Initial-boundary value problems for Einstein’s field equations

GMR Satellite meeting
Southampton, 19 August 2005

Olivier Sarbach

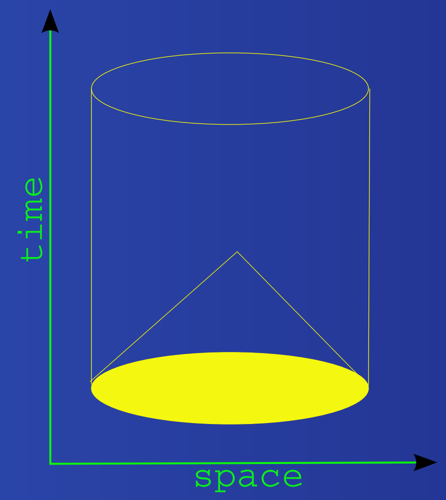
Collaborators: M. Tiglio, O. Reula

Outline

- Introduction
- Experiments
- Model problem
- Concluding remarks

Introduction

Solve Einstein's equations on a spatially compact domain with smooth boundaries.



Boundary conditions should

(i) be compatible with the constraints (constraint-preserving)

(ii) be physically reasonable (e.g. minimize reflections)

(iii) yield a well posed initial-boundary value formulation

Introduction

CALTECH

- A well posed initial-boundary value formulation was given by Friedrich & Nagy, 1999 in terms of a tetrad-based formulation involving the Weyl tensor as a dynamical variable.
- Numerical implementation for related formulations is underway (Reula, Bardeen, Buchman, S,...)
- For metric-based approaches, only partial results are available (reflection symmetry, linearization about Minkowski space).
- Relevant for: Outer/interface boundary conditions; Cauchy characteristic/perturbative matching, constraint projection, elliptic gauge conditions, boundary of a star,...

Introduction

CALTECH

The IBVF for Einstein's equations

- Possible to write Einstein evolution equations in first order symmetric hyperbolic (FOSH) form $\dot{u} = \mathcal{A}(u)u + \mathcal{F}(u)$, where $\mathcal{A}(u) = A^i(u)\partial_i = -\mathcal{A}(u)^\dagger$.

Introduction

The IBVF for Einstein's equations

- Possible to write Einstein evolution equations in first order symmetric hyperbolic (FOSH) form $\dot{u} = \mathcal{A}(u)u + \mathcal{F}(u)$, where $\mathcal{A}(u) = A^i(u)\partial_i = -\mathcal{A}(u)^\dagger$.
- For such systems the specification of maximally dissipative boundary conditions ($u_{in} = S u_{out} + data$) leads to well posed initial-boundary value problems.

Introduction

The IBVF for Einstein's equations

- Possible to write Einstein evolution equations in first order symmetric hyperbolic (FOSH) form
$\dot{u} = \mathcal{A}(u)u + \mathcal{F}(u)$, where $\mathcal{A}(u) = A^i(u)\partial_i = -\mathcal{A}(u)^\dagger$.
- For such systems the specification of maximally dissipative boundary conditions ($u_{in} = Su_{out} + data$) leads to well posed initial-boundary value problems.
- Standard discretization techniques which guarantee numerical convergence of the linearized system.

Introduction

CALTECH

The IBVF for Einstein's equations

- Possible to write Einstein evolution equations in first order symmetric hyperbolic (FOSH) form $\dot{u} = \mathcal{A}(u)u + \mathcal{F}(u)$, where $\mathcal{A}(u) = A^i(u)\partial_i = -\mathcal{A}(u)^\dagger$.
- For such systems the specification of maximally dissipative boundary conditions ($u_{in} = S u_{out} + data$) leads to well posed initial-boundary value problems.

Introduction

CALTECH

The IBVF for Einstein's equations

- Possible to write Einstein evolution equations in first order symmetric hyperbolic (FOSH) form
  $\dot{u} = \mathcal{A}(u)u + \mathcal{F}(u)$, where $\mathcal{A}(u) = A^i(u)\partial_i = -\mathcal{A}(u)^\dagger$.
- For such systems the specification of maximally dissipative boundary conditions ($u_{in} = Su_{out} + data$) leads to well posed initial-boundary value problems.
- Differential constraints:
  $C \equiv L^i(u)\partial_i u + B(u) = 0$. Constraint variables C satisfy a linear evolution system. If this system is FOSH, the specification of homogeneous maximally dissipative boundary conditions for this system leads to constraint preservation.

