Critical phenomena in gravitational collapse

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Bouncing Cosmologies, Sesto, 2 July 2019

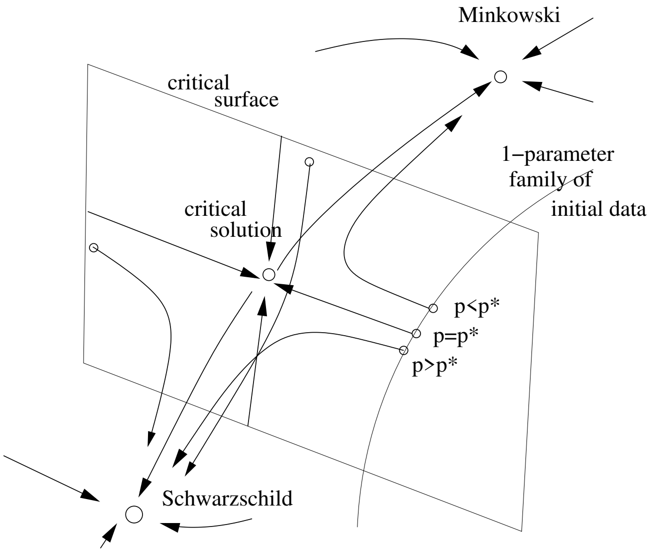
Weak and strong data

- Take regular, asymptotically flat initial data for GR with matter or in vacuum and evolve in time to see what happens
- For weak data, the solution stays regular (Christodoulou-Klainermann '93 for vacuum GR, Christodoulou '93 for spherical Einstein-scalar)
- For strong data, a black hole is formed (Christodoulou '87 for spherical Einstein-scalar)
- Probe the space of initial data with 1-parameter families spanning weak and strong data
- Typically, there is a single p_* such a black hole forms if and only if $p > p_*$

The black hole threshold

- Think of the evolution of initial data sets in time (in some gauge choice) as a dynamical system
- If the black hole threshold really is a hypersurface then it must be a dynamical system in its own right
- If it has attractive fixed points, these are critical points of the full system ("critical solution")
- Static/stationary/time-periodic: "type I" critical phenomena
- Continuously/discretely self-similar: "type II"
- Type II critical collapse naturally generates large curvature as $|p - p_*| \to 0$, **naked singularities** in the limit $p = p_*$
- Beautiful, too

Phase space picture



Continuous and discrete self-similarity

CSS:

$$\mathcal{L}_\xi \, g_{ab} = -2\, g_{ab}$$

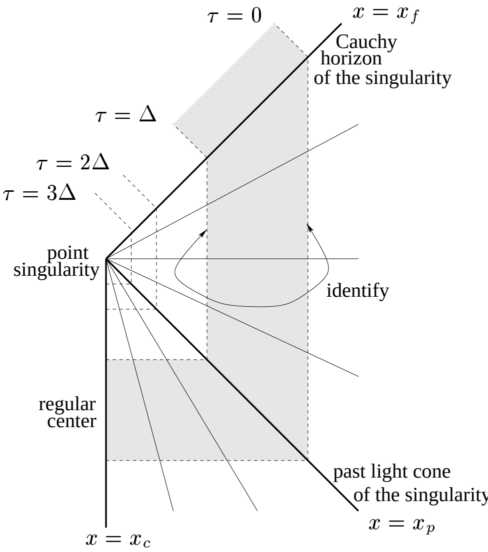
CSS in adapted coordinates $x^\mu := (\tau, x^i)$:

$$\xi = \frac{\partial}{\partial \tau}, \qquad g_{\mu\nu} = e^{-2\tau} \bar{g}_{\mu\nu}$$

$$\bar{g}_{\mu\nu}(\tau, x^i) = \bar{g}_{\mu\nu}(x^i)$$

DSS in adapted coordinates:

$$\bar{g}_{\mu\nu}(\tau + \Delta, x^i) = \bar{g}_{\mu\nu}(\tau, x^i)$$



Mass and curvature scaling

Near-critical initial data, intermediate phase of evolution:

$$\begin{aligned}
\phi(x,\tau) &\simeq \phi_*(x) + \sum_{i=0}^{\infty} C_i(p)\, e^{\lambda_i \tau}\, \phi_i(x) \\
&\simeq \phi_*(x) + \frac{dC_0}{dp}(p_*)\,(p-p_*)\, e^{\lambda_0 \tau}\, \phi_0(x) \\
&\simeq \phi_*(x) + (\text{some constant})\, \phi_0(x) \qquad \text{when AH forms}
\end{aligned}$$

