10}$$

(The equivalence of quantising f on a FRW background and F on Minkowski space-time should not conceal the ambiguities in defining a vacuum state in curved spaces. In the context considered here however, the choice of the vacuum will not be crucial, as long as it belongs to the family defined by (3.4), (3.7) and (3.10). For more details see e.g. [7].)

The quantity of interest to us will be the two-point function of the field F :

$$<0\,|\hat{F}(\eta,\vec{x})\hat{F}(\eta,\vec{x}')|\,0>= \frac{1}{(2\pi)^3}\int d^3k |F_k(\eta)|^2 e^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{x}')} \tag{3.11}$$

which we shall identify with its equal time correlation function, as defined in classical statistics (see e.g. [14]). The power spectrum $P(k)$, or Fourier transform of the correlation function of F, is hence given by :

$$P(k) = |F_k(\eta)|^2. \tag{3.12}$$

(We shall not dwell here on the problem of defining in a satisfactory manner the transition from the quantum to the classical regime. We refer for these matters to e.g. [13] [15-16].)

Consider now the gauge invariant perturbation v whose equation of motion (2.18) derives from the action (2.16). We can quantise it by considering it as a scalar field on Minkowski space-time, and hence identify it with F. Then its mass squared is $M^2 \equiv -z''/z$ ([7]). Equivalently we can identify v/a with f and consider it as a scalar field on the physical background (2.1). Then we have : $\mu^2 a^2 - \frac{a''}{a}(1-\xi) \equiv -z''/z$. Since $z''/z \simeq a''/a$ during the inflationary stage, we see that v/a can be considered, during inflation, as a massless ($\mu = 0$), minimally coupled ($\xi = 0$), scalar field on a (quasi-)de Sitter background, as already mentioned.

Once we have quantised v along the preceeding lines, where $F \to v$ and $f \to v/a$, the quantisation of all the other gauge invariant perturbations follows, through the constraints which relate them to v. In particular the quantity Y becomes the operator :

$$\hat{Y} = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left[\hat{a}_k Y_k(\eta) \exp(\mathrm{i}\vec{k}\cdot\vec{x}) + \hat{a}_k^+ Y_k^*(\eta) \exp(-\mathrm{i}\vec{k}\cdot\vec{x})\right] \tag{3.13}$$

where the normalisation of the eigenfunctions $Y_k(\eta)$ follows from (2.21) and (3.7) (where $F_k \to v_k$), and reads :

$$Y_k \partial_\eta Y_k^* - Y_k^* \partial_\eta Y_k = \mathrm{i}k^{-2}. \tag{3.14}$$

At this stage the pitfall of a naive quantisation based on equations of motion rather than the actions they derive from (see e.g. [17]), should become transparent. In view of the similarity between the equations of evolution (2.13) for Y and (2.18) for v, one could indeed be tempted to consider Y, just as we did for v, as a scalar field on Minkowski space-time, with a mass squared $M^2 \equiv -w''/w$. Equivalently, this would mean identifying Y/a with a scalar field on the background (2.1), with : $\mu^2 a^2 - \frac{a''}{a}(1-\xi) \equiv -w''/w$. Since $w''/w \simeq \frac{2}{3}m^2 a^2$ during the inflationary stage, we would then consider Y/a, during inflation, as a conformally coupled field ($\xi = 1$), endowed with a small imaginary mass ($\mu^2 = -\frac{2}{3}m^2$). A quantisation along the preceeding lines, with $F \to Y$ and $f \to Y/a$, would then lead to a normalisation of the modes Y_k of the field $\hat{Y}$ given by (3.7) with $F_k \to Y_k$, in flat contradiction with (3.14).

The reason for the contradiction is that this naive quantisation of Y amounts to assuming incorrectly that its equation of motion derives from the action (3.1-2) (with $F \to Y$ and $f \to Y/a$) or, equivalently, to assuming that its commutation relations are given by (3.8-9). We cannot however choose the action for Y : it must be deduced from the Einstein-Hilbert action of the theory. A calculation, similar to that of ref. [7], or, more quickly, a substitution of v as given by (2.19) into (2.16), yields, with $M^2 \equiv w''/w$:

$$\delta S = \tfrac{1}{2}\int d^4x \left\{ Y''^2 - \left(\vec{\nabla}Y'\right)^2 + M^2\left[2Y'^2 - \left(\vec{\nabla}Y\right)^2\right] + Y^2\left[M^4 - M^{2''}\right]\right\}. \tag{3.15}$$

Variation of (3.15) gives (2.20), and hence (2.13), but (3.8-9) do *not* implement its canonical quantisation. The correct way to quantise (3.15) is, in fact, to redefine Y as a function of v and rewrite it as the standard action (2.16) for v. Thus we recover (3.14).

