Dark Matter, Time-varying G and the Dilaton

T. Damour

D.A.R.C, Observatoire de Paris - CNRS, 92195 Meudon Cedex, France
and I.H.E.S., 91440 Bures sur Yvette, France

and

G. W. Gibbons and C. Gundlach

D.A.M.T.P., University of Cambridge
Silver Street, Cambridge CB3 9EW, UK

11 July 1989

The value of the Jordan-Brans-Dicke coupling parameter ω is at present much more tightly bounded below by radar time delay experiments than it is by the predicted time variation of Newton's constant G. We point out the possibility of having a Jordan-Brans-Dicke-type scalar (i.e. a scale-free massless scalar field or "dilaton") couple with different strengths to visible and to conjectured "dark" matter. This would allow a larger time variation of G (and a small value of ω for dark matter) while ω is still large for visible matter. $\dot{G}$ estimations (e.g. from the binary pulsar) may thus provide nontrivial bounds on the coupling of the dilaton to dark matter.

1. Introduction

Recent experimental data provide relatively high precision information about any possible time variation of Newton's constant of gravitation. E.g. in recent work the determination [1]

$$\dot{G}/G = (1.2 \pm 1.3) \times 10^{-11}\,\mathrm{yr}^{-1} \tag{1}$$

(consistent with zero) was phenomenologically derived from binary pulsar data. The simplest theoretical framework in which to consider a time-variation of G is a "scalar-tensor" theory of the sort originally developed by Jordan [2] and elaborated upon by Brans and Dicke [3]. In effect one regards $G(x)$, or some function of it, as a scalar field additional to the spacetime metric. The converse is to some extent true— a theory with a neutral scalar field coupled to gravity can give rise to spatio-temporal variations of Newton's "constant". The condition that such variations be observable on some length scale L is that the field be effectively massless on the scale L, i.e. its mass must be $\ll \hbar/L$ (we henceforth use units where $c = 1$).

Neutral scalar fields of this kind may arise in Kaluza-Klein theory (where the scalar gives the size of the compactified dimensions) or in superstring theories (the dilaton for instance). Since the effects of quantum corrections are generally expected to give contributions to the mass of such scalar fields of the order of the Planck mass or some modest fraction thereof it is generally supposed that any scalar fields in nature must be very massive (the Higgs fields of the standard model while not of Planck mass are of course in this sense also very massive).

Nevertheless it is not self-evident that one or more massless scalar fields could not survive down to the lowest energy, i.e. largest distance, scales we know and it becomes an interesting question to ask what its effects might be and how it might be detected. Consider a scalar field $\tilde{\phi}$ coupled to ordinary matter in the way envisaged by Brans and Dicke. The action S (ignoring boundary terms and using signature $-{+}{+}{+}$) is

$$S = \int \left(\tilde{\phi}\tilde{R} - \frac{\omega}{\tilde{\phi}}\, \tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} \right) \sqrt{\tilde{g}}\, d^4x \; + \; S_m[\, \psi, \tilde{g}_{\mu\nu} \,], \tag{2}$$

where $\tilde{g}_{\mu\nu}$ is the spacetime metric (in what we shall call the Jordan conformal frame), $\tilde{R}$ its curvature scalar and ω a dimensionless coupling constant. (Note that $\tilde{\phi}$ as defined here is $1/16\pi$ times ϕ as defined in [3].) The last term in (2) denotes the action of the matter, which is a functional of some matter variables, collectively denoted by ψ, and of the metric $\tilde{g}_{\mu\nu}$. The functional $S_m[\psi, \tilde{g}]$ should reduce to the corresponding Minkowski matter action, $S_m[\psi, \eta]$, when $\tilde{g}_{\mu\nu} \to \eta_{\mu\nu}$, and, therefore, should contain, besides ψ, $\tilde{g}_{\mu\nu}$ and their derivatives, only some <u>constant</u> parameters : masses, coupling constants, ... In the simple case where the matter is phenomenologically represented as a set of point particles of mass $\tilde{m}$ and worldlines $x^\mu = z^\mu(\lambda)$ the matter action reads

$$S_m[\, z, \, \tilde{g}_{\mu\nu} \,] = - \sum_{\text{particles}} \int \tilde{m}\, d\tilde{s},$$

where $d\tilde{s} = (-\tilde{g}_{\mu\nu}(z)dz^\mu dz^\nu)^{1/2}$ is the $\tilde{g}$-proper-time along their worldlines.

