

Bogoliubov Transformations, The Unruh Effect and Hawking Radiation

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1 Massless scalar Field Theory in a General Metric

The Lagrange action describing a free real massless scalar field in a 4-dimensional space-time described by a metric $g_{\mu\nu}(x)$ is :

$$S = \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x), \quad (1)$$

where g denotes the determinant of the metric tensor $g_{\mu\nu}$. The corresponding equation of motion obeyed by the field $\phi(x)$ is

$$\partial_\mu \sqrt{-g(x)} g^{\mu\nu}(x) \partial_\nu \phi(x) = 0.. \quad (2)$$

Henceforth, we restrict ourselves to metrics which are time independent, so that the space-time possesses a time-like Killing vector and we can label the solutions by a frequency ω :

$$\phi_{\omega,\mathbf{k}}(x) = e^{-i\omega x_0} f_{\omega,\mathbf{k}}(x_i). \quad (3)$$

The label $\mathbf{k}$ labels all the degenerate solutions (i.e. with the same frequency). For example in Minkowski space $\mathbf{k}$ labels the angles of the momenta whose magnitudes are equal to the frequency, and in a space which is flat in the transverse directions, $\mathbf{k}$ labels the transverse momentum, plus a binary label *in/out* indicating whether the wave is moving along the positive or negative longitudinal direction.

The functions $f_{\omega,\mathbf{k}}(x_i)$ obey the equation

$$[\partial_i \sqrt{-g} [g^{ij} \partial_j - i\omega g^{0i}] - \sqrt{-g} g^{00} \omega^2] f_{\omega,\mathbf{k}}(x_i) = 0 \quad (4)$$

The solutions obey the orthormality relation

$$\begin{aligned} (\phi_{\omega,\mathbf{k}}, \phi_{\omega',\mathbf{k}'})_{KG} &= (2\pi)(2\omega)\delta(\omega - \omega') \boldsymbol{\delta}(\mathbf{k} - \mathbf{k}') \\ (\phi_{\omega,\mathbf{k}}, \phi_{\omega',\mathbf{k}'}^*)_{KG} &= 0 \end{aligned} \quad (5)$$

where the Klein-Gordon overlap is defined as

$$(\phi_{\omega,\mathbf{k}}, \phi_{\omega',\mathbf{k}'})_{KG} \equiv i \int n \cdot dS [\phi_{\omega,\mathbf{k}}^*(x) n \cdot \partial(\phi_{\omega',\mathbf{k}'}(x)) - \phi_{\omega',\mathbf{k}'}(x) n \cdot \partial(\phi_{\omega,\mathbf{k}}^*(x))] \quad (6)$$

n^μ is a time-like unit vector ($n \cdot n = 1$) and the space-like surface element dS_μ is given by

$$dS_\mu \equiv \frac{\sqrt{-g}}{3!} \epsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma,$$

($\epsilon_{\mu\nu\rho\sigma}$ is the Levy-Civita tensor in Minkowski space.)

It is often convenient to select the vector n_μ to be in the time-direction. In this case the Klein-Gordon overlap is written as

$$(\phi_{\omega,\mathbf{k}}, \phi_{\omega',\mathbf{k}'})_{KG} = -i \int \sqrt{h} d^3x [\phi_{\omega,\mathbf{k}}^*(x) \partial_0 \phi_{\omega',\mathbf{k}'}(x) - \phi_{\omega',\mathbf{k}'}(x) \partial_0 \phi_{\omega,\mathbf{k}}^*(x)]. \quad (7)$$

$h \equiv \det(g_{ij})$, the determinant of the spatial part of the metric.

Another representation for the Klein-Gordon overlap integral, which is particularly appropriate if the metric-space is flat in two-dimensional transverse directions, is the light-cone representation for which the metric can be written

$$ds^2 = f(u, v) du dv - d\mathbf{x}_T^2,$$

where

$$v \equiv (x^0 + x^3), \quad u \equiv (x^0 - x^3).$$

For a two-dimensional space, it is *always* possible to find a transformation of coordinates for which the metric can be written in this term. This means that for a space which is flat in the transverse dimensions, the overlap integral, in terms of light-cone coordinates, is

$$(\phi_{\omega,\mathbf{k}}, \phi_{\omega',\mathbf{k}'})_{KG} = i \int dv d^{d-2}\mathbf{x}_T [\phi_{\omega,\mathbf{k}}^*(u, v, \mathbf{x}_T) \partial_v \phi_{\omega',\mathbf{k}'}(u, v, \mathbf{x}_T) - \phi_{\omega',\mathbf{k}'}(u, v, \mathbf{x}_T) \partial_v \phi_{\omega,\mathbf{k}}^*(u, v, \mathbf{x}_T)]. \quad (8)$$