This happens at some τ defined by

$$(p-p_*)\, e^{\lambda_0 \tau_\sharp} \simeq (\text{some constant})$$

Because $ds^2 = e^{-2\tau} g_{\mu\nu}(x)\, dx^\mu\, dx^\nu$,

$$M_{\rm AH}(p) \sim e^{-\tau_\sharp(p)} \sim |p-p_*|^{1/\lambda_0}, \qquad \mathrm{Ric}_{\max}(p) \sim |p-p_*|^{-2/\lambda_0}$$

Spherical symmetry example

Massless scalar field matter, polar-radial coordinates (Choptuik '93)

$$ds^2 = -\alpha^2(t,r)\,dt^2 + a^2(t,r)\,dr^2 + r^2\,d\Omega^2$$

DSS-adapted coordinates based on this, for example

$$x := \frac{r}{t_* - t}, \qquad \tau := -\ln(t_* - t)$$

Metric becomes

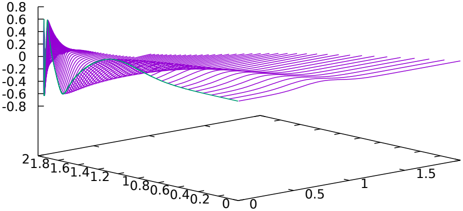
$$ds^2 = e^{-2\tau}\left[-\alpha^2\,d\tau^2 + a^2\,(dx - x\,d\tau)^2 + x^2\,d\Omega^2\right]$$

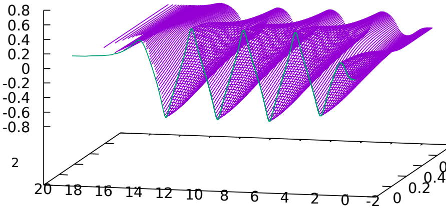
Spacetime is DSS if and only if $Z := (a, \alpha, \phi)$ obey

$$Z(x, \tau + \Delta) = Z(x, \tau) \quad \Leftrightarrow \quad Z(e^{-\Delta} t, e^{-\Delta} r) = Z(t, r)$$

Choptuik solution

- Numerical time evolution with $1 - p/p_* \simeq 10^{-15}$ (method of Garfinkle '95):





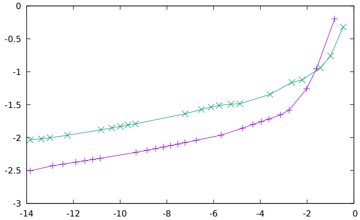
- Critical solution and Δ from a nonlinear boundary value problem in (x, τ) between regular centre and past lightcone (CG '95)
- Perturbation spectrum and γ from a linear boundary value problem (CG '97)

More spherical symmetry

- Massless real scalar field: DSS with $\Delta \simeq 3.44$, $\gamma = 0.37$
- Perfect fluid with $P = k\rho$: CSS with γ depending on k
- $SU(2)$ Yang-Mills: DSS with $\Delta \simeq 0.6$ and $\gamma \simeq 0.2$
- Massive real scalar field: both type II (mass becomes irrelevant on small scales) and type I (critical solution is unstable real boson star)
- Scalar electrodynamics: Choptuik, with a separate scaling law for black hole charge
- Real scalar field in $d > 4$: DSS with $\Delta(d)$
- Real scalar field in $d = 2 + 1$ with $\Lambda < 0$: asymptotically CSS
- Squashed spheres in $d = 4 + 1$: DSS
- Other fields, two competing fields, more than two possible outcomes, ...

2+1 dimensions with $\Lambda < 0$

- Λ should become irrelevant on small scales, when $|Ric| \gg \Lambda$
- But without it, there are no black hole solutions
- Spherically symmetric scalar field
  - Time evolutions (Pretorius-Choptuik '00, Jałmużna-CG-Chmaj '15): CSS, AH mass and curvature scaling
  - Candidate critical solution (Garfinkle '01) correct only inside lightcone, and has 3 growing perturbations (CG-Garfinkle '02)
  - Proposed resolution gives $\gamma = 1/\lambda_0$, $\gamma_M = 2/(2+\lambda_0)$ (Jałmużna-CG-Chmaj '15)
- Perfect fluid $P = k\rho$: CSS, $\gamma_M = 2\gamma$? (CG-Davey-Bourg '19)



Let's pause and take stock

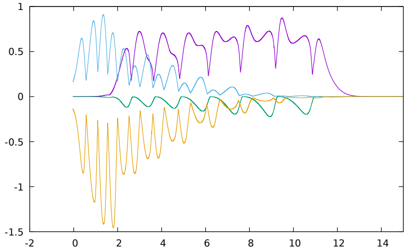
- Are critical phenomena in gravitational collapse a feature of spherical symmetry?
- Or do they occur only for special types of matter?
- What about angular momentum?
- Do we really get naked singularities at the threshold?
- Why self-similarity? Why DSS?
- What if the field equations are not scale-invariant?