A few remarks now about the two-point function (3.11). During the inflationary stage, v/a behaves, as we have mentioned, as a massless, minimally coupled field on a quasi-de Sitter space-time. Its two-point function (3.11) (with $F_k \to v_k$) therefore exhibits an infra-red divergence (see e.g. [18] and references therein). This can be easily seen, as the solutions of (2.16), during the inflationary stage and in the adiabatic approximation, which satify (3.7), are :

$$v_k(\eta) \approx \frac{c_1}{\sqrt{2k}}\, e^{-\mathrm{i}k\eta}\left(-\mathrm{i}-\frac{1}{k\eta}\right)+\frac{c_2}{\sqrt{2k}}\, e^{+\mathrm{i}k\eta}\left(+\mathrm{i}-\frac{1}{k\eta}\right) \tag{3.16}$$

with : $|c_1|^2 - |c_2|^2 = 1$. (Imposing $c_2 = 0$, as we shall henceforth do, means choosing the so-called Bunch-Davies vacuum : see e.g. [18].) Now, Y and v are related by (2.21), which, in the inflationary stage and in the adiabatic approximation, yields :

$$Y_k \approx -\frac{1}{\sqrt{2}}\frac{1}{k^{\frac{3}{2}}}\, e^{-\mathrm{i}k\eta}, \tag{3.17}$$

in keeping with (3.14). Hence the two-point function of Y exhibits the same infra-red divergence as v. A first remark is that if we had naively quantised Y, its modes would have been, according to (3.7), k times the expression given by (3.17) and its two-point function would have been free of infra-red divergence ! A second remark is that this infra-red divergence should not be of too much concern, as we are interested in the correlation function of the density fluctuation δ. Indeed δ is related to Y by (2.23), so that, during inflation and in the adiabatic approximation :

$$\delta_k = -\frac{\dot{\varphi}}{3\sqrt{2}}\frac{\sqrt{k}}{(Ha)^2}\, e^{-\mathrm{i}k\eta}. \tag{3.18}$$

For completeness, let us recall how (3.18) leads to a scale invariant spectrum of energy density fluctuations ([1-3] [7-8] and [17]). The power spectrum (3.12) of δ during inflation is proportional to $\dot{\varphi}^2(aH)^{-4}k$. The average value of the fluctuations on a given scale λ is defined as (see e.g. [14] for precise definitions) :

$$<\delta^2>_\lambda \;\propto\; \int_0^{1/\lambda} d^3k\, P(k) \tag{3.19}$$

and is therefore :

$$<\delta^2>_\lambda \;\propto\; m^2(aH)^{-4}\lambda^{-4}. \tag{3.20}$$

Hence when, during inflation, a physical scale $a\lambda$ becomes larger than the Hubble radius H^{-1}, we see that (3.20) does not depend on the scale λ, in keeping with the intuitive idea that in a maximally symmetric de Sitter space-time no scale can be privileged. Once the perturbations are well outside the Hubble radius or horizon, a k-independent amplification of Φ takes place, of the order of $(H/\dot{\varphi})^2$, where this quantity is to be evaluated at horizon crossing (HC)([1-3] [7-8] and [17]). Hence, when the perturbations reenter the Hubble radius in the post-inflationary era, their spectrum is still almost scale-independent and given by :

$$< \delta^2 >_\lambda \ \propto \ m^2 \left(\frac{H^4}{\dot{\varphi}^2} \right)_{HC} \ \propto \ m^2 \varphi_{HC}^2 \tag{3.21}$$

where the factor of proportionality is generically of order unity. Since $\varphi_{HC} = \mathcal{O}(1-10)$ for galactic scales (see e.g. [19]) the spectrum (3.21) is compatible with the observed isotropy of the microwave background radiation if $m = \mathcal{O}(10^{-6} - 10^{-7})$.