Then the coupling of $\tilde{\phi}$ to ordinary matter in the solar system (and to some extent the binary pulsar) gives a contradiction with present observations unless [4]

$$\omega^{-1} < 0.002 \quad (1\sigma \text{ limit}). \tag{3}$$

In a zero-pressure zero-curvature Friedmann-Lemaître cosmological model Weinberg [5] has shown that (2) leads to a decrease of Newton's constant

$$\dot{G}_0/G_0 = -\frac{H_0}{\omega+1}\ , \tag{4}$$

where H_0 is the Hubble constant (say $7.67 \times 10^{-11}\,\mathrm{yr}^{-1}$, corresponding to $75\,\mathrm{km\,s}^{-1}/\mathrm{Mpc}$). Given the limit (3) we can see that (4) predicts a much smaller variation of Newton's constant than the upper limit (1) which would thus seem to provide us with no better information than that giving the bound (3). However, (4) is derived supposing that all the matter in the universe including the dark matter, which is required to reconcile our assumption that $k = 0$ with the observed cosmological matter density, couples in the same way to the $\tilde{\phi}$ field. This is far from obvious and becomes even less so if we perform a Weyl scaling of the metric $\tilde{g}_{\mu\nu}$ to transform the action (2) to a more useful form [6]. We define a new metric $g_{\mu\nu}$, which defines what we will call the Einstein conformal frame, and a new scalar field σ by

$$g_{\mu\nu} = (16\pi\mathcal{G}\tilde{\phi})\tilde{g}_{\mu\nu} = (2\kappa^2\tilde{\phi})\tilde{g}_{\mu\nu}, \tag{5}$$

$$\sigma = -\sqrt{\omega + 3/2}\,\ln(2\kappa^2\tilde{\phi}), \tag{6}$$

where $\mathcal{G}$ is a constant with the same dimensions as Newton's constant and $\kappa^2 = 8\pi\mathcal{G}$ to obtain (modulo boundary terms) the equivalent action

$$S = \int \left(\frac{R}{2\kappa^2} - \frac{g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma}{2\kappa^2}\right) g^{1/2}\, d^4x + \ S_m\left[\psi, e^{2\beta\sigma} g_{\mu\nu}\right], \tag{7}$$

where R denotes the curvature scalar of $g_{\mu\nu}$ and where

$$\beta = \frac{1}{2\sqrt{\omega+3/2}}, \qquad \tilde{\phi} = \frac{1}{2\kappa^2}\, e^{-2\beta\sigma}.$$

In the case of point particles, the matter action now reads

$$S_m[z, e^{2\beta\sigma} g_{\mu\nu}] = -\sum \int \tilde{m}\, e^{\beta\sigma} ds\ ,$$

which may be recast in terms of a space-time dependent mass m given by $m = \tilde{m}\ e^{\beta\sigma}$.

It is important to realize that by means of suitable Weyl rescalings and redefinitions of the scalar field any reasonable scalar-tensor theory has a kinetic term given by the first two terms in (7). What distinguishes different scalar-tensor theories is the coupling to matter and any possible mass or self-interaction terms. The matter-scalar coupling chosen by Brans and Dicke has three important features:

1) The exponential form, meaning the absence of a privileged scale in the theory (shifting σ by a constant is equivalent to rescaling the mass $\tilde{m}$).

