Critical phenomena in gravitational collapse: codimension-one stability of naked singularity formation?

Carsten Gundlach

School of Mathematical Sciences
University of Southampton

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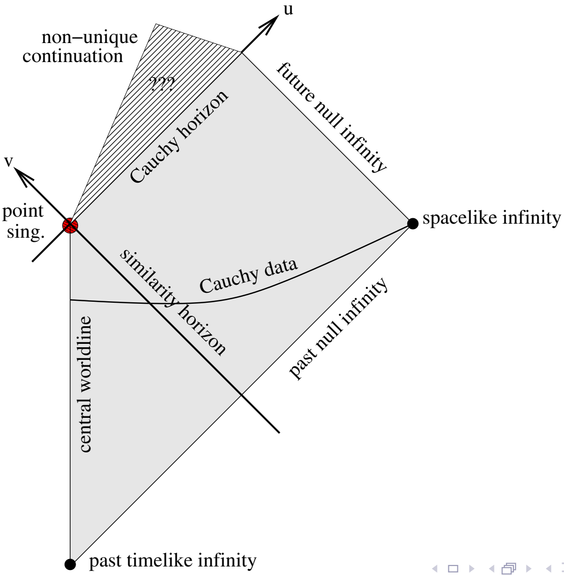
Plan of the talk

1. Critical collapse as a natural mechanism for forming naked singularities in general relativity
2. Mathematical work on naked singularity formation
3. Numerical work on vacuum critical collapse

Naked singularities in critical collapse

Here I am interested in this kind of naked singularity:



Self-similarity in general relativity

- Continuous self-similarity (CSS)
  - Homothetic vector field X, $\mathcal{L}_X g_{ab} = -2g_{ab}$
  - In adapted coordinates $x^a := (\tau, x, \theta^A)$:

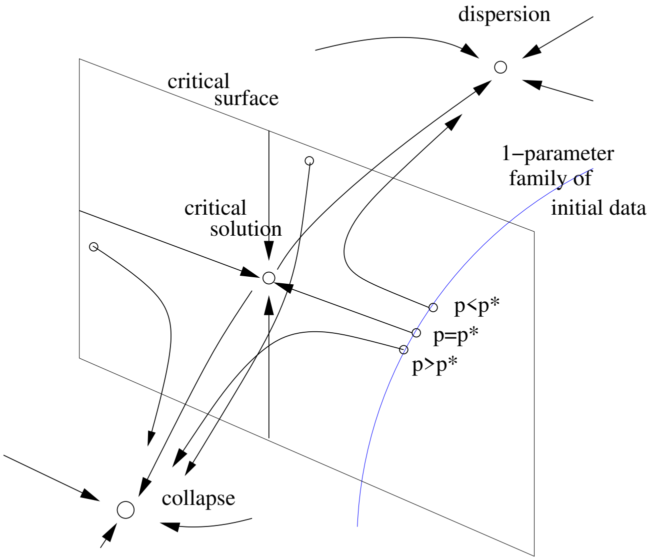
$$g_{ab} = e^{-2\tau}\tilde{g}_{ab}(x,\theta^A) \quad \Rightarrow \quad R^{ab}{}_{cd} = e^{2\tau}\tilde{R}^{ab}{}_{cd}(x,\theta^A).$$

  - If there is matter, $G_{ab} = 8\pi T_{ab}$, then $T^a{}_b = e^{2\tau}\tilde{T}^a{}_b$
  - Familiar from Newtonian fluids (Riemann problem, Sedov-Taylor blast wave): $\tau := \ln t$, $x := r/t$ (for $t > 0$)
- Discrete self-similarity (DSS)
  - Discrete conformal isometry Φ, $\Phi_* g_{ab} = e^{-2\Delta} g_{ab}$
  - In adapted coordinates: $\tilde{g}_{ab}$, $\tilde{R}^{ab}{}_{cd}$, $\tilde{T}^a{}_b$ now depend periodically on τ with period Δ
  - Essentially unknown elsewhere in physics
- Requires scale-invariant physics (massless scalar field, electromagnetism, ultrarelativistic fluid, **vacuum GR**, ...)