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Appendix A : The electromagnetic analogy

The 4 components (A_0, A_i) of the electromagnetic potential are the analog of the 10 perturbations of the metric tensor. Decomposing A_i as $A_i = \partial_i A + \bar{A}_i$ where $\bar{A}_i$ is divergence-free ($\partial_i \bar{A}^i = 0$), we can classify them, in the terminology of ref. [5], into 2 "scalars" (A_0, A) and 2 "vector" components $(\bar{A}_i)$. The two scalars are the analog of the four scalar metric perturbations appearing in (2.6).

In a gauge transformation $A_\alpha \to A_\alpha + \partial_\alpha \zeta$, A_0 goes over to $A_0 + \zeta'$ (where the prime denotes the time derivative), A to $A + \zeta$, while the vectors $\bar{A}_i$ remain unchanged. One can therefore construct 3 gauge invariant quantities : a scalar one, analogous to Φ and Ψ in the text, defined as : $\Phi \equiv A_0 - A'$; and two vector ones, which can be taken to be $\bar{A}_i$. The equivalent of the longitudinal gauge of the text is the gauge $A = 0$ where Φ reduces to A_0 (and $\bar{A}_i$ to A_i).

The equations of motion in vacuo for Φ and $\bar{A}_i$ can then be obtained by writing Maxwell's equations ($\partial_\alpha F^{\alpha\beta} = 0$ with $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$) in the gauge $A = 0$ and then replacing A_0 by Φ. They read :

$$\Phi = 0 \qquad \Longleftrightarrow \qquad \vec{\nabla} \cdot \vec{E} = 0 \quad \text{with} \quad E_i \equiv F_{0i} = \bar{A}_i' - \partial_i \Phi, \tag{A.1}$$

and :

$$\Box \bar{A}_i = 0 \qquad \Longleftrightarrow \qquad \vec{\nabla} \wedge \vec{H} = \vec{E}' \quad \text{with} \quad H_i \equiv \epsilon_i^{\ jk} F_{jk} = \epsilon_i^{\ jk} \partial_j \bar{A}_k. \tag{A.2}$$

(The two remaining equations : $\vec{\nabla} \cdot \vec{H} = 0$ and $\vec{\nabla} \wedge \vec{E} = -\vec{H}'$ follow from the definitions of E_i and H_i in terms of F_{ij}.) Equation (A.1) is a dynamical constraint, similar to (2.9-10). As we have not included matter, there is here no equivalent of v. Therefore equation (A.2) is not the analog of the evolution equation (2.11-13) : it would rather be the equivalent of the equation of motion for the tensor perturbations of the metric (the gravitons), had we considered them.

An alternative route to Maxwell's equations is to vary the Lagrangian : $L \equiv \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$. Again we can express it as a function of gauge invariant quantities by going to the gauge $A = 0$ and then replacing A_0 by Φ. One obtains, up to a total divergence :

$$L = \tfrac{1}{2} \left[-\Phi \triangle \Phi + \partial_\alpha \bar{A}_i \partial^\alpha \bar{A}^i \right]. \tag{A.3}$$

The Euler variation of L with respect to Φ and $\bar{A}_i$ readily yields Maxwell's equations (A.1-2). Equation (A.3) is the analog of the Einstein-Hilbert Lagrangian, quadratic in the gauge invariant perturbations, which was considered in the main text. Hence the claim that its variation with respect to Φ, Ψ and χ should yield the constraint equations (2.9-10) as well as the evolution equation (2.11). Finally a Lagrangian for the true dynamical degrees of freedom is obtained by eliminating Φ from (A.3) using the dynamical constraint (A.1). The result :

$$L = \tfrac{1}{2} \partial_\alpha \bar{A}_i \partial^\alpha \bar{A}^i \tag{A.4}$$

is the analog of the Lagrangian (2.16) for v. Since its variation yields (A.2), the use of the constraint before variation was licit.

The economical way to arrive at (A.3-4), respectively (2.16), is to calculate the Lagrangian in the gauge $A = 0$, respectively the longitudinal gauge. This mathematical trick should not be confused with a fixing of the gauge, such as $A_0 = 0$, respectively a choice of lapse and shift, which, when performed in the action before variation, implies an irretrievable loss of the dynamical constraint (A.1), respectively (2.9-10).