2) All matter couplings are metric, e.g. σ appears in the matter Lagrangian only in the combination $e^{2\beta\sigma} g_{\mu\nu}$.

3) All material systems couple to the same metric, i.e. are "freely falling" in this universal metric.

The first feature is typical of "dilaton couplings" and also arises naturally in the Kaluza-Klein picture. We shall comment upon it no further here. The second feature is motivated by the very precise experiments which support the "weak equivalence principle" and the "Einstein equivalence principle" [4]. It is certainly a very plausible assumption for all conventional, visible matter that we have knowledge of at present. It is however not at all obvious that the invisible matter in the universe need couple to σ with the same strength as the visible matter. We propose here to investigate a new class of scalar-tensor models in which the visible matter couples to the metric $e^{2\beta_V \sigma} g_{\mu\nu}$, while the invisible one couples to the different, but conformally related, metric $e^{2\beta_I \sigma} g_{\mu\nu}$. In other words, we shall investigate the consequences of Lagrangians of the type

$$\begin{aligned} S = \int \left(\frac{R}{2\kappa^2} - \frac{g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma}{2\kappa^2} \right) g^{1/2} d^4x \quad + \\ + S_V \left[\psi_V, e^{2\beta_V \sigma} g_{\mu\nu}\right] + S_I \left[\psi_I, e^{2\beta_I \sigma} g_{\mu\nu}\right], \end{aligned} \tag{8}$$

where now the basic dimensionless coupling constants, β_i with $i = I, V$, can be positive or negative (actually only the relative sign matters). Then the corresponding ω_i's are defined by $\omega_i + 3/2 \equiv \left(4\beta_i^2\right)^{-1}$ (so that $\omega_i > -3/2$).

As discussed below, in the approximation of a smooth distribution of dark matter over solar system scales, the present experimental limits, eq.(3), on the Jordan-Brans-Dicke theory concern only the coupling to visible matter and give at the 2σ level (95 % confidence)

$$\omega_V > 250 \;\Leftrightarrow\; |\beta_V| < 0.032. \tag{9}$$

In this paper, we shall investigate to what extent cosmological observations and limits on $\dot{G}_0/G_0$, e.g. eq.(1), give us a handle on β_I (and ω_I). It is worth pointing out at this stage that it has been claimed [7] that naively reducing the superstring gives $\omega = -1$, i.e. $\beta = \pm 1/\sqrt{2}$.

2. The Model

In the Einstein conformal frame $g_{\mu\nu}$ the field equations derived from the Lagrangian (8) are

$$G_{\mu\nu} = \sigma_{,\mu}\sigma_{,\nu} - \frac{1}{2}\, g_{\mu\nu}(\nabla\sigma)^2 + \kappa^2 \left(T^V_{\mu\nu} + T^I_{\mu\nu}\right), \tag{10}$$

$$\sigma = -\kappa^2 \left(\beta_V T^V + \beta_I T^I\right), \tag{11}$$

where $G_{\mu\nu}$ is the Einstein tensor of the metric $g_{\mu\nu}$, ∇_μ its Levi-Civita connection (with $\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$), and

$$T^{\mu\nu}_i \equiv 2g^{-1/2}\ \delta S_i\left[\psi_i, e^{2\beta_i\sigma} g_{\mu\nu}\right]/\delta g_{\mu\nu} \tag{12}$$

the stress-energy tensor of the i-type of matter ($i = V, I$) in the Einstein frame (all indices are moved by the g-metric and T^i denotes the g-trace of $T^i_{\mu\nu}$). The invariance of $S_i\left[\psi_i, e^{2\beta_i\sigma} g_{\mu\nu}\right]$ under coordinate transformations leads, when the ψ_i-equations of motion are satisfied, to

$$\nabla^\lambda T^i_{\lambda\mu} = \beta_i T^i \nabla_\mu \sigma \quad \text{(no summation on } i). \tag{13}$$

The equations of motion in the Jordan frame are given in an appendix.

