The interplay of mathematics and numerics in understanding naked singularity formation in critical collapse

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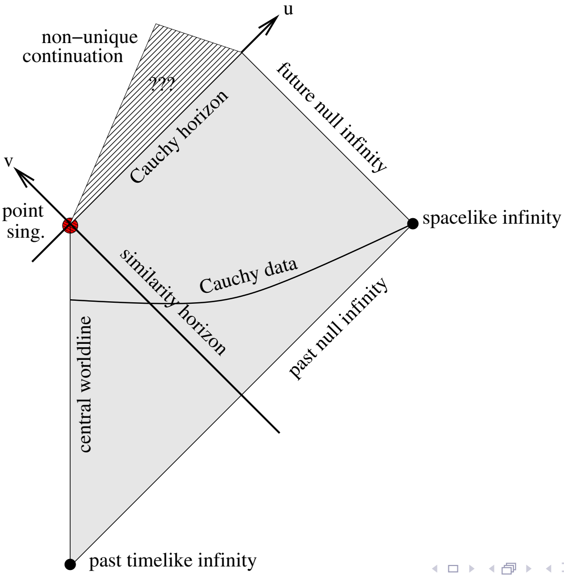
Plan of the talk

- Critical collapse as a natural mechanism for forming naked singularities in general relativity
- Recent numerical work on vacuum critical collapse
- Recent mathematical work on naked singularity formation and their stability

Naked singularities in critical collapse

Here I am interested in this kind of naked singularity:



Self-similarity in general relativity

- Continuous self-similarity (CSS)
  - Homothetic vector field $\mathcal{L}_\xi g_{ab} = -2g_{ab}$
  - In adapted coordinates $x^\mu := (\tau, x, \theta^A)$

$$g_{\mu\nu} = e^{-2\tau}\tilde{g}_{\mu\nu}(x,\theta^A) \quad \Rightarrow \quad R^{\mu\nu}{}_{\kappa\lambda} = e^{2\tau}\tilde{R}^{\mu\nu}{}_{\kappa\lambda}(x,\theta^A).$$

  - Familiar from Newtonian physics
- Discrete self-similarity (DSS)
  - Conformal isometry $\Phi : M \to M$ with $\Phi_* g_{ab} = e^{-2\Delta} g_{ab}$
  - **DSS-adapted vector field** ξ^a: Φ given by $\tau \to \tau + \Delta$ along integral curves of ξ^a with parameter τ
  - In adapted coordinates

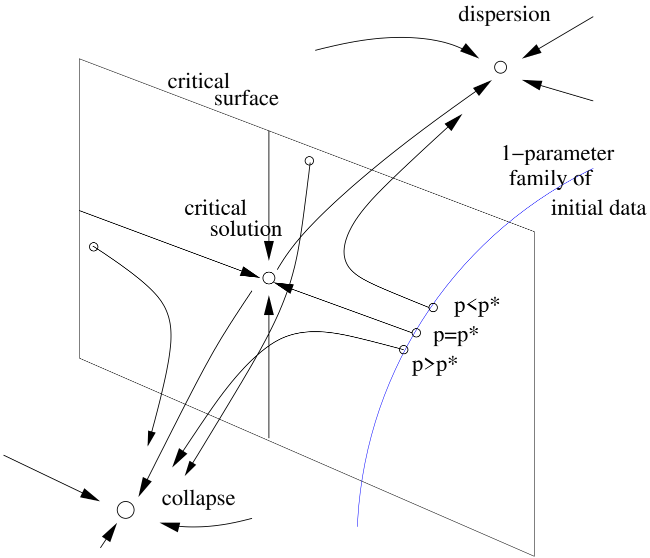
$$g_{\mu\nu} = e^{-2\tau}\tilde{g}_{\mu\nu}(x,\tau,\theta^A), \quad g_{\mu\nu}(x,\tau+\Delta,\theta^A) = g_{\mu\nu}(x,\tau,\theta^A)$$

  - Essentially unknown elsewhere in physics (Eggers-Fontelos08)