Appendix B : The Hamiltonian for v

We start from δS given in equation 19 of reference [7], which is the quadratic action of the metric (2.6) and of the matter field $\varphi + \delta\varphi$ with potential $V(\varphi)$, but we set $4\pi G = 1$ (instead of $8\pi G = 1$ as in the main text). We add a total derivative such that B' and A', the time derivatives of the shift and lapse perturbations, do not appear. (We use conformal time η.) Their canonical momenta π_B and π_A vanish therefore identically. The situation is then exactly equivalent to that of full gravity. We can make use of the flatness of the background space slices to Fourier-analyse all fields in δS following the model of $A(x) = (2\pi)^{-3/2} \int d^3k e^{i\vec{k}\vec{x}} A_k(\eta)$. The canonical Hamiltonian then becomes :

$$\mathcal{H} = \int d^3k \, \left(\mathcal{H}_{0_k} + B_k \mathcal{H}_{B_k} + A_k \mathcal{H}_{A_k}\right), \tag{B.1}$$

where

$$\begin{aligned}
\mathcal{H}_{0_k} =& \tfrac{1}{2} a^{-2} \pi^2_{\delta\varphi_k} + \varphi' \delta\varphi_k \pi_{C_k} + \tfrac{1}{2}\left(a^2 V_{,\varphi\varphi} - 3\varphi'^2 + k^2\right) \delta\varphi_k^2 \\
& - \tfrac{1}{2} a^2 k^2 C_k^2 + a^{-4} \pi_{D_k} \left(\tfrac{3}{2} k^{-4} \pi_{D_k} - k^{-2} \pi_{C_k}\right) \qquad (B.2) \\
\mathcal{H}_{A_k} =& \varphi' \pi_{\delta\varphi_k} - \alpha \pi_{C_k} + a^2 \left(a^2 V_{,\varphi} + 3\alpha\varphi'\right) \delta\varphi_k + a^2 k^2 C_k \qquad (B.3) \\
\mathcal{H}_{B_k} =& \pi_{D_k} \qquad (B.4)
\end{aligned}$$

where $\alpha \equiv a'/a$. The secondary constraints $\dot{\pi}_B = -\partial\mathcal{H}/\partial B = \mathcal{H}_B = 0$ and $\dot{\pi}_A = -\partial\mathcal{H}/\partial A = \mathcal{H}_A = 0$ are linearised versions of the shift and lapse constraints of general relativity. As they are linear they can be solved explicitly, in contrast to the nonlinear theory. By solving the constraints we shall understand giving a Hamiltonian (and hence a Lagrangian) that contains only as many variables as there are true dynamical degrees of freedom. By a lucky accident one of the constraints, $\mathcal{H}_{B_k}$, is already one of the momenta, π_{D_k}. This is what we must achieve, by means of a canonical transformation, for all the constraints. We define a new variable y by $\mathcal{H}_{A_k} = \pi_{y_k}$. The remaining new variable is then uniquely defined as Mukhanov's v (up to overall multiplication by a function of time). The generating function of the canonical transformation is, in terms of the new positions and old momenta,

$$\begin{aligned}
F_k\left(\pi_{C_k}, \pi_{\delta\varphi_k}, v_k, y_k, \eta\right) =& -\tfrac{1}{2}\alpha^{-1} a^{-2} E_k^{-1} y_k^2 - E_k^{-1} \Big(a^{-2} \frac{\varphi'}{\alpha} \pi_{\delta\varphi_k} y_k \\
& - a^{-1} k^2 \pi_{\delta\varphi_k} v_k - a^{-2} \pi_{\delta C_k} y_k + a^{-1}(a^2 V_{,\varphi} + 3\alpha\varphi') \pi_{\delta C_k} v_k \Big), \qquad (B.5)
\end{aligned}$$

where $E_k = -k^2 + \alpha^{-1}\varphi'(a^2 V_{,\varphi} + 3\alpha\varphi')$. In the new Hamiltonian $\tilde{\mathcal{H}}$ we can then enforce the secondary constraints $\pi_{D_k} = 0$ and $\pi_{y_k} = 0$. As their time derivative must also vanish, we have $\partial\mathcal{H}/\partial D = 0$ and $\partial\mathcal{H}/\partial y = 0$, which means that $\tilde{\mathcal{H}}$ depends only on v and π_v when the constraints are enforced. It is then

$$\begin{aligned}
\tilde{\mathcal{H}}_k =& \tfrac{1}{2}\pi^2_{v_k} + \left(\frac{\varphi'^2}{\alpha} + \alpha - \alpha^{-1} E_k^{-1} (\alpha\varphi' - \varphi'')^2\right) \pi_{v_k} v_k \\
& + \tfrac{1}{2} E_k^{-2} \left(k^6 + (\alpha^2 + \alpha' - \frac{\varphi'''}{\varphi'}) k^4 + (\alpha\varphi' - \varphi'')^2 k^2\right) v_k^2. \qquad (B.6)
\end{aligned}$$