Perfect-fluid distributions, i.e.

$$T^{\mu\nu}_i = (\rho_i + p_i)\ u^\mu_i\, u^\nu_i + p_i g^{\mu\nu}, \tag{14}$$

with $g_{\mu\nu}u^\mu_i\, u^\nu_i = -1$, can act as sources of a Robertson-Walker metric (in the Einstein-frame)

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\ell^2, \tag{15}$$

with

$$d\ell^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{16}$$

if $u^\mu_V = u^\mu_I$, and if all physical quantities depend only on time. Then the field equations (10)-(11) give :

$$3\,\frac{\dot a^2 + k}{a^2} = \kappa^2 \sum_i \rho_i + \frac{1}{2}\dot\sigma^2 \quad , \tag{17}$$

$$-3\,\frac{\ddot a}{a} = \frac{1}{2}\,\kappa^2 \sum_i (\rho_i + 3p_i) + \dot\sigma^2 \quad , \tag{18}$$

$$-a^{-3}(a^3\dot\sigma)^{\cdot} = \kappa^2 \sum_i \beta_i\,(\rho_i - 3p_i) \quad , \tag{19}$$

where $\dot{} \equiv d/dt$.

We will now assume that the universe is dynamically dominated by some kind of invisible matter, so that we can neglect ρ_V and p_V compared to ρ_I and p_I on very large distance scales. For the moment, we shall also assume that the pressure is zero. Introducing the dynamical variables $H = \dot{a}/a$, $y = \dot{\sigma}$, and adopting units in which $\kappa^2 = 1$, we obtain a constrained dynamical system out of (17)–(19). The constraint can be eliminated by defining $F = ka^{-2}H^{-2}$, obtaining

$$\begin{cases} \dot{y} = -3Hy - 3\beta_I(F+1)H^2 + \frac{1}{2}\,\beta_I y^2, \\ \dot{H} = -\frac{3}{2}\,H^2 - \frac{1}{4}\,y^2 - \frac{1}{2}\,FH^2, \\ \dot{F} = (F+F^2)H + \frac{1}{2}\,\frac{y^2 F}{H}\ . \end{cases} \tag{20}$$

$F = 0$ ($k = 0$) is an invariant plane in phase space. In view of the existence of convincing arguments (e.g. inflation) for $k = 0$ (or more exactly, for F being exponentially small) we shall limit ourselves to this subset of solutions. (A similar but not identical treatment of Brans-Dicke cosmology as a dynamical system has recently been given by Romero et al. [8].)

The simple dynamical system (20) is written in the Einstein frame, which is however not directly accessible to observation (one would need, e.g., a system of two black holes, for which $T^i_{\mu\nu} = 0$, and $\sigma =$ const. [9], to make up a clock ticking the Einstein time). Observations are rather made with objects made of visible matter (this includes atomic clocks as well as the binary pulsar), coupled to, and therefore measuring, only the "visible metric",

$$d\tilde{s}^2 = e^{2\beta_V\sigma}ds^2\ . \tag{21}$$

In this "Jordan conformal frame" (or "atomic frame") the Robertson-Walker metric (15) becomes

$$\begin{aligned} d\tilde{s}^2 &= e^{2\beta_V\sigma}[-dt^2 + a^2(t)d\ell^2] \\ &= -d\tilde{t}^{\,2} + \tilde{a}^2(\tilde{t})d\ell^2 \end{aligned} \tag{22}$$

