Naked singularity formation in vacuum gravitational collapse

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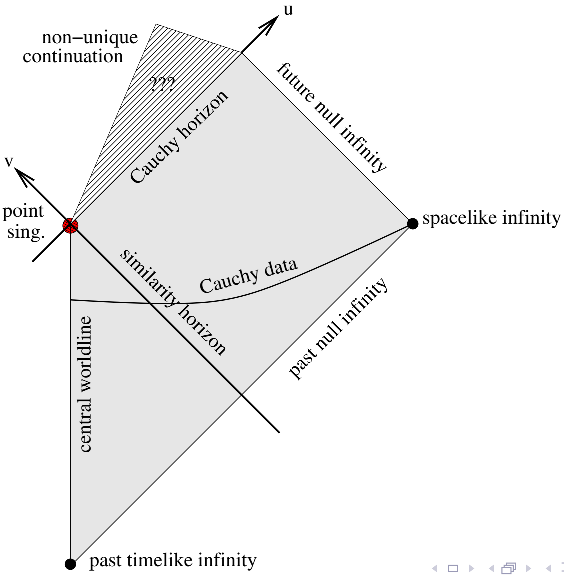
London PDE seminar, 21 January 2021

Plan of the talk

- 1. Critical collapse as a natural mechanism for forming naked singularities in general relativity
- 2. Mathematical work on naked singularity formation
- 3. Numerical work on vacuum critical collapse
- This talk is not in historical order
- Can I assume you know: metric and line element, Riemann tensor, simple spacetime diagrams (null curves at 45 degrees)?
- Upcoming update of my article in Living Reviews in Relativity

Naked singularities in critical collapse

Here I am interested in this kind of naked singularity:



Self-similarity in general relativity

- Continuous self-similarity (CSS)
  - Homothetic vector field X, $\mathcal{L}_X g_{ab} = -2g_{ab}$
  - In adapted coordinates $x^a := (\tau, x, \theta^A)$:

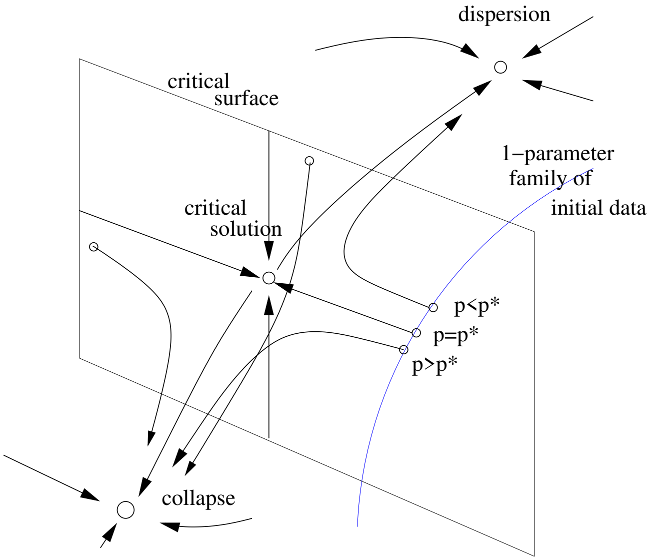
$$g_{ab} = e^{-2\tau} \tilde{g}_{ab}(x, \theta^A) \quad \Rightarrow \quad R^{ab}{}_{cd} = e^{2\tau} \tilde{R}^{ab}{}_{cd}(x, \theta^A).$$

  - If there is matter, $G_{ab} = 8\pi T_{ab}$, then $T^a{}_b = e^{2\tau} \tilde{T}^a{}_b$
  - Familiar from Newtonian fluids (Riemann problem, Sedov-Taylor blast wave): $\tau := -\ln t$, $x := r/t$
- Discrete self-similarity (DSS)
  - Discrete conformal isometry Φ, $\Phi_* g_{ab} = e^{-2\Delta} g_{ab}$
  - In adapted coordinates: $\tilde{g}_{ab}$, $\tilde{R}^{ab}{}_{cd}$, $\tilde{T}^a{}_b$ now depend periodically on τ with period Δ
  - Essentially unknown elsewhere in physics
- Requires scale-invariant physics (massless scalar field, ultrarelativistic fluid, **vacuum GR**, ...)