Introduction

The IBVF for Einstein's equations

- Possible to write Einstein evolution equations in first order symmetric hyperbolic (FOSH) form $\dot{u} = \mathcal{A}(u)u + \mathcal{F}(u)$, where $\mathcal{A}(u) = A^i(u)\partial_i = -\mathcal{A}(u)^\dagger$.
- For such systems the specification of maximally dissipative boundary conditions ($u_{in} = S u_{out} + data$) leads to well posed initial-boundary value problems.
- Differential constraints: $C \equiv L^i(u)\partial_i u + B(u) = 0$. Constraint variables C satisfy a linear evolution system. If this system is FOSH, the specification of homogeneous maximally dissipative boundary conditions for this system leads to constraint preservation.
- FOSH system with *differential* boundary conditions.

Experiments

CALTECH

Einstein evolution equations (Einstein-Christoffel formulation)
(Frittelli & Reula, Anderson & York, ...)

$$\begin{aligned}
\pounds_n \alpha &= -\alpha K, \\
\pounds_n g_{ij} &= -2K_{ij} \ , \\
\pounds_n K_{ij} &= \frac{1}{2} g^{ab} \left(-\partial_a d_{bij} + 2\partial_{(i} d_{|ab|j)} - \partial_{(i} d_{j)ab} - 2\partial_{(i} A_{j)}\right) + \gamma\, g_{ij} H + \text{l.o} \\
\pounds_n d_{kij} &= -2\partial_k K_{ij} + \eta\, g_{k(i} M_{j)} + \chi\, g_{ij} M_k + \text{l.o.} \\
\pounds_n A_i &= -K A_i - g^{ab} \partial_i K_{ab} + \xi M_i + \text{l.o.}
\end{aligned}$$

with some parameters γ, η, χ ξ.
Constraints: $H = 0$, $M_j = 0$ (Hamiltonian and momentum),
$d_{kij} = \partial_k g_{ij}$, $A_i = \partial_i \alpha / \alpha$.

Experiments

Boundary conditions:

- Three constraint-preserving (differential) boundary conditions.
- One boundary condition that is related to a gauge degree of freedom.
- Two boundary conditions that control the “physical” degrees of freedom.

Experiments

CALTECH

Boundary conditions:

- Three constraint-preserving (differential) boundary conditions.
- One boundary condition that is related to a gauge degree of freedom.
- Two boundary conditions that control the “physical” degrees of freedom.

Numerical implementation in spherically symmetry with scalar field seems stable (Calabrese, Lehner, Tiglio, PRD **65**, 104031 (2002)).

Experiments

Boundary conditions:

- Three constraint-preserving (differential) boundary conditions.
- One boundary condition that is related to a gauge degree of freedom.
- Two boundary conditions that control the “physical” degrees of freedom.

Numerical implementation in spherically symmetry with scalar field seems stable (Calabrese, Lehner, Tiglio, PRD **65**, 104031 (2002)).
Also used for the numerical evolution of bubble spacetimes (Lehner & S).

Experiments

Consider linear hyperbolic system with constant coefficients (high-frequency limit),

$$\partial_t u = \mathcal{A}u,\ t > 0,\ x > 0,$$

where $\mathcal{A}u \equiv A^x \partial_x u + A^y \partial_y u + A^z \partial_z u$ with differential boundary conditions

$$M(\partial_x, \partial_y, \partial_z)u = h(t, y, z).$$

Look for solutions of the form $u(t, x, y, z) = e^{st+i(w_y y + w_z z)} f(x)$, where $Re(s) > 0$, w_y, w_z real.

Test: If $h = 0$ there should be no such solutions. Otherwise the system is ill posed: Because if there is such a solution for some s, $Re(s) > 0$, then there is also a solution u_α for αs, $\alpha > 0$ and for each fixed t

$$|u_\alpha(t, x, y, z)| / |u_\alpha(0, x, y, z)| = e^{\alpha Re(s) t} \to \infty.$$

(i.e. the operator $s - \mathcal{A}$ is not invertible for all $Re(s) > 0$.)