)
where

$$\tilde{a} = e^{\beta_V\sigma}a \quad , \quad d\tilde{t} = e^{\beta_V\sigma}dt\ , \tag{23}$$

are the cosmological variables directly accessible to observation. Generalizing the system (20) to a gamma law equation of state, $p_I = (\gamma-1)\rho_I$, and rewriting it in terms of $\tilde{a}$ and $\tilde{t}$, with the definitions $\tilde{y} = d\sigma/d\tilde{t}$, $\tilde{H} = (d\tilde{a}/d\tilde{t})/\tilde{a}$, $r = 4-3\gamma$, $\dot{\tilde{}} = d/d\tilde{t}$, one obtains ($k = 0$)

$$
\begin{cases}
\dot{\tilde{y}} = -3r\beta_I\tilde{H}^2 + (6r\beta_I\beta_V - 3)\tilde{H}\tilde{y} + (2\beta_V - 3r\beta_I\beta_V^2 + \frac{1}{2}r\beta_I)\tilde{y}^2, \\
\dot{\tilde{H}} = (-\frac{3}{2}\gamma - 3r\beta_I\beta_V)\tilde{H}^2 + (3\gamma\beta_V - 4\beta_V + 6r\beta_I\beta_V^2)\tilde{H}\tilde{y}, \\
\qquad + (\frac{1}{4}\,\gamma - \frac{1}{2} - \frac{3}{2}\,\gamma\beta_V^2 + 3\beta_V^2 - 3r\beta_I\beta_V^3 + \frac{1}{2}\,r\beta_V\beta_I)\tilde{y}^2.
\end{cases} \tag{24}
$$

Its phase space is sketched in figure 1. There are two repellers and one attractor, which happen to be straight lines through the origin, corresponding to power-law solutions. All the other solutions transform into one another by a change of scale. The two repellers are

$$
\begin{aligned}
\tilde{H} &= \frac{\beta_V \pm \sqrt{1/6}}{\beta_V \pm 3\sqrt{1/6}}\,\tilde{t}^{-1}, \\
\tilde{y} &= \frac{1}{\beta_V \pm \sqrt{1/6}}\,\tilde{H},
\end{aligned} \tag{25}
$$

and the attractor is

$$
\tilde{H} = \frac{2 - \gamma - 2r\beta_I\beta_V}{3\gamma - \frac{3}{2}\gamma^2 + r^2\beta_I^2 - 2r\beta_I\beta_V}\;\tilde{t}^{-1}\;, \tag{26}
$$

$$
\tilde{y} = \frac{r\beta_I}{r\beta_I\beta_V - 1 + \frac{\gamma}{2}}\;\tilde{H}\;. \tag{27}
$$

These correspond to results in [5] for the special case $\beta_V = \beta_I = \beta,\ \gamma = 1$.

To connect the result (27) to the observations of a time-varying G, let us make the approximation that in the study of a localized system of visible matter (e.g. a binary pulsar) the dark-matter source terms are locally negligible, and contribute only through their imposing a spatially asymptotic value, $\sigma_c(t)$, to the scalar field. As shown e.g. in [4] this leads for the value of the observable gravitational constant to

$$
\tilde{G} = \frac{2\omega_V + 4}{2\omega_V + 3}\,\frac{1}{16\pi\tilde{\phi}_c} = \mathcal{G}(1 + 2\beta_V^2)e^{2\beta_V\sigma_c}. \tag{28}
$$

The reason for the factor $1 + 2\beta_V^2$ in (28) is that in Jordan-Brans-Dicke theory the inverse square law attraction between two static masses is due not only to graviton exchange as in conventional General Relativity but also to the exchange of σ quanta which gives an additional attraction. Thus in our model visible matter might attract visible matter more weakly than invisible matter attracts invisible matter by a factor $\left(1 + 2\beta_V^2\right) / \left(1 + 2\beta_I^2\right)$. Eq. (28) implies for the observable (Jordan-time) variation of Newton's constant

$$
\dot{\tilde{G}}/\tilde{G} = 2\beta_V\tilde{y}\;. \tag{29}
$$

If we assume that the world is now in a state very close (by angle) to the attractor this becomes

