The two-body problem in 2+1 dimensions

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BTZ solution

Bañados-Teitelboim-Zanelli 1992: axistationary ansatz gives

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2(d\phi + \beta dt)^2,$$

$$f(r) = -M + (-\Lambda)r^2 + \frac{J^2}{4r^2}, \qquad \beta(r) = -\frac{J}{2r^2}$$

Set $\Lambda = -1$ from now on

- $M = -1$, $J = 0$ is adS3
- Solutions with $M > 0$, $|J| < M$ are black holes
- Solutions with $-1 < M < 0$, $|J| < M$ are point particles
- Also overspinning solutions $|J| > |M$ (not well understood)
- Locally spacetime is always adS3 (no Weyl curvature)

Black holes

Solutions with $M > 0$, $|J| < M$ resemble 3+1 Kerr

- Event horizon at $r = r_+$ [larger zero of $f(r)$]
- For $J = 0$, $r = 0$ is spacelike
- For $J \neq 0$, Cauchy horizon at $r = r_-$, $r = 0$ is timelike

But if spacetime is locally adS3, how is $r = 0$ a singularity?

- Spacetime can be analytically continued beyond $r = 0$ but then contains closed timelike curves (CTCs)
- Matter will generically turn $r = 0$ into a curvature singularity

Point particles

- Global adS3 is given by

$$ds^2 = -\cosh^2\chi\, dt^2 + \underbrace{d\chi^2 + \sinh^2\chi\, d\phi^2}_{\text{hyperbolic 2-space}}, \quad (t,\chi,\phi) \sim (t,\chi,\phi+2\pi)$$

- Spinning point particle has defect angle and time shift

$$(t,\chi,\phi) \sim (t+\Delta t,\chi,\phi+2\pi-2\nu)$$

- Where is the (spinning) particle? Two interpretations
  - point particle at $\chi = 0$ and CTCs (Deser-Jackiw-t'Hooft 1984)
  - brane at $r = 0$ and no CTCs (Mišković-Zanelli 2009)

Black hole exteriors

- Global adS3 is also given by

$$ds^2 = -\sinh^2\chi\, dt^2 + \underbrace{d\chi^2 + \cosh^2\chi\, d\phi^2}_{\text{hyperbolic 2-space again}}$$

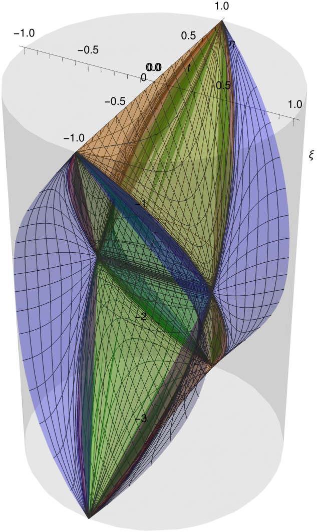
with $-\infty < \phi < \infty$ and $-\infty < \chi < \infty$

- Black hole exterior by identifying

$$(t, \chi, \phi) \sim (t + \Delta t, \chi, \phi + \Delta\phi)$$

- $\chi = 0$ is the bifurcation point of the Killing horizon

Non-spinning Kruskal BH as an identification of adS3



- Grey cylinder: global adS3
- Blue: exterior identification surfaces
- Orange: interior identification surfaces
- Green: bifurcate horizon
- Top and bottom edges: $r = 0$ singularity

BTZ as an identification of adS3: group theory approach

- Instead of cutting and pasting in coordinates, treat the identification as an abstract isometry of adS3
- adS3 restricted to $0 \leq t < 2\pi$ is the hypersurface

$$-x_3^2 - x_0^2 + x_1^2 + x_2^2 = -1$$

in $\mathbb{R}^{2,2}$ equipped with the metric induced by

$$ds^2 = -dx_3^2 - dx_0^2 + dx_1^2 + dx_2^2$$

- Spacetime points $\mathbf{x} \in SL(2,\mathbb{R})$
- Isometries $(u, v) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$ act by matrix conjugation