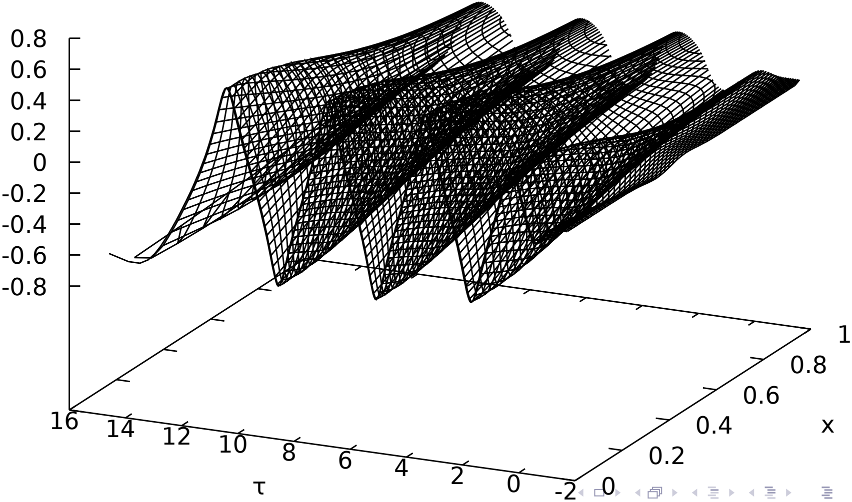
Naked singularity formation in critical collapse

- A "critical solution" exists, defined by the properties
  - CSS or DSS with $\tilde{g}_{ab}$ smooth
    $\Rightarrow$ pointlike curvature singularity at $\tau = \infty$
  - future lightcone of this singularity has finite curvature
    $\Rightarrow$ locally and globally naked
  - attractor of codimension one $\Rightarrow$ single-mode unstable
- Take a generic one-parameter family of **smooth**, asymptotically flat initial data that interpolates between dispersing (for $p > p_*$) and collapsing (for $p < p_*$) solutions
  - ...fine-tune the parameter p to the threshold p_* of collapse
  - ...which is also the attracting manifold of the critical solution
  - ...and we get a naked singularity for $p = p_*$
  - ...and for $p \simeq p_*$ scalings such as $M_{\mathrm{BH}} \simeq (p - p_*)^\gamma$

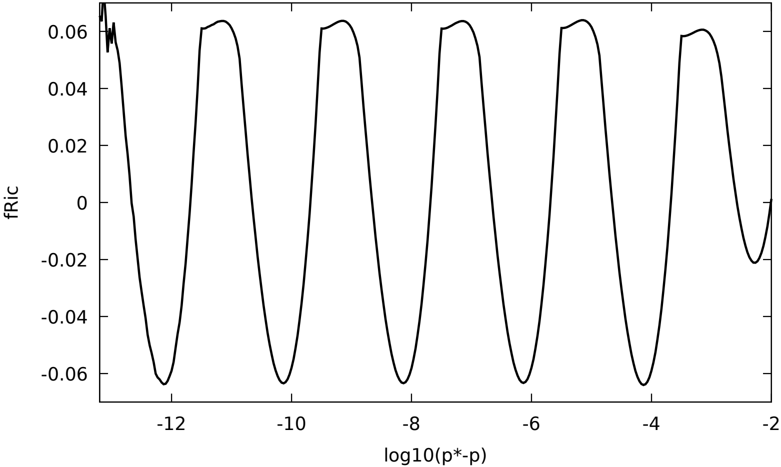
Phase space picture



near-critical $\psi(u, r)$ against $\tau := -\ln(u_* - u)$ and $x := r/(u_* - u)$



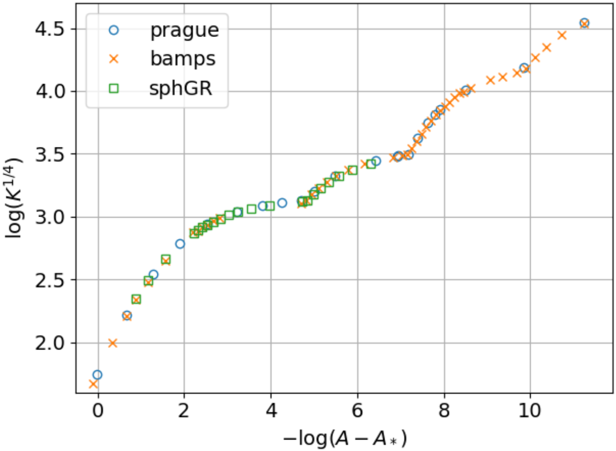
$$-\frac{1}{2}\ln\max|\mathrm{Ric}| \simeq \gamma\ln(p_*-p) + f_{\mathrm{Ric}}(\gamma\ln(p_*-p))$$