Critical collapse: a natural mechanism for naked singularity formation

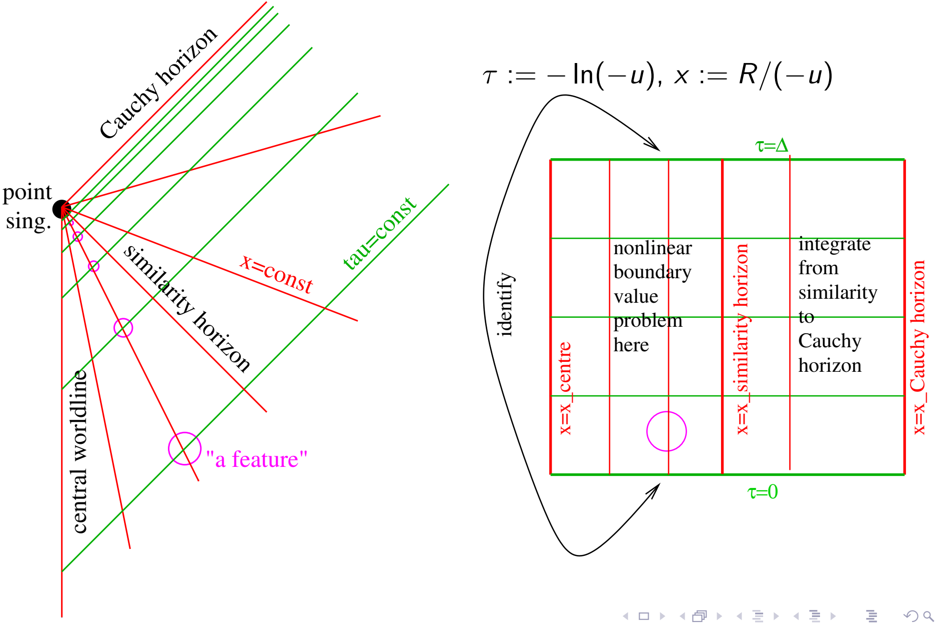
- A **critical solution** exists with the following properties
  - CSS or DSS with $\tilde{g}_{ab}$ regular
    $\Rightarrow$ pointlike curvature singularity at $\tau = \infty$
  - future lightcone of this singularity (almost) regular
    $\Rightarrow$ locally and globally naked
  - single-mode unstable $\Leftarrow$ attractor of codimension one
- Take a generic one-parameter family of smooth, asymptotically flat initial data that goes from dispersing to collapsing data...
- ... fine-tune the parameter p to the threshold p_* of black-hole formation...
- ...which is also the attracting manifold of the critical solution...
- ...and we get a naked singularity from smooth initial data

Phase space picture: scaling and universality



How to construct the critical solution

$$\tau := -\ln(-u), \; x := R/(-u)$$



Critical scaling

- While the time evolution is near the critical solution (here, assumed to be CSS)

$$\tilde{g}(x,\tau) \simeq \tilde{g}_*(x) + (p-p_*)e^{\lambda_0\tau}(\delta\tilde{g})_0(x) + \text{decaying modes}$$

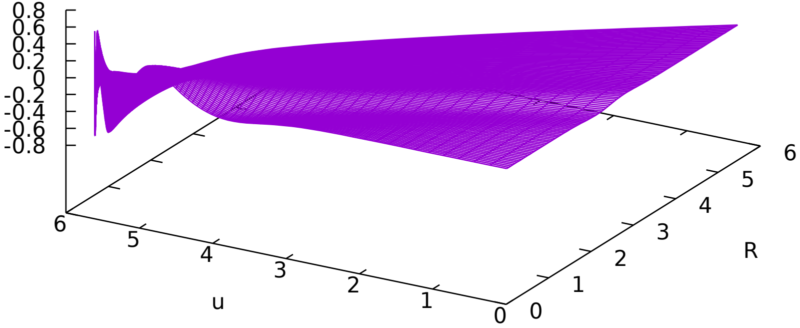
  with $\lambda_0 > 0$, all other $\mathrm{Re}\lambda_i < 0$
- Setting $(p-p_*)e^{\lambda_0\tau} \stackrel{!}{=} 1$ and (all length scales) $\propto e^{-\tau}$ gives

$$\begin{aligned} M_{\text{black hole}} &\sim (p-p_*)^{1/\lambda_0} \\ \text{maximal curvature} &\sim (p_*-p)^{-2/\lambda_0} \end{aligned}$$