Experiments

- This leads to a determinant condition.

Experiments

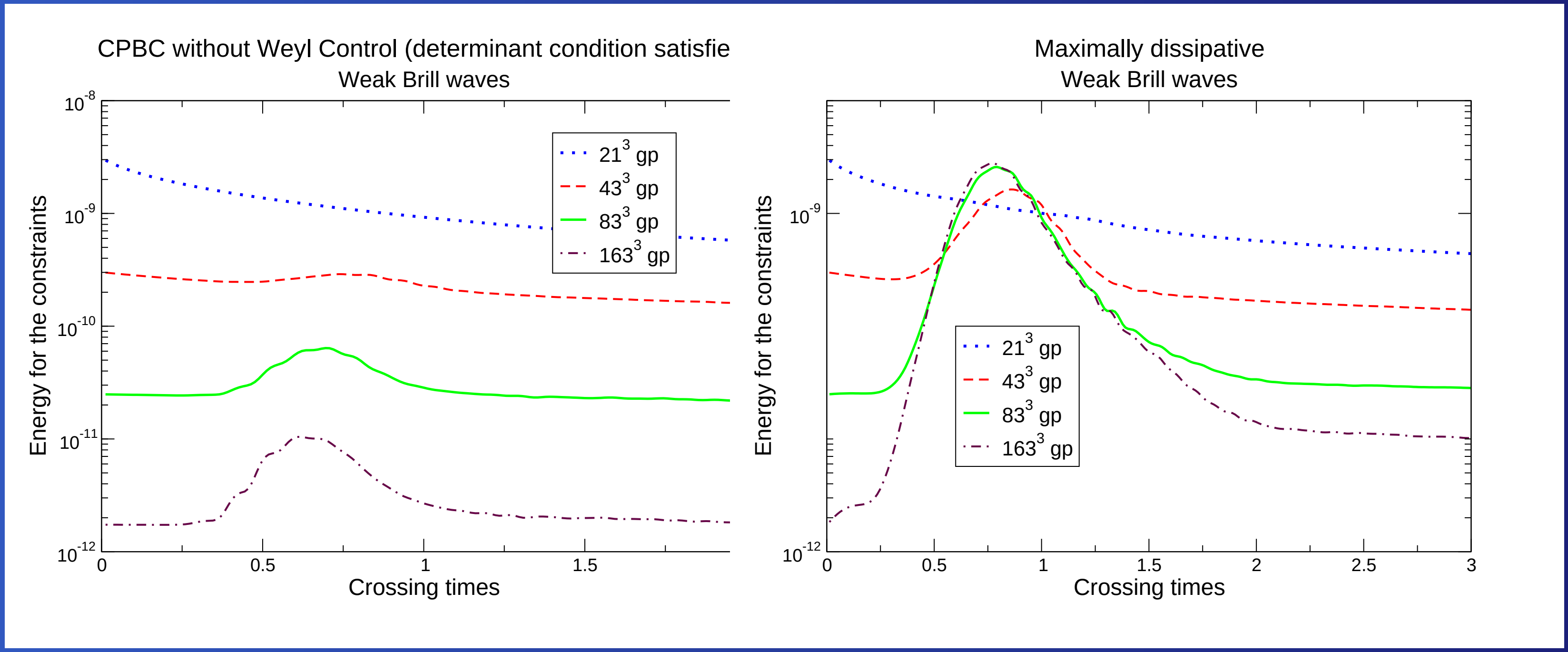
- This leads to a determinant condition.
- Very effective in ruling out "candidate" constraint-preserving boundary conditions (Calabrese, OS, *J. Math. Phys.* **44**, 3888 (2003)).

Experiments

- This leads to a determinant condition.
- Very effective in ruling out "candidate" constraint-preserving boundary conditions (Calabrese, OS, *J. Math. Phys.* **44**, 3888 (2003)).
- Ill posed modes have non-trivial dependency in the directions tangential to the boundary.