$$\mathbf{x} \sim u^{-1}\mathbf{x}v$$

Group theory approach to multi-object spacetimes

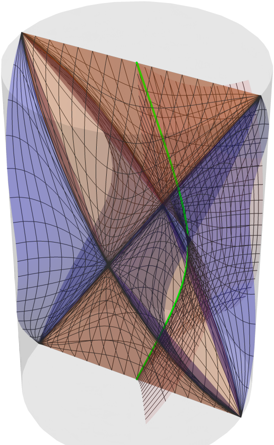
- (u_1, v_1) and (u_2, v_2) represent holonomies around each of two objects, or the isometries near them
- (g, h) represents another isometry ("gauge transformation"), and can be used to move objects
- $(u_1 u_2, v_1 v_2)$ represents the holonomy going around both, or common isometry seen from further away
- $(u_2 u_1, v_2 v_1)$ represents the same two-object spacetime in a different $SL(2, \mathbb{R})$ gauge
- M and J are given by traces of u and v (or their products), and are (the only) $SL(2, \mathbb{R})$ gauge-invariants

The programme: solve the two-body problem

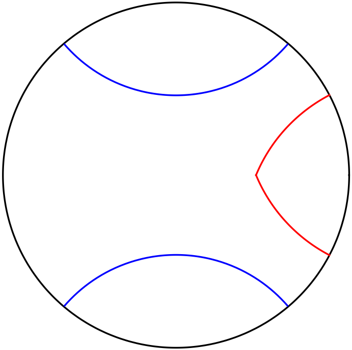
- Represent two objects with given masses M_i, spins J_i, two orbital parameters by their $SL(2,\mathbb{R})$ generators
- If their product generators have $M > 0$ and $|J| < 0$, these objects will form a black hole in their time evolution
- Otherwise they are on an eternal elliptic orbit (two massive particles) or fly-by (one or both particles massless), or space closes
- The exterior of any compact object is BTZ, so if we can do particles and black holes we can do any compact objects
- But the group-theoretical shortcut can be misleading: we need to construct a fundamental domain to show that the relevant infinities exist. This is hard!

Fundamental domain: particle and black hole, headon, nonspinning

Spacetime picture



The moment of time symmetry



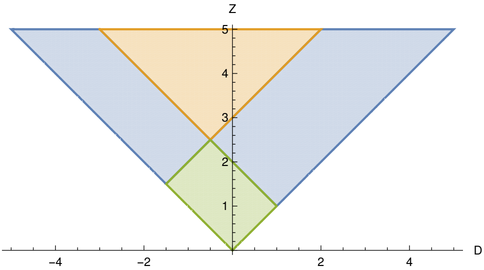
Previous work

- Effective spinning particle from two non-spinning ones in $\Lambda = 0$ (Deser-Jackiw-t'Hooft 1984, Louko-Matschull 2000)
- Spinning black hole from the collision of two non-spinning massless particles with sufficiently large energy and sufficiently small impact parameter (Holst-Matschull 1999)
- Two or more massive particles colliding at a single point (Lindgren 2016)
- Time-symmetric initial date for black holes... (Brill 1996)
- ...and for BH-BH, BH-PP, PP-PP (Steif 1996)

My results so far

- Total M and J for two objects, which can be spinning black hole, spinning massive particle, spinning massless particle
- Characterisation of the rest frame and centre of mass
- Arbitrary (signed) impact parameter D and relative rapidity Z in rest frame at moment of closest approach
- For eternal particle orbits, D is the apogee distance and Z the perigee proper distance in the rest frame
- Preliminary classification of final state as black hole, space closes, fly-by or eternal binary (in progress)
- There are indications that eternal black hole binaries and particle-black hole binaries exist

Outcomes for two massive spinning particles



D: signed impact parameter

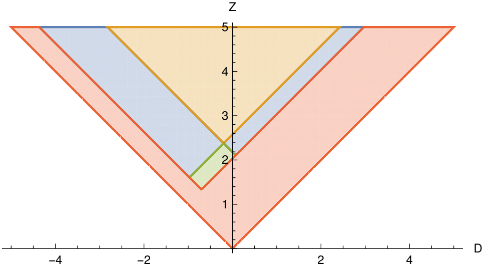
Z: relative rapidity at closest approach

Green: effective spinning particle (eternal binary) or space closes

Blue: overspinning (eternal binary)

Orange: black hole

Outcomes for two spinning black holes



D: signed impact parameter

Z: relative rapidity at closest approach

Green: effective spinning particle (eternal binary?) or space closes

Blue: overspinning (eternal binary?)

Orange: (single) black hole

Red: no solution