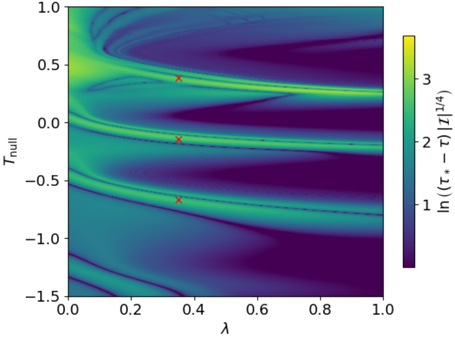
Axisymmetric vacuum critical collapse

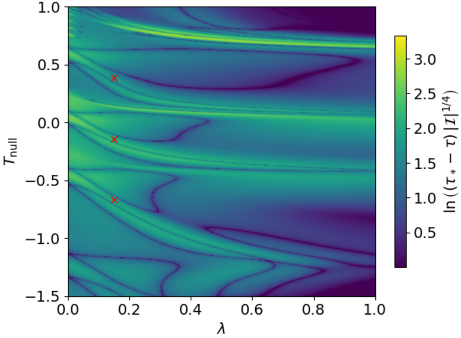
- Twist-free axisymmetry ($\partial_\varphi \to -\partial_\varphi$ symmetry)
- Abrahams-Evans93
  - initial data built on ingoing Teukolsky waves
  - tentative $\gamma \simeq 0.36$, $\Delta \simeq 0.6$
  - tentative universality for different one-parameter families
- 30 years later, three codes roughly agree with each other...
  - Suárez-Fernández+22 (Cartesian, GH, spectral, Brill waves)
  - Ledvinka-Khirnov22 (Cartesian, BSSN, max slicing, Teuk/Brill)
  - Baumgarte-G-Hilditch23 (polar, BSSN, shock-avoiding slicing, Brill/time-symm Teuk)
- ...but not with Abr-Ev93
  - curvature scaling (but with large irregular wiggles)
  - only approximate DSS, or unclear symmetry
  - Δ and γ not universal

Brill wave initial data, $\ln\max(\mathrm{Riem}^2)$ against $\ln(p_* - p)$, results from three codes (Baumgarte+23 and in prep.)



Vertical axis $T_{\text{null}} = -\ln(u_* - u)$
Horizontal axis $\lambda = (\text{affine parameter})/(u_* - u)$
Colour map $(u_* - u)(\text{Riem}^2)^{\frac{1}{4}}$

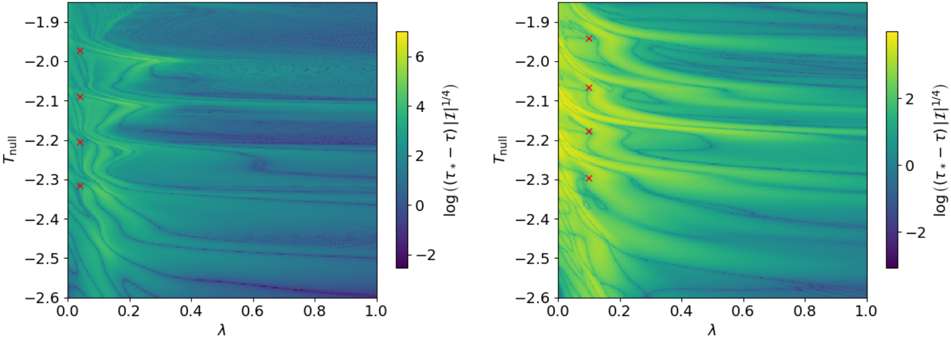




on the rotation axis (z-axis, left) and on the equator (x-axis, right)

Evidence for lack of universality: compare $l = 4$

As before, but different family of initial data



$\Delta \simeq 0.53$ for $l = 2$ but $\Delta \simeq 0.1$ for $l = 4$

CSS, non-smooth, naked singularities in spherical Einstein-scalar (Christodoulou 86-99)