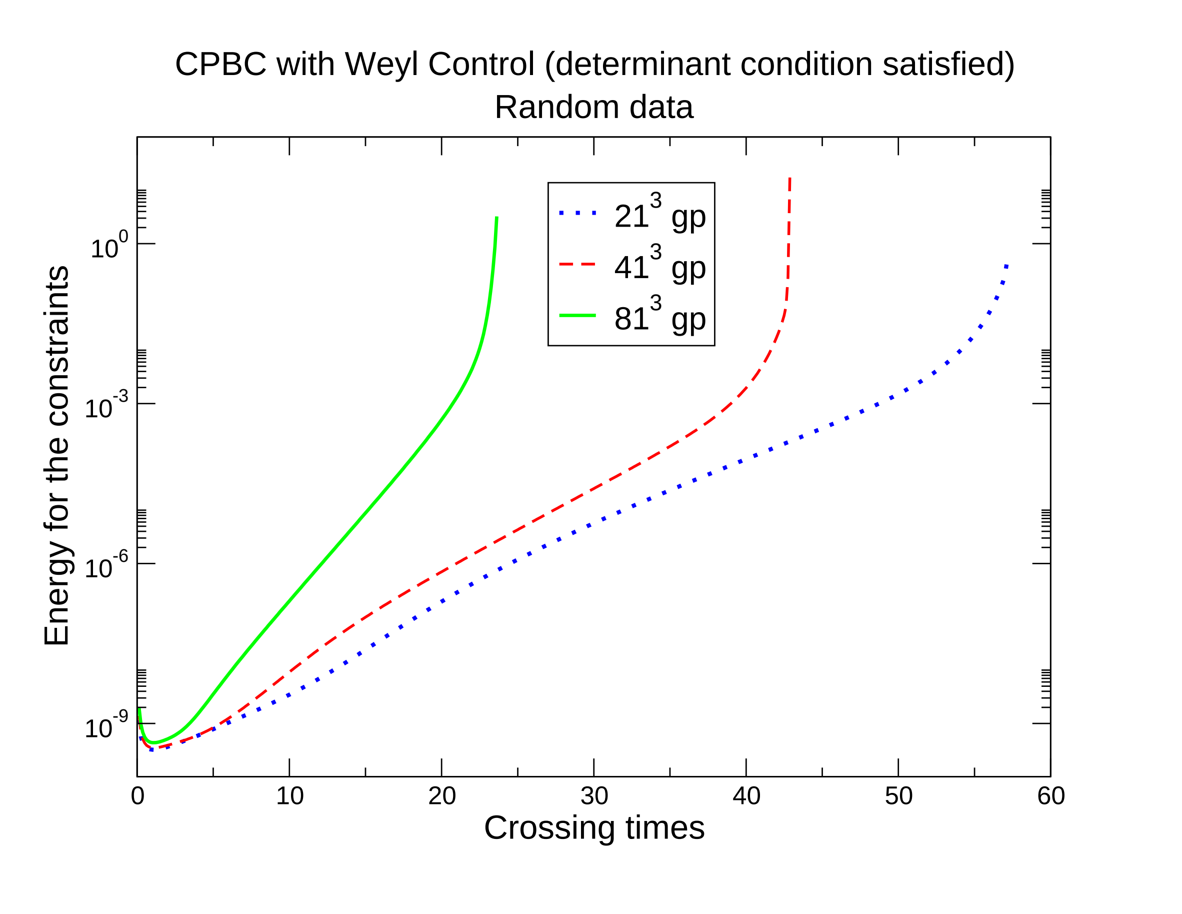
Experiments

3D Brill wave evolutions (OS and M. Tiglio, gr-qc/0412115) with and without constraint-preserving boundary conditions.



similar results by Kidder, Lindblom, Scheel, Pfeiffer, Phys.Rev.D71, 064020 (2005).