These functions obey the completeness relation

$$i \sum_{\mathbf{k}} \int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} (\phi_{\omega,\mathbf{k}}^*(x) n \cdot \partial \phi_{\omega,\mathbf{k}}(y) - \phi_{\omega,\mathbf{k}}(y) n \cdot \partial \phi_{\omega,\mathbf{k}}^*(x)) = \frac{\delta^4(x - y)}{\sqrt{-g}}. \quad (9)$$

In the case where the off-diagonal elements of the metric between space-like and time-like coordinates (g_{i0}) vanish, the orthonormality relation can be expressed in terms of the spatial functions $f_{\omega,\mathbf{k}}$ as

$$\int d^3x_i \sqrt{h} f_{\omega,\mathbf{k}}^*(x_i) f_{\omega',\mathbf{k}'}(x_i) = \delta(\omega - \omega') \delta(\mathbf{k} - \mathbf{k}') \quad (10)$$

and the completeness relation becomes

$$\sum_{\mathbf{k}} \int_0^\infty d\omega f_{\omega,\mathbf{k}}^*(x_i) f_{\omega,\mathbf{k}}(y_i) = \frac{\delta^3(x_i - y_i)}{\sqrt{h}}. \quad (11)$$

1.1 Quantum Fields

The free massless scalar quantum field, $\hat{\Phi}(x)$, is an operator which may be expanded in terms of the solutions, $\phi_{\omega,\mathbf{k}}(x)$ to the Klein-Gordon equation with coefficients which are creation and annihilation operators of a particle of energy ω , with other properties denoted by $\mathbf{k}$:

$$\hat{\Phi}(x) = \int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} \sum_{\mathbf{k}} \left(a_{\mathbf{k}}(\omega) \phi_{\omega,\mathbf{k}}(x) + a_{\mathbf{k}}^\dagger(\omega) \phi_{-\omega,\mathbf{k}}(x) \right). \quad (12)$$

This annihilates a particle with positive energy or creates a particle with negative energy. In the case of a scalar field for a particle with vacuum quantum numbers, the sum over $\mathbf{k}$ means integrating over all momenta whose magnitude is ω

$$\sum_{\mathbf{k}} \equiv \int d^3\mathbf{k} \delta(|\mathbf{k}| - \omega),$$

but in other cases it may also include sums over helicity, charge, etc.

The canonical momentum, $\hat{\Pi}(x)$ is defined as

$$\hat{\Pi}(x) \equiv \frac{\delta L}{\delta(\partial_0 \hat{\Phi})} = \sqrt{-g} g^{0\mu} \partial_\mu \hat{\Phi}(x) \quad (13)$$

The equal time commutation relation

$$\left[\hat{\Phi}(\mathbf{x}, t), \hat{\Pi}(\mathbf{y}, t), \right] = i \delta^3(\mathbf{x} - \mathbf{y}),$$

are satisfied provided the creation and annihilation operators obey the commutation relations

$$\left[a_{\mathbf{k}}(\omega), a_{\mathbf{k}'}^\dagger(\omega') \right] = (2\pi)(2\omega) \delta(\omega - \omega') \delta(\mathbf{k} - \mathbf{k}') \quad (14)$$

Using the orthonormality relations (5) the creation and annihilation operators can be extracted from (12) yielding:

$$\begin{aligned} a_{\mathbf{k}}^\dagger(\omega) &= \left(\phi_{\omega, \mathbf{k}}, \hat{\Phi} \right)_{KG} \\ a_{\mathbf{k}}(\omega) &= \left(\phi_{-\omega, \mathbf{k}}, \hat{\Phi} \right)_{KG} \end{aligned} \quad (15)$$