- Ansatz $\psi(x,\tau) = \tilde{\psi}(x) + k\tau$ compatible with CSS metric
- Impose analyticity at centre $x = 0$
- 1-parameter family of non-unique continuations through SH, all $C^{1,\epsilon}$ at $x = x_{\mathrm{SH}}$ with $\epsilon \sim k^2 \ll 1$
- Open set of solutions has a naked singularity
- These **and any other** naked singularity solutions have some codimension **in BV**
- Explicit unstable perturbations at most $C^{1,\delta}$ at SH with $\delta \sim k^2/4$, zero inside SH, form a BH for either sign
- Scalar test fields unstable if less than $C^{1,\epsilon}$ (Singh24)

CSS, non-smooth, vacuum naked singularities (Rodnianski-Shlapentokh-Rothmann 17-22)

- Bondi-like double null coordinates

$$ds^2 = -4\Omega^2\, du\, dv + g_{AB}(d\theta^A - b^A\, du)(d\theta^B - b^B\, du)$$

- Existence of a class of **small-data**, **CSS** naked singularity solutions with regular centre
- Homothetic vector field ξ^a must wind around SH generators n^a
- Metric is only $C^{1,\epsilon}$ on SH

DSS Choptuik solution

- Numerical construction of solution as nonlinear PDE boundary value problem
- Computer-assisted proof of analyticity (Reiterer-Trubowitz19): convergence of Chebyshev-Fourier
- Numerical demonstration of codimension-one mode stability (conjectured to be in C^∞)
- Numerical extension to Cauchy horizon and (non-uniquely) beyond, $C^{1,\epsilon}$ at **CH**, $\epsilon \simeq 10^{-6}$
- Rei-Tru19 methods could be extended to prove analyticity up to Cauchy horizon and codim-1 mode stability

CSS critical solutions

- Numerically known CSS critical solution for Einstein-wavemap: proof of analyticity via shooting methods (Bizón-Wasserman02)
- Method could be extended to prove analyticity up to Cauchy horizon and codim-1 mode stability
- Perfect fluid $P = k\rho$, $0 < k \ll 1$: proof of analyticity and naked singularity of CSS solution (Guo-Hadzic-Yang23), proof of nonlinear (codim-0) stability (in spherical symmetry) in progress (post-talk correction)
- Sonic point replaces SH $\Rightarrow$ SH and CH analytic
- Extend proof to numerically known (codim-1) critical solutions for $0 < k < 1$?
- Analytically known rotation instability of any CSS solution for $0 < k < 1/9$ saves cosmic censorship (G02)

Surface gravity of CH and SH

- Let n^a be tangent to the affinely parameterised generators of SH. Let ξ^a be homothetic, or DSS-adapted and tangent to SH (or CH). Define A, m^a by

$$\xi^a = An^a + m^a, \qquad n^a m_a = 0$$

Then ν defined by

$$\Phi_* A = e^{-\nu\Delta} A \quad \text{(DSS)}, \qquad \mathcal{L}_\xi A = -\nu A \quad \text{(CSS)}$$

- In spherical symmetry

$$\nu = 1 - \int_0^\Delta \frac{2m}{r-2m}\, d\tau$$

- Choptuik solution $\nu_{\text{SH}} \simeq 0.6138$ (to 80 digits, Rei-Tru19), $\nu_{\text{CH}} \simeq 1 - 1.471 \cdot 10^{-6}$ (to 7 digits, G–Martín-García03)
- Christodoulou CSS-Einstein scalar solutions $\nu_{\text{SH}} = 1 - k^2$
- RodShl CSS vacuum solutions $\nu_{\text{SH}} = 1 - 2\kappa$

Things to do

- Vacuum numerics: better fine-tuning, more resolution, more families, beyond axisymmetry
  - Time evolution on null cones would be ideal (G24+)
  - Really no exact DSS?
  - How much universality?
  - Find vacuum DSS as eigenvalue problem?
- Mathematical
  - CSS perfect fluid: prove existence and **codim-1** stability
  - DSS spherical scalar: prove naked singularity and and codim-1 stability (in some Sobolev space)
  - CSS Einstein-wavemap (Biz-Was02) as a stepping stone?
  - Codim-1 stability desirable/possible in smooth?
  - Why DSS?
  - CH always $C^{1,\epsilon}$ (except for spherical perfect fluid)?