Experiments



OS and M. Tiglio, gr-qc/0412115

Model problem

Model problem for the Einstein-Christoffel type of formulations of Einstein's field equations ($u = (\phi, A_i, E_j, W_{ij}) \leftrightarrow (\beta_i, g_{ij}, K_{ij}, \Gamma_{kij})$)

$$
\begin{aligned}
\partial_t A_i &= E_i + \nabla_i \phi, \\
\partial_t E_j &= \nabla^i (W_{ij} - W_{ji}) + \alpha C_j, \\
\partial_t W_{ij} &= \nabla_i E_j + \nabla_i \nabla_j \phi + \frac{\beta}{2} \delta_{ij} C,
\end{aligned}
$$

with constraints $C \equiv -\nabla^k E_k = 0$, $C_j \equiv \delta^{kl} (\nabla_j W_{kl} - \nabla_k W_{jl}) = 0$.
Constraints propagate according to

$$
\begin{aligned}
\partial_t C &= -\alpha \nabla^j C_j \ , \\
\partial_t C_j &= -\beta \nabla_j C.
\end{aligned}
$$

Strongly hyperbolic if $\alpha\beta > 0$.

Model problem

Cauchy problem well posed in $L^2(\mathbb{R}^3)$ if we adopt, for example, the temporal gauge $\phi = 0$.

Model problem

CALTECH

Next, consider the case where we want to solve the equations on an open subset $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$.

- Constraint preservation:

$$\partial_t C = -\alpha \nabla^j C_j \ ,$$
$$\partial_t C_j = -\beta \nabla_j C. \qquad C = \frac{\sqrt{\alpha\beta}}{\beta} n^k C_k \ ,$$

where n denotes the unit outward normal to the boundary.

Model problem

Next, consider the case where we want to solve the equations on an open subset $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$.

- Constraint preservation:

$$\partial_t C = -\alpha \nabla^j C_j \ ,$$
$$\partial_t C_j = -\beta \nabla_j C. \qquad C = \frac{\sqrt{\alpha\beta}}{\beta} n^k C_k \ ,$$

  where n denotes the unit outward normal to the boundary.

- Nonincreasing of total energy flux through the boundary:

$$\mathbf{E}_{||} + (W_{n||} - W_{||n}) = d\left[\mathbf{E}_{||} - (W_{n||} - W_{||n})\right] + h_{||},$$

  where $|d| < 1$ and $h_{||}$ is some boundary data (controls normal component of Poynting vector).

Model problem

Choose the gauge condition $\phi = 0$ (temporal gauge $\leftrightarrow$ fixed shift).

- Is the resulting problem well posed in L^2 ($u = (A_i, E_j, W_{ij})$)? In particular, are there constants $a > 0$ and $b \in \mathbb{R}$ such that

$$\|u(t,.)\|_{L^2(\Omega)} \leq ae^{bt}\left[\|u_0\|_{L^2(\Omega)} + \int_0^t \|h(s)\|_{L^2(\partial\Omega)}\, ds\right].$$

  for solutions with initial data $u(t=0) = u_0$ and boundary data h?

Model problem

Choose the gauge condition $\phi = 0$ (temporal gauge $\leftrightarrow$ fixed shift).

- Is the resulting problem well posed in L^2 ($u = (A_i, E_j, W_{ij})$)? In particular, are there constants $a > 0$ and $b \in \mathbb{R}$ such that

$$\|u(t,.)\|_{L^2(\Omega)} \leq a e^{bt} \left[\|u_0\|_{L^2(\Omega)} + \int_0^t \|h(s)\|_{L^2(\partial\Omega)} \, ds \right].$$

  for solutions with initial data $u(t=0) = u_0$ and boundary data h?

- Let f be a smooth, time-independent, harmonic function and set

$$A_i = t \nabla_i f, \qquad E_j = \nabla_i f, \qquad W_{ij} = t \nabla_i \nabla_j f.$$

  Evolution and constraint equations are satisfied. Initial and boundary data only depend on first derivatives of f whereas the solution depends on second derivatives of f.

Model problem

- Physically, these solutions represent an electrostatic solution with nontrivial electric charge density at the boundary. They are represented in a “bad” gauge because we adopted the temporal gauge.

Model problem

- Physically, these solutions represent an electrostatic solution with nontrivial electric charge density at the boundary. They are represented in a “bad” gauge because we adopted the temporal gauge.
- In asymptotically flat space, these solutions do not exist.

Model problem

- Physically, these solutions represent an electrostatic solution with nontrivial electric charge density at the boundary. They are represented in a "bad" gauge because we adopted the temporal gauge.
- In asymptotically flat space, these solutions do not exist.
- This motivates the following gauge condition:

$$\Delta\phi = -\nabla^k E_k, \qquad \text{with boundary condition } \partial_n \phi = -E_n \; .$$

In this gauge, the above solutions are $\phi = -f + const$, $A_i = 0$.