Critical collapse: a natural mechanism for naked singularity formation

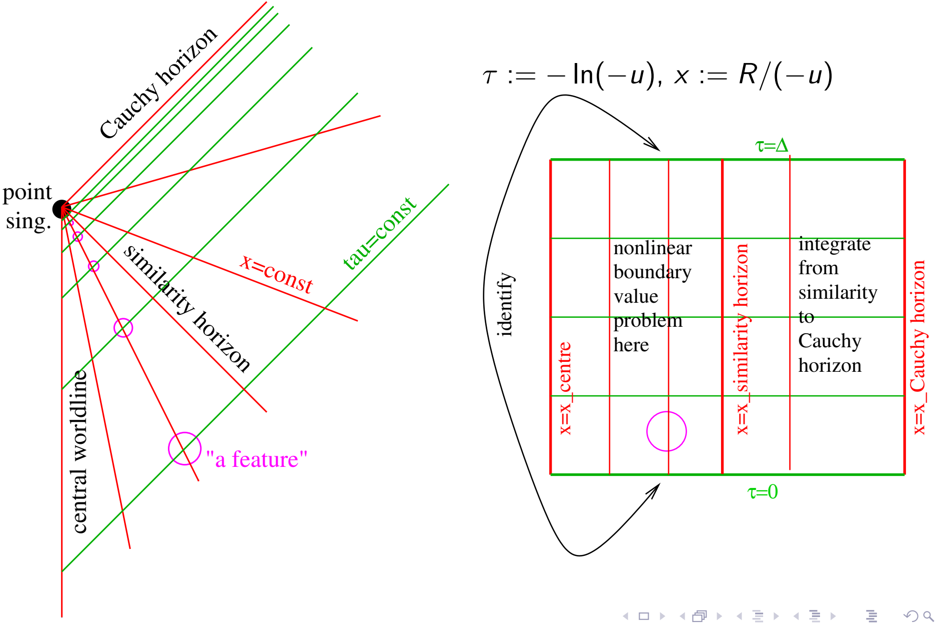
- A "critical solution" exists with the following properties
  - CSS or DSS with $\tilde{g}_{ab}$ regular
    $\Rightarrow$ pointlike curvature singularity at $\tau = \infty$
  - future lightcone of this singularity is regular
    $\Rightarrow$ locally and globally naked
  - single-mode unstable $\Leftarrow$ attractor of codimension one
- Take a generic one-parameter family of smooth, asymptotically flat initial data...
- ... fine-tune the parameter p to the threshold p_* of black-hole formation...
- ...which is also the attracting manifold of the critical solution...
- ...and we get a naked singularity from smooth initial data

Phase space picture



How to construct the critical solution

$$\tau := -\ln(-u), \; x := R/(-u)$$



Critical scaling

- While the time evolution is near the critical solution (here, assumed to be CSS)

  $$\tilde{g}(x,\tau) \simeq \tilde{g}_*(x) + (p-p_*)e^{\lambda_0\tau}(\delta\tilde{g})_0(x) + \text{decaying modes}$$

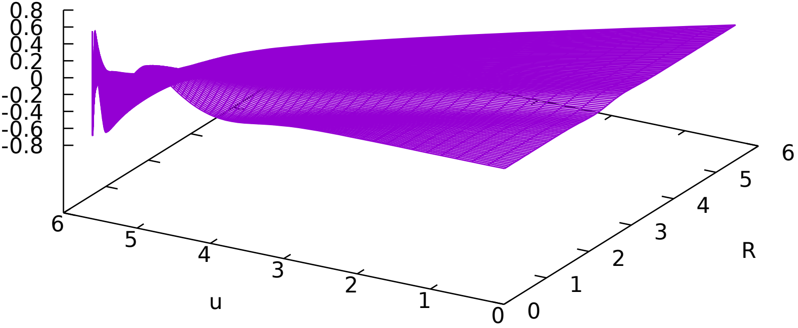
  with $\lambda_0 > 0$, all other $\mathrm{Re}\lambda_i < 0$
- Essentially dimensional analysis gives

  $$\begin{aligned} M_{\text{black hole}} &\sim (p-p_*)^{1/\lambda_0} \\ \text{maximal curvature} &\sim (p_*-p)^{-2/\lambda_0} \end{aligned}$$

  for any one-parameter family of initial data
- Critical exponents for black hole spin and charge, universality classes...