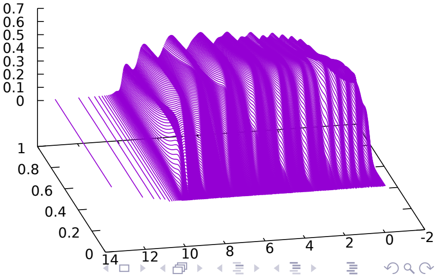
What if the field equations are not scale-invariant?

- In general, we will get type I for some initial data (the critical solution is an unstable "star", with mass fixed by the field equations) and type II for others
- For type II, write the field equations in DSS-compatible coordinates and dependent variables. Dimensionful constants K in the field equations appearing as $Ke^{-n\tau}$ typically become irrelevant at small scale/large curvature ($\tau \to \infty$)
  - Mass and self-interaction potential of a scalar field
  - Yang-Mills self-interaction
  - Coupling between matter and EM or YM
  - Particle rest mass in perfect fluid
  - Λ in 3+1 and higher, but not in 2+1
- We have a notion of universality classes

Why self-similarity? Why DSS?

- Many examples where continuous similarity solutions are attractors/intermediate attractors in nonlinear time evolution PDEs (Eggers-Fontelos '09), but non-trivial DSS **only in GR**
- Rodniansky-Shlapentokh-Rothman '18: CSS approached asymptotically just to the **future** of the past lightcone of the singularity, Hawking mass zero on lightcone
- DSS (only) where CSS is not possible, e.g. scalar field?
- DSS does not even have to be periodic, e.g. massless scalar gradually replacing Yang-Mills:



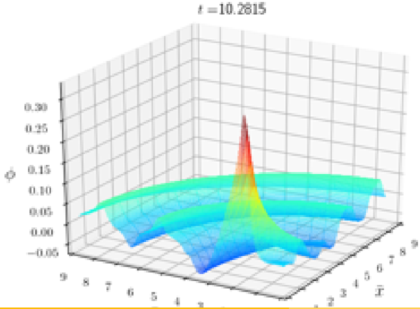


Do we really get naked singularities at the threshold?

- Numerical simulations out to future null infinity of near-subcritical initial data (Hamadé-Stewart '95)
- Continuation of Choptuik solution to Cauchy horizon: scalar field C^ϵ there, regular null data exist (Martín-García-CG '03)
- To do: Continuation of linear perturbation modes
- Proof of naked singularity formation (via CSS, but only C^0 at past lightcone of singularity) (Christodoulou '94)
- Proof of instability (i.e. some codimension) of naked singularities (including the Choptuik solution, but with the instability only C^0 at past lightcone of singularity) (Christodoulou '99)
- Proof that the Choptuik solution exists (from regular centre to beyond past lightcone) as a real **analytic** solution (Reiterer-Trubowitz '19)

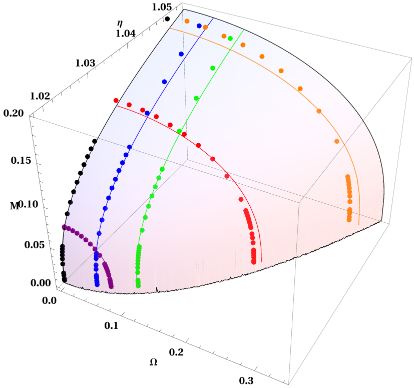
Axisymmetric scalar field

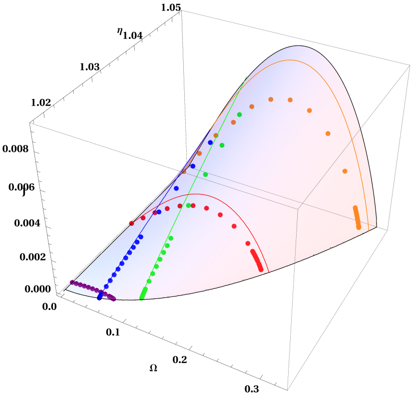
- All nonspherical perturbations of Choptuik decay (Martín-García-CG '99)...
- ... but in axisymmetric time evolutions Choptuik solution splits into two centres (Choptuik-Hirschmann-Liebling-Pretorius '03)
- This is a nonlinear instability, small perturbations **do** decay (Baumgarte '18)