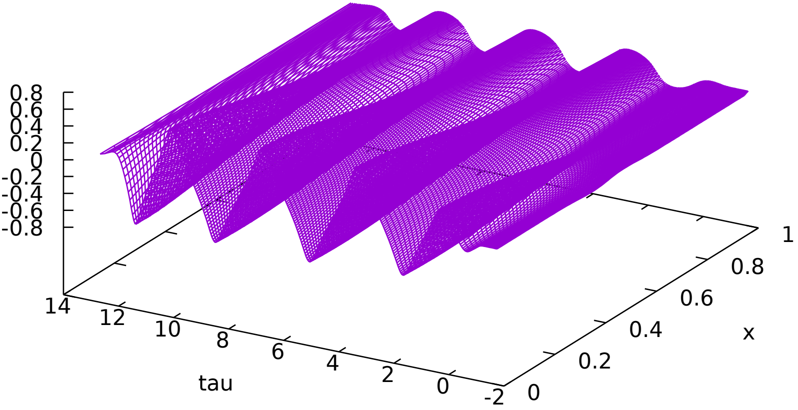
  for any one-parameter family of initial data
- Critical exponents for black hole spin and charge, universality classes...

Numerical example: spherical scalar field ψ

- $ds^2 = -4\Omega^2(u,v)\,du\,dv + R^2(u,v)(d\theta^2 + \sin^2\theta\,d\varphi^2)$
- Gaussian initial data for ψ at $u = 0$, amplitude fine-tuned to just below black hole threshold
- Plot ψ against u and $R(u,v)$

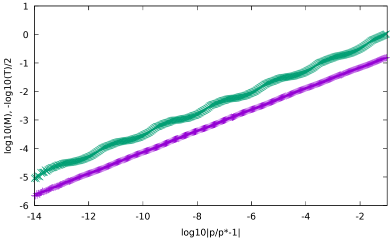


- Plot ψ again, now against $\tau(u) := -\ln(u_* - u)$ and $x := R(u,v)/(u_* - u)$, we have fitted $u_* \simeq 5.60924$
- We can read off $\Delta \simeq 3.44$ as the period in τ

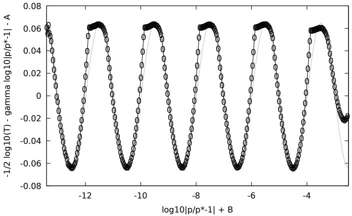


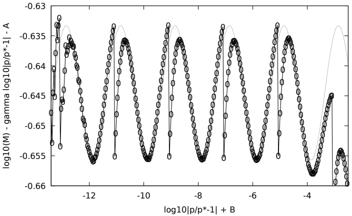
Same example: scaling laws

Mass and curvature scaling over 13 orders of magnitude in $|p - p_*|$



Ricci and mass scaling laws with factors $|p - p_*|^{0.374}$ taken out





Examples of critical collapse in twist-free axisymmetry

- Massless scalar field (DSS with $\Delta \simeq 3.44$)

$$R_{ab} = 8\pi \nabla_a \psi \nabla_b \psi, \qquad \nabla_a \nabla^a \psi = 0$$

(Spherical symmetry: Choptuik 1993, CG 1995, 1997, Martín-García & CG 1999, Reiterer & Trubowitz 2019) Axisymmetry: Choptuik+ 2003, Baumgarte 2018, CG in prep