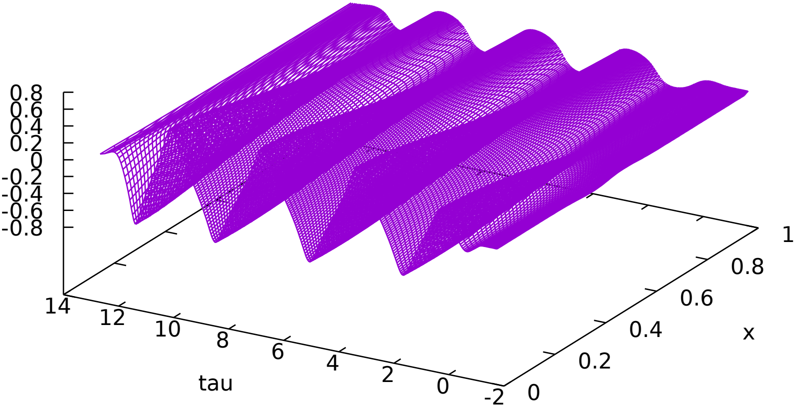
Numerical example: spherical scalar field

- $ds^2 = -4\Omega^2(u,v)\,du\,dv + R^2(u,v)(d\theta^2 + \sin^2\theta\,d\varphi^2)$
- Gaussian initial data for ψ at $u = 0$, amplitude fine-tuned to just below black hole threshold
- Plot ψ against u and $R(u,v)$



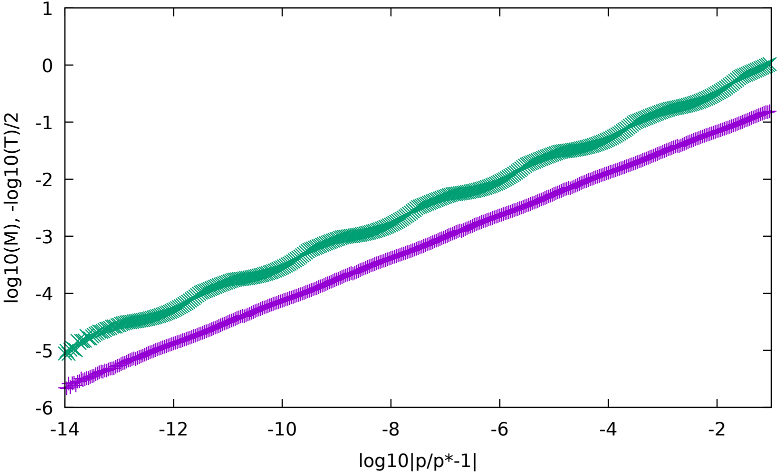
Same example: scale echoing

- Plot ψ again, now against $\tau(u) := -\ln(u_* - u)$ and $x := R(u,v)/(u_* - u)$, we have fitted $u_* \simeq 5.60924$
- We can read off $\Delta \simeq 3.44$ as the period in τ



Same example: scaling laws

Mass and curvature scaling over 13 orders of magnitude in $|p - p_*|$



Examples of critical collapse

- Massless scalar field (DSS with $\Delta \simeq 3.44$)

$$R_{ab} = 8\pi \nabla_a \psi \nabla_b \psi, \qquad \nabla_a \nabla^a \psi = 0$$

(Choptuik 1993, CG 1995, 1997, Martín-García & CG 1999, Baumgarte 2018, Reiterer & Trubowitz 2019)

- Ultrarelativistic perfect fluid (CSS)

$$T_{ab} = (\rho + P) u_a u_b + P g_{ab}, \qquad P = k\rho$$

(Evans & Coleman 1994, Maison 1996, CG 2002)

- Vacuum $R_{ab} = 0$ in polarised axisymmetry (see later)
- Electrovacuum in polarised axisymmetry (Baumgarte, CG & Hilditch 2019)

Rigorous results on the spherical scalar field (Christodoulou 1986-1999)

- Work in Bondi coordinates
- Generic regular initial data: Sufficient conditions for dispersal and collapse (in class of bounded variation)
- Restrict to CSS:
  - Ansatz $\psi(x,\tau) = f(x) + k\tau$ compatible with CSS $\Rightarrow$ ODE system in x
  - Impose regularity at centre $x = 0$
  - 1-parameter family of non-unique continuations through similarity horizon $x = x_{\rm SH}$
  - Open set has naked singularities
- Shows explicitly that these **and any other** naked singularity solutions have **some** codimension (in class of bounded variation): a family of perturbations of the initial data forms a black hole