$$\dot{\tilde{G}}/\tilde{G} = \frac{2r\beta_V\beta_I}{r\beta_V\beta_I - 1 + \frac{\gamma}{2}}\ \tilde{H}\ , \tag{30}$$

or, if $p_I = 0$ ($\gamma = 1 = r$)

$$\dot{\tilde{G}}/\tilde{G} = -\frac{4\beta_V\beta_I}{1 - 2\beta_V\beta_I}\ \tilde{H}\ = -\frac{4\beta_V\beta_I}{\frac{3}{2}\ + \beta_I^2 - 2\beta_I\beta_V}\ \tilde{t}^{-1}, \tag{31}$$

which generalizes (4). For highly relativistic matter ($\gamma = 4/3$) we find $\dot{\tilde{G}} = 0$, which was to be expected from the fact that σ couples to the trace of the energy-momentum tensor only.

3. Observational Constraints

We know of three independent constraints on the dimensionless coupling constants β_V and β_I of our model.

(i) From radar time delay measurements $|\beta_V|$ is independently constrained to be smaller than 0.032 ($\omega_V > 250,\ 2\sigma$ limit of [10]).

(ii) A firm lower limit on $\tilde{H}_0\tilde{t}_0$, where $\tilde{H}_0$ is the Hubble constant and $\tilde{t}_0$ the age of the Universe, is 0.4 (corresponding to the 2σ lower limits $\tilde{H}_0^{\rm min} = 48$km/s/Mpc, $\tilde{t}_0^{\rm min} = 7.8$Gyr quoted in [11]). For the above small values of β_V equation (26) , with $\gamma = 1 = r$, then constrains $|\beta_I|$ to be smaller than 1.0 nearly independently of β_V. This allows for instance the "string value" $\omega_I = -1$ ($|\beta_I| = 1/\sqrt{2}$).

(iii) The combination of these first two constraints allows $|\dot{\tilde{G}}_0/\tilde{G}_0|$, according to equation (31), to be as large as $0.66\times10^{-11}{\rm yr}^{-1}$ (for the consistently extreme values $|\beta_I|^{\rm max} = 1$, $\tilde{t}_0^{\rm min} = 7.8$Gyr). This upper limit is within the reach of $\dot{G}$ measurements. The latest binary pulsar measurements of $\dot{G}/G$ [1], taken at their 2σ limits, give $-1.4\times10^{-11}{\rm yr}^{-1} < \dot{\tilde{G}}_0/\tilde{G}_0 < 3.8\times10^{-11}{\rm yr}^{-1}$, while the latest solar system 2σ limits [12] give $-2.2\times10^{-11}{\rm yr}^{-1} < \dot{\tilde{G}}_0/\tilde{G}_0 < 1.8\times10^{-11}{\rm yr}^{-1}$ — see figure 2. (Note that the sign of $\dot{\tilde{G}}$ is opposite to the relative sign of β_V and β_I).

We hope that the extended framework for gravitational interactions presented here will give an added motivation for improving the precision of $\dot{G}$ experiments.

Appendix

When transforming to the Jordan frame, $\tilde{g}_{\mu\nu} = \exp(2\beta_V\sigma)g_{\mu\nu}$ (adapted to the visible matter), the action (8) reads

$$S = \int \left(\tilde{\phi}\tilde{R} - \frac{\omega_V}{\tilde{\phi}}\,\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}\right)\sqrt{\tilde{g}}d^4x + $$
$$+ S_V\left[\psi_V, \tilde{g}_{\mu\nu}\right] + S_I\left[\psi_I, (2\kappa^2\tilde{\phi})^{\frac{\beta_V-\beta_I}{\beta_V}}\,\tilde{g}_{\mu\nu}\right],$$