- Ultrarelativistic perfect fluid (CSS)

$$T_{ab} = (\rho + P) u_a u_b + P g_{ab}, \qquad P = k\rho$$

(Spherical symmetry: Evans & Coleman 1994, Maison 1996, CG 2002) Axisymmetry: Baumgarte & CG 2016+

- Electrovacuum: Baumgarte, CG & Hilditch 2019
- Vacuum: see later

Rigorous results on the spherical scalar field (Christodoulou 1986-1999)

- Work in Bondi coordinates
- Generic regular initial data: Sufficient conditions for dispersal and collapse (in class of bounded variation)
- Restrict to CSS:
  - Ansatz $\psi(x,\tau) = f(x) + k\tau$ compatible with CSS $\Rightarrow$ ODE system in x
  - Impose regularity at centre $x = 0$
  - 1-parameter family of non-unique continuations through similarity horizon $x = x_{\mathrm{SH}}$
  - Open set of such solutions has naked singularities
- These **and any other** naked singularity solutions have **some** codimension (in class of bounded variation): a family of perturbations of the initial data changes naked singularity to BH formation (for either sign of the perturbation)

Choptuik solution is analytic (Reiterer & Trubowitz 2019)

- First-order formulation of the Einstein equations with only quadratic nonlinearities (using tetrad and connection as variables)
- Null coordinates, Chebychev series in x, Fourier series in τ, quadratic terms by convolution
- Start from a very accurate approximate solution (truncated series) in **rational** arithmetic
- Contraction argument to show the full series converges with finite convergence radius $\Rightarrow$ solution is analytic from centre to slightly beyond similarity horizon
- Computer-aided proof

Comments on Christodoulou and R&T

- All CSS spherical scalar naked singularity solutions are not regular ($C^{1,\epsilon}$) at the similarity horizon
- C's explicit perturbation that destroys **any** naked singularity solution is not regular ($C^{1,\epsilon'}$) at the singularity horizon
- By contrast, critical solution seen in critical collapse is DSS and analytic (R&T)...
- ...and believed to be an attractor of codimension one (from critical collapse experiments, numerical perturbation spectrum of critical solution)
- What can one hope to prove? Example: Glogić & Schörkhuber 2021: CSS blowup solution of $\Box u = -u^3$ in 7D is codimension-one stable in its past lightcone (without symmetries). Namely, after subtracting the unique unstable mode, convergence to the critical solution in some Sobolev space in the backward lightcone

CSS vacuum (Rodnianski & Shlapentokh-Rothmann 2017, 2019, S-R 2022)

- Double null coordinates

$$ds^2 = -4\Omega^2 \, du \, dv + g_{AB}(d\theta^A - b^A \, du)(d\theta^B - b^B \, du)$$

  Lines of constant (u, θ^A) are outgoing null geodesics
- CSS data on $v = 0$ (similarity horizon) and "admissible" data on $u = -1$. No other symmetries
- Existence of a class of naked singularity solutions for $0 \leq v < \infty$, $-1 \leq u < \infty$ (to the future of the similarity horizon)
- Later extended to the past, from the singularity horizon to a regular centre
- Essential for naked singularities: X^a winds around generators $\nabla^a v$ of similarity horizon because $b^A \neq 0$

Comments on R&S-R

- Homothetic vector field is (in my notation)

$$X = \nu_u \, u \frac{\partial}{\partial u} + \nu_v \, v \frac{\partial}{\partial v}$$

  ν_u and ν_v are geometric invariants if coordinates are regular at $v = 0$ (similarity horizon) and $u = 0$ (Cauchy horizon) (previously noted by C, R&T for spherical scalar field)
- Naked singularities require $\nu_v < 1$ (previously R&T found $\nu \simeq 0.6138$ for scalar field critical solution)
- Metric is only $C^{1,\epsilon}$ on similarity horizon (just like Christodoulou's naked singularity solutions)
- CSS is easier but not quite regular. Critical solutions must be analytic and therefore DSS?
- DSS vacuum critical solution conjectured but not yet known even approximately