¿From this Hamiltonian we can go back and derive a Lagrangian $\mathcal{L}_k$ for v_k. If we add a total derivative $(f_k(\eta)v_k^2)'$ to $\mathcal{L}_k$ such that the $v_k v_k'$ term vanishes, the high powers of k^2 disappear and the integral over the $\mathcal{L}_k$ gives (2.16). It seems to be an accident of the symmetry of the background spacetime, however, that the action for the gauge-invariant perturbations can be written in the form of covariant scalar field action. If one had substituted the constraints (B.3-4) directly into (B.1-2), the result would not have been a functional of only one of the original variables, while the Hamiltonian detour to (2.16) leads uniquely to v as the true degree of freedom. The method applied here to scalar perturbations of flat FRW spacetime is described for arbitrary backgrounds in reference [13].

References

[1] R.Brandenberger in *Physics of the Early Universe*, J.A. Peacock, A.F. Heavens, A.T. Davies Eds., SUSSP Publ., Edinburgh, 1990
K.A.Olive, Phys. Reports 190 (1990) 6
A.Linde, *Particle Physics and Inflationary Cosmology*, Gordon and Breach, 1990
G.Börner, *The Early Universe*, Springer 1988
E.W.Kolb and M.S.Turner, *The Early Universe*, Addison-Wesley, 1990
V.N.Lukash and I.D.Novikov, *Lectures on the Early Universe*, April 1991, Paris Observatoire, preprint 91012

[2] S.W.Hawking, Phys. Lett. 115B (1982) 339
A.Linde, Phys. Lett. 116B (1982) 335
A.A.Starobinski, Phys. Lett. 117B (1982) 175
A.H.Guth and S.Y.Pi, Phys. Rev. Lett. 49 (1982) 1110

[3] J.M.Bardeen, P.J.Steinhardt and M.S.Turner, Phys. Rev. D28 (1983) 679

[4] V.A.Belinski, L.P.Grishchuk, I.M.Khalatnikov and Ya.B.Zel'dovich, JETP 62 (1985) 195

[5] J.M.Bardeen, Phys. Rev. D22 (1980) 1882

[6] J.M.Stewart, Class. and Quant. Grav. 7 (1990) 1169

[7] V.F.Mukhanov, Sov. Phys. JETP 68 (1988) 1

[8] V.F.Mukhanov, H.A.Feldman and R.H.Brandenberger, *Theory of Cosmological Perturbations*, Brown-HET-780 preprint, December 1990

[9] M. Sasaki, Prog. Theor. Phys. 70 (1983) 394
Jai-chan Hwang, Class. and Quant. Grav. 8 (1991) 195

[10] P. Ramond, *Field Theory: A Modern Primer*, Addison Wesley 1988

[11] N.D.Birrell and P.C.W.Davies, *Quantum Fields in Curved Space*, Cambridge University Press, 1982

[12] N.Makino, M.Sasaki, *The Density Perturbation in the Chaotic Inflation with Non-minimal Coupling*, YITP/U-91-8 preprint, February 1991

[13] C.Gundlach, *Gauge Invariant Quantisation of Cosmological Perturbations*, in preparation

[14] G.Efstathiou, in *Physics of the Early Universe*, J.A. Peacock, A.F. Heavens, A.T. Davies Edts, SUSSP Publ., Edinburgh, 1990

[15] R.Brandenberger, R.Laflamme and M.Mijic *On the Origin of Classical Density Perturbations in the Inflationary Universe*, preprint, to appear in the Proceedings of the Fifth Seminar of Quantum Gravity, Moscow 1990.

[16] A.A.Starobinski, in *Current Topics in Field Theory, Quantum Gravity and Strings*, H.J. de Vega and N.Sanchez Edts, Lecture Notes in Physics, 246, Springer 1986
L.P.Grishchuk and Yu.V.Sidorov, Class. and Quant. Grav. L162 (1989) 6.

[17] M.Sasaki, Prog. Theor. Phys. 76 (1986) 1036
Y.Nambu and M.Sasaki, Prog. Theor. Phys. 83 (1990) 37

[18] D.Polarski, Phys. Rev. D43 (1991) 1892

[19] L.A.Kofman and A.D.Linde, Nucl. Phys. B282 (1987) 555