Choptuik solution is analytic (Reiterer & Trubowitz 2019)

- First-order formulation of the Einstein equations with only quadratic nonlinearities (using tetrad and connection as variables)
- Null coordinates, Chebychev series in x, Fourier series in τ, quadratic terms by convolution
- Start from a very accurate approximate solution (truncated series) in rational arithmetic
- Contraction argument to show the full series converges with finite convergence radius $\Rightarrow$ solution is analytic from centre to slightly beyond similarity horizon
- Computer-aided proof

Comments on Christodoulou and R&T

- All CSS naked singularity solutions are not regular ($C^{1,\epsilon}$) at the similarity horizon
- C's explicit perturbation that destroys *any* naked singularity solution is not regular ($C^{1,\epsilon'}$) at the singularity horizon
- By contrast, critical solution seen in critical collapse is DSS and analytic (R&T)...
- ...and believed to be an attractor of codimension one (from critical collapse experiments, numerical perturbation spectrum of critical solution)
- But in what function space can one hope to prove this?
  - Analytic, smooth, energy norm, bounded variation?
  - Nonlinear, linear, mode stability?
  - Attractor only locally in spacetime (full critical solution is not asymptotically flat)

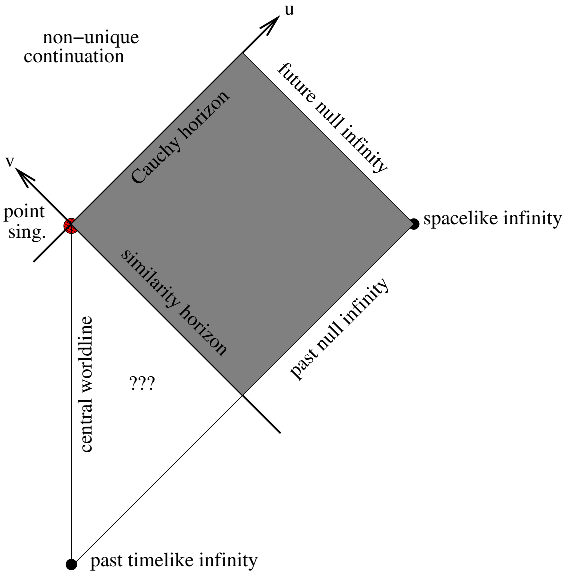
CSS vacuum (Rodnianski & Shlapentokh-Rothmann 2017, 2019)

- Double null coordinates

$$ds^2 = -4\Omega^2\, du\, dv + g_{AB}(d\theta^A - b^A\, du)(d\theta^B - b^B\, du)$$

  Lines of constant (u, θ^A) are outgoing null geodesics
- CSS data on $v = 0$ (similarity horizon) and "admissible" data on $u = -1$. No other symmetries
- Existence of a class of naked singularity solutions for $0 \leq v < \infty$, $-1 \leq u < \infty$ (to the future of the similarity horizon)
- Future work to extend to the past, from the singularity horizon to a regular centre
- Essential for naked singularities: X^a winds around generators $\nabla^a v$ of similarity horizon because $b^A \neq 0$

Rodnianski & Shlapentokh-Rothmann's naked singularities



Comments on R&S-R

- Homothetic vector field is (in my notation)

$$X = \nu_u \, u \frac{\partial}{\partial u} + \nu_v \, v \frac{\partial}{\partial v}$$

  ν_u and ν_v are geometric invariants if coordinates are regular at $v = 0$ (similarity horizon) and $u = 0$ (Cauchy horizon) (previously noted by C, R&T for spherical scalar field)
- Naked singularities require $\nu_v < 1$ (previously R&T found $\nu \simeq 0.6138$ for scalar field critical solution)
- Metric is only $C^{1,\epsilon}$ on similarity horizon (just like Christodoulou's naked singularity solutions)
- CSS is easier but not quite regular. Critical solutions must be analytic and therefore DSS?
- DSS vacuum critical solution conjectured but not yet known even approximately