Model problem

CALTECH

- Physically, these solutions represent an electrostatic solution with nontrivial electric charge density at the boundary. They are represented in a “bad” gauge because we adopted the temporal gauge.
- In asymptotically flat space, these solutions do not exist.
- This motivates the following gauge condition:

$$\Delta\phi = -\nabla^k E_k, \qquad \text{with boundary condition } \partial_n\phi = -E_n \ .$$

  In this gauge, the above solutions are $\phi = -f + const$, $A_i = 0$.
- We are allowed to set $n^k A_k = 0$ at the boundary.

Model problem

CALTECH

Theorem (Reula & S, JHDE, Vol. 2, 2005)

"The resulting initial-boundary value problem is well posed in a Hilbert space that controls the L^2 norm of the main variables *and* the constraint variables.

Furthermore, solutions satisfying the constraints initially automatically satisfy the constraints at later times."

Model problem

CALTECH

Idea of the proof:

- Estimate L^2 norm $\mathcal{E}_{cons}$ of constraint variables.

Model problem

Idea of the proof:

- Estimate L^2 norm $\mathcal{E}_{cons}$ of constraint variables.
- Estimate the *physical* energy,

$$\mathcal{E}_{phys} = \frac{1}{2}\int_\Omega \left(E^j E_j + \frac{1}{2} F^{ij} F_{ij} \right) d^3x, \qquad F_{ij} = W_{ij} - W_{ji} \;:$$

$$\dot{\mathcal{E}}_{phys} = \alpha \int_\Omega E^j C_j d^3x \leq const(\mathcal{E}_{phys} + \mathcal{E}_{cons}).$$

Model problem

CALTECH

Idea of the proof:

- Estimate L^2 norm $\mathcal{E}_{cons}$ of constraint variables.
- Estimate the *physical* energy,

$$\mathcal{E}_{phys} = \frac{1}{2}\int_\Omega \left(E^j E_j + \frac{1}{2} F^{ij} F_{ij} \right) d^3x, \qquad F_{ij} = W_{ij} - W_{ji} \ :$$

$$\dot{\mathcal{E}}_{phys} = \alpha \int_\Omega E^j C_j d^3x \le const(\mathcal{E}_{phys} + \mathcal{E}_{cons}).$$

- Estimate the symmetric part of W_{ij} using the constraints, the boundary condition $n^i A_i = 0$ and the inequality

$$\int_\Omega \nabla^i A^j \cdot \nabla_i A_j d^3x \le \int_\Omega \left[2\nabla^{[i} A^{j]} \cdot \nabla_{[i} A_{j]} + (\nabla_i A^i)^2 \right] d^3x.$$

Model problem

Idea of the proof:

- For existence, rewrite the problem as an abstract Cauchy problem

$$\frac{d}{dt}u(t) = \mathcal{A}u(t), \qquad u(0) = u_0 \in H,$$

where $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is a densely-defined linear operator on a Hilbert space.

Model problem

Idea of the proof:

- For existence, rewrite the problem as an abstract Cauchy problem

$$\frac{d}{dt}u(t) = \mathcal{A}u(t), \qquad u(0) = u_0 \in H,$$

  where $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is a densely-defined linear operator on a Hilbert space.

- Show that this operator (or its closure) defines a strongly continuous semigroup $P(t) = \exp(t\mathcal{A})$ on H.

Concluding remarks

CALTECH

- IBVP for metric based formulations of Einstein's equations not yet understood Friedrich & Nagy solved the problem for a formulation based on tetrads and the Weyl tensor.
- One can propose constraint-preserving boundary conditions for hyperbolic formulations of Einstein's equations and perform analytic (determinant condition) and numerical tests.
- Determinant condition is not sufficient.
- Model problem shows: Careful with the gauge choice near the boundary. Problem solved with elliptic gauge condition and use of "physical energy"; symmetrizer energy irrelevant.
- Ongoing work with G. Nagy for the IBVP for Einstein's equations, linearized about stationary solutions.