Numerical experiments in vacuum collapse: initial data

- 3-metric can be written as

$$\gamma_{ij}\,dx^i\,dx^j = \psi^4\left(e^q(dr^2 + r^2\,d\theta^2) + r^2\sin^2\theta\,d\varphi^2\right)$$

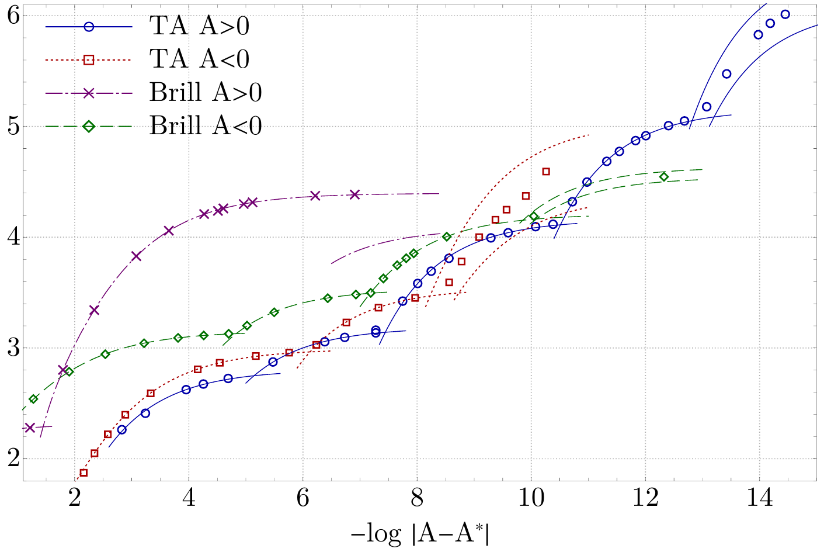
- "Teukolsky data": q and $K^r{}_\theta$ taken from linear wave, itself given by one freely specified function $I(v)$, then solve constraints
  - approximately **ingoing** at $t = 0$, in the sense that $I(u)$ small there
  - "**time-antisymmetric**": $q = 0$ at $t = 0$ (so γ_{ij} conf. flat)
- **Brill** waves: time symmetric, $K_{ij} = 0$, and q at $t = 0$ freely specified, for example

$$q = A\,r^2 e^{-r^2}$$

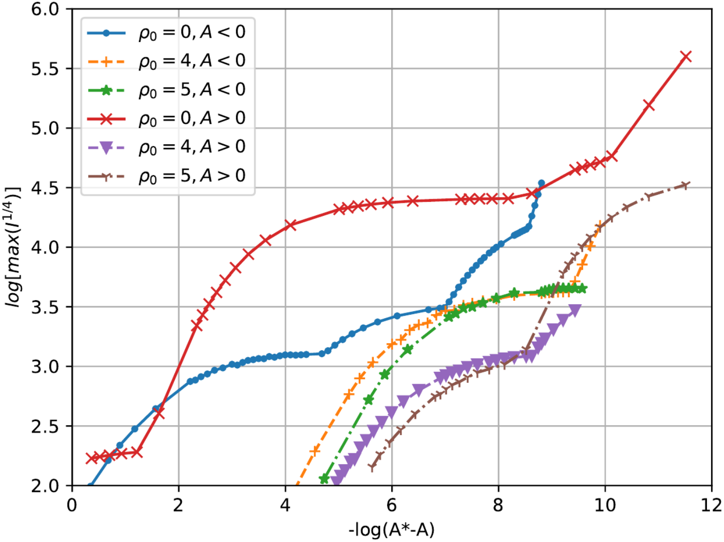
Numerical experiments in vacuum collapse: overview

- Abrahams & Evans 1993
  - ingoing and time-antisymmetric initial data
  - tentative scaling with $\gamma \simeq 0.36$ down to 0.2 ADM mass
  - tentative scale echoing with $\Delta \simeq 0.6$ and 3 echos
  - tentative universality for these two families
- Later attempts cannot fine-tune well enough
- bamps group (Brügmann, Hilditch+) 2013, 2017, 2022
  - Brill data ($A > 0$ and $A < 0$, centred and off-centred Gaussians)
- Ledvinka & Khirnov 2019, 2021
  - time-antisymmetric and Brill data ($A > 0$ and $A < 0$, centred)
  - could not reproduce A&E time-antisymmetric initial data
- Both groups (current comparison effort, also Baumgarte, CG)
  - irregular scaling laws without clear universality
  - irregular scale echoing, clear only around peaks of $(\text{Riem})^2$
  - no relation between periods of scaling laws and echoing
  - possible global DSS seen only for $A < 0$ Brill data, otherwise "two centres of collapse"?