Numerical experiments in vacuum collapse

- All using 3+1 formulations of the Einstein equations, restricted to polarised axisymmetry
- Abrahams & Evans 1993
  - tentative scaling with $\gamma \simeq 0.36$ down to 0.2 ADM mass
  - tentative scale echoing with $\Delta \simeq 0.6$ and 3 echos
  - tentative universality for different one-parameter families
- Later attempts by other groups cannot fine-tune well enough
- Hilditch, Weyhausen & Brügmann 2017
  - tentative scaling with $\gamma \simeq 0.36$
  - wiggle in scaling law suggests $\Delta \simeq 3$
  - tentative universality for different one-parameter families
- Ledvinka & Khirnov 2021
  - $\gamma \simeq 0.33, 0.37, 0.19$ for three one-parameter families
  - irregular scale echoing (see next slide)

From Ledvinka & Khirnov 2021

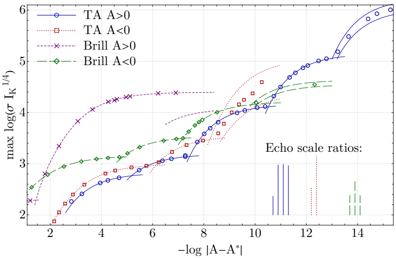


FIG. 1. Extremes of Kretschmann invariant I_K for $A \prec A^*$ correspond to spikes of curvature in subcritical spacetimes. Because the echo amplitudes exhibit smooth dependence on A, we show curve segments representing typically linear fits of $\log I_K^{\max}$ vs. A through respective points. Effect of the uncertainty of A^* is also shown for the last segments. Echo scale ratios at A^* are also depicted in logarithmic scale – in DSS case they would universally approach DSS factor e^{Δ}.

$\gamma_{\mathrm{ta}+} \doteq 0.33$, $\gamma_{\mathrm{ta}-} \doteq 0.37$ and $\gamma_{\mathrm{Brill}-} \doteq 0.19$. According to [4] $\gamma_{\mathrm{Brill}+} \doteq 0.37$. In our fits we excluded data points corresponding to the first echo so that a direct influence of the form of initial data is suppressed. Similar approach applied to results [4] seems to result in $\gamma_{\mathrm{Brill}+} > 0.5$.

These differences of the exponent γ seem to be significant, but we cannot decide if the slopes in Fig. 1 really settle towards a specific value for a given family or

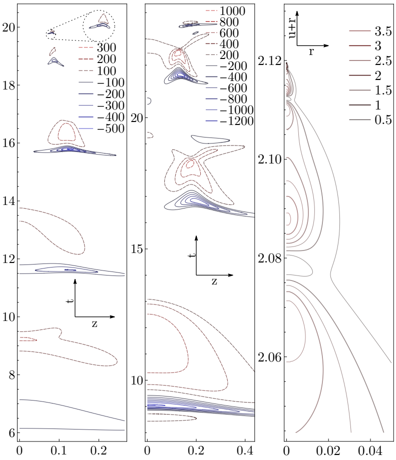


FIG. 2. The spacetime curvature in a sub-critical collapse shown using a dimensionless quantity $(\tau^* - \tau)^2\zeta$ (see text). Left: $A_{\mathrm{ta}+} = \overline{1.30080828}$, $z_0 = 0.08415\sigma$, $\tau^* = 3.88\sigma$; center: $A_{\mathrm{Brill}-} = -3.5090625$, $z_0 = 0.161\sigma$, $\tau^* = 5.9\sigma$ (coordinates t, z in units of σ); right: near-critical massless scalar field collapse.

Using null coordinates for collapse simulations

$$ds^2 = -2G\,du\,dx - H\,du^2 + R^2\left[e^{2F}(d\theta + \beta\,du)^2 + e^{-2F}\sin^2\theta\,d\varphi^2\right]$$

- Lines of constant (u, θ, φ) are outgoing null geodesics emanating from $R = 0$, forming null cones of constant u
- H can be chosen freely and controls coordinate x along null geodesics
  - $H = 0 \Rightarrow x$ is a null coordinate
  - $R = x$ Bondi coordinates $\Rightarrow$ ODE for H
  - $G = \text{const} \Rightarrow x$ is affine parameter, ODE for H
- G, β, $R_{,u}$, $f_{,u}$, $\psi_{,u}$ found by integration along null rays
- My gauge choice $R = R(u, x)$ only ("local shifted Bondi"), with $x = x_0$ roughly ingoing null
  - Work on domain of dependence
  - Grid shrinks to a point
  - x and $-\ln(u_* - u)$ are potentially adapted to DSS

My null coordinates adapted to DSS