and leads to Jordan-frame field equations of the form

$$\tilde{G}_{\mu\nu} = \frac{1}{2\tilde{\phi}}\left[\tilde{T}^V_{\mu\nu} + \tilde{T}^I_{\mu\nu}\right] + \frac{\omega_V}{\tilde{\phi}^2}\left(\tilde{\phi}_{,\mu}\,\tilde{\phi}_{,\nu} - \frac{1}{2}\,\tilde{g}_{\mu\nu}(\tilde{\nabla}\tilde{\phi})^2\right) + \frac{1}{\tilde{\phi}}\left(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\phi} - \tilde{g}_{\mu\nu}\tilde{\Box}\tilde{\phi}\right) ,$$

$$\tilde{\Box}\tilde{\phi} = \beta_V^2\left[\tilde{T}^V + \frac{\beta_I}{\beta_V}\,\tilde{T}^I\right] ,$$

where $\omega_V + 3/2 \equiv (4\beta_V^2)^{-1}$, where all metric operations are performed with $\tilde{g}_{\mu\nu}$, and where $\tilde{T}^{\mu\nu}_i \equiv 2\tilde{g}^{-1/2}\delta S_i[\psi_i, \tilde{\phi}, \tilde{g}_{\mu\nu}]/\delta\tilde{g}_{\mu\nu}$ denotes the stress-energy tensor of the i-type of matter in the Jordan frame. The latter satisfy (from the invariance of S_V and S_I under coordinate transformations) the conservation laws

$$\tilde{\nabla}_\lambda\tilde{T}^{\lambda\mu}_V = 0 ,$$
$$\tilde{\nabla}_\lambda\tilde{T}^{\lambda\mu}_I = \frac{1}{2}\,\frac{\beta_V - \beta_I}{\beta_V}\,\tilde{T}_I\tilde{\nabla}^\mu \ln\tilde{\phi},$$

and are related to their Einstein-frame counterparts by

$$T^{\mu\nu}_i = e^{6\beta_V\sigma}\tilde{T}^{\mu\nu}_i .$$

Hence the density, pressure and four-velocity (for any type of matter) in the Einstein conformal frame are related to those in the Jordan (atomic) frame by

$$\rho = e^{4\beta_V\sigma}\tilde{\rho} ,\ p = e^{4\beta_V\sigma}\tilde{p},\ u^\mu = e^{\beta_V\sigma}\tilde{u}^\mu .$$

References

[1] T. Damour, G.W. Gibbons and J.H. Taylor, Phys. Rev. Lett. **61** (1988) 1151 ;
J.H. Taylor and J.M. Weisberg, Further Experimental Tests of Relativistic Gravity Using the Binary Pulsar PSR 1913+16, unpublished.

[2] P. Jordan, Nature **164** (1949) 637 ;
Schwerkraft und Weltall, Fried. Vieweg & Sohn, Braunschweig (1955) ;
Z. Phys. **157** (1959) 112.

[3] C. Brans and R. H. Dicke, Phys. Rev. **124** (1961) 925.

[4] C.M. Will, Theory and Experiment in Gravitational Physics, Cambridge University Press (1981).

[5] S. Weinberg, Gravitation and Cosmology, John Wiley (1972).

[6] W. Pauli, quoted in §28 of Jordan's book, ref. [2] ;
M. Fierz, Helv. Phys. Acta **29** (1956) 128 ;
R. H. Dicke, Phys. Rev. **125** (1962) 2163.

[7] J. Scherk and J.H. Schwarz, Nucl. Phys. **B81** (1974) 118.

[8] C. Romero et al., Exact Solutions in Brans-Dicke theory : A Dynamical System Approach, Rio de Janeiro preprint CBPF-NF-045/88.

[9] S.W. Hawking, Commun. Math. Phys. **25** (1972) 167.

[10] R.D. Reasenberg et al. Astrophys. J. **234** (1979) L219.

[11] W.A. Fowler, Quart. J. Roy. Astr. Soc. **28** (1987) 87.

[12] I.I. Shapiro, plenary talk given at the 12th international conference on General Relativity and Gravitation (Boulder, July 1989), unpublished.