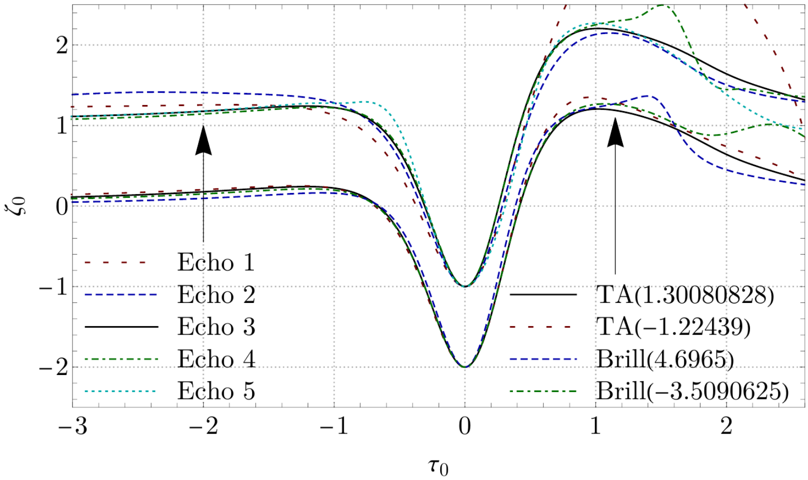
Ledvinka & Khirnov 2021: scaling laws



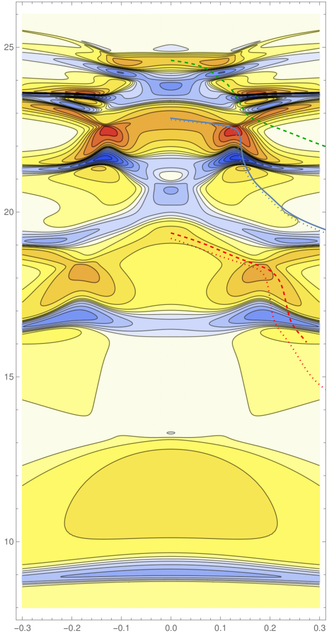
Suárez Fernández, Renkhoff, Cors Agulló, Brügmann, Hilditch 2022: scaling laws

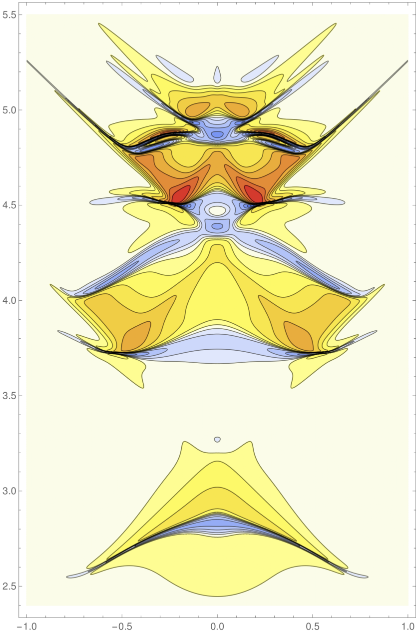


Ledvinka & Khirnov 2021: local DSS?



Ledvinka & Khirnov 2021: global DSS? (L, private comm)





Conclusions

- Critical solutions need to be DSS (except for perfect fluid)
- A proof of codimension-one stability of a critical solution is at least conceivable
- What would this mean?
- Vacuum critical collapse is still confusing: why?
- But numerical relativity is finally providing more data to work with