Rotating perfect fluids with $P = k\rho$

- For $k > 1/9$ all nonspherical perturbations of CSS spherical critical solution decay. Prediction of critical exponent for angular momentum... (CG '02)
- ...agrees with time evolutions (Baumgarte-CG '16):





Universal scaling functions

- For $k < 1/9$ one $l = 1$ angular momentum perturbation is also unstable (CG '02)

$$Z \simeq Z_*(x) + P(p,q)e^{\lambda_0\tau} Z_0(x) + Q(p,q)e^{\lambda_0\tau} Z_0(x)$$

For 2-parameter families of initial data such that

$$J(p,-q) = -J(p,q), \quad M(p,-q) = M(p,q)$$

we must have

$$P \sim p - p_{*0} - Kq^2, \qquad Q \sim q$$

When the growing perturbations become nonlinear, $\tau = \tau_\sharp$ fixes the overall length scale, but M and J also depend on

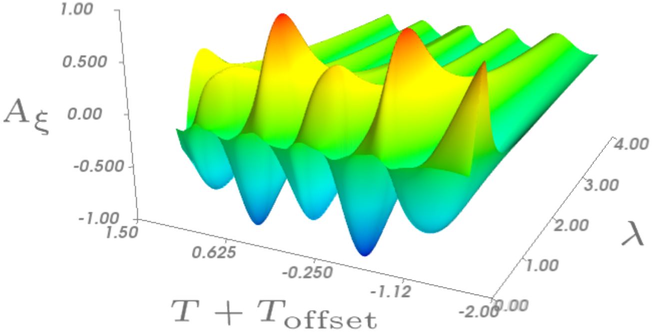
$$\delta := Q|P|^{-\frac{\lambda_1}{\lambda_0}}$$

for example $J/M^2 = F(\delta)$ where $F(-\delta) = -F(\delta)$ (CG '02)

- But nonlinearity seems to make $l = 1$ stable so we do not see large δ even as $P \to 0$ (Baumgarte-CG '17)

Axisymmetric Einstein-Maxwell

- In axisymmetry, with $F =: dA$ and $F^* =: d\tilde{A}$, can write the Maxwell equations and T_{ab} in terms of A_φ and $\tilde{A}_\varphi$
- Time evolutions for $\tilde{A}_\varphi = 0$ ($\Rightarrow$ zero twist) show DSS with $\Delta \simeq 0.6$ and $\gamma \simeq 1.4$ (Baumgarte-Hilditch-CG)
- DSS seems to be quasiperiodic only: could this be subdominant DSS gravitational waves?



Axisymmetric Vacuum

- Axisymmetry, zero twist, spatial metric:

$$ds^2_{(3)} = \phi^4 \left[e^{2\eta/3}(dr^2 + r^2\, d\theta^2) + e^{-4\eta/3} r^2 \sin^2\theta\, d\varphi^2 \right]$$

- Abrahams-Evans '93: Teukolsky waves

$$\eta(t, r, \theta) \simeq r^{-1} \left[f(t + r) - f(t - r) + \dots \right] \sin^2\theta,$$

  constrained evolution, find DSS with $\Delta \simeq 0.6$ and mass scaling with $\gamma \simeq 0.36$, both ingoing and $\eta = 0$ initial data, down to $|p/p_* - 1| \simeq 10^{-6}$
- Several other attempts with Brill waves $K_{ij} = 0$ do not get close enough
- Hilditch-Weyhausen-Brügmann '17, Brill waves, generalized harmonic free evolution, find same γ in curvature scaling but $\Delta \simeq 3$ from scaling wiggle

Things to do

- In rotating perfect fluids with $P = k\rho$:
  - Understand nonlinear effects that stabilise angular momentum modes even for $k < 1/9$
  - Compute universal scaling functions from $\phi_* + \phi_0 + \delta \cdot \phi_1$
- In 2+1 dimensions: understand role of $\Lambda < 0$
- In higher dimensions: $D \to \infty$?
- In vacuum:
  - Improve resolution in vacuum evolutions
  - Double null coordinates
  - Is there a single critical solution? Evolve Abrahams-Evans data
  - Construct the critical solution as a boundary value problem
  - Prove existence
  - Prove codimension-one stability
  - Allow for twist (two polarisations, angular momentum)
- Stability in a function space versus analytic mode stability