2 Bogoliubov Transformations

The expansion of the quantum field given in (12) relates to a set of coordinates, x^μ , (frame A) for which the metric tensor is $g_{\mu\nu}(x)$. One could instead express this in a set of coordinates, y^μ , (frame B) for which the metric is transformed to $\tilde{g}_{\mu\nu}$, given by (expressing x^μ as functions of y^ν) by

$$\tilde{g}_{\mu\nu}(y) = \frac{\partial x^\rho}{\partial y^\mu} \frac{\partial x^\sigma}{\partial y^\nu} g_{\rho\sigma}(x) \quad (16)$$

In this metric the solutions to the monochromatic wave equation are written

$$\tilde{\phi}_{\Omega, \mathbf{l}}(y) = e^{-i\Omega y^0} \tilde{f}_{\Omega, \mathbf{l}}(y_i) \quad (17)$$

Note that “monochromatic” means monochromatic in frame B , with coordinates y and these are, in general, *not* monochromatic in frame A , with coordinates x , since $y^0 \neq x^0$. This means that the monochromatic waves (frequency Ω) in frame B are a superposition

of monochromatic waves in frame A of frequency ω . Importantly, it is possible that such superpositions contain waves with *negative* as well as positive values of ω . In Special Relativity where the transformation of coordinates is linear, the frequencies in the two frames are related by the Doppler effect, but for a more general transformation y_0 is a non-linear function of x^0 and the relation between monochromatic waves in the two coordinate systems is more complicated. A monochromatic wave (which has infinite length) in one coordinate system is a localized wavepacket in another coordinate system.

A scalar quantum field is invariant under a coordinate transformation, so that instead of (12), we can expand the quantum field in terms of the monochromatic solutions in frame B

$$\hat{\Phi}(y) = \int_0^\infty \frac{d\Omega}{(2\pi)(2\Omega)} \sum_{\mathbf{l}} \left(b_{\mathbf{l}}(\Omega) \phi_{\Omega, \mathbf{k}}(y) + b_{\mathbf{l}}^\dagger(\Omega) \phi_{-\Omega, \mathbf{l}}(y) \right). \quad (18)$$

Here $b_{\mathbf{l}}^\dagger(\Omega)$, $b_{\mathbf{l}}(\Omega)$ are creation and annihilation operators for a particle of energy Ω (and other properties denoted by $\mathbf{l}$) in frame B .

Equating the two expressions for the quantum field (18) and (12), we see that the creation and annihilation operators are not the same in the two frames, but the creation/annihilation operators in one frame can be expanded as a superposition of creation/annihilation operators in the other frame.

$$\begin{aligned} b_{\mathbf{l}}(\Omega) &= \int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} \sum_{\mathbf{k}} \left[\alpha_{\mathbf{k}, \mathbf{l}}(\omega, \Omega) a_{\mathbf{k}}(\omega) + \beta_{\mathbf{k}, \mathbf{l}}(\omega, \Omega) a_{\mathbf{l}}^\dagger(\omega) \right] \\ b_{\mathbf{l}}^\dagger(\Omega) &= \int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} \sum_{\mathbf{k}} \left[\beta_{\mathbf{k}, \mathbf{l}}^*(\omega, \Omega) a_{\mathbf{l}}(\omega) + \alpha_{\mathbf{k}, \mathbf{l}}^*(\omega, \Omega) a_{\mathbf{l}}^\dagger(\omega) \right] \end{aligned} \quad (19)$$

Making use of (15) which projects out the creation and annihilation operators, we obtain

$$\begin{aligned} \alpha_{\mathbf{k}, \mathbf{l}}(\omega, \Omega) &= \left(\phi_{\omega, \mathbf{k}}, \tilde{\phi}_{\Omega, \mathbf{l}} \right)_{KG} \\ \beta_{\mathbf{k}, \mathbf{l}}(\omega, \Omega) &= \left(\phi_{\omega, \mathbf{k}}, \tilde{\phi}_{-\Omega, \mathbf{l}} \right)_{KG} = \left(\phi_{-\omega, \mathbf{k}}, \tilde{\Phi}_{\Omega, \mathbf{l}} \right)_{KG}^*, \end{aligned} \quad (20)$$

where in the last step, we have used

$$\phi_{\omega, \mathbf{k}} = \phi_{-\omega, \mathbf{k}}^*, \quad \text{and} \quad \tilde{\phi}_{-\Omega, \mathbf{l}} = \tilde{\phi}_{\Omega, \mathbf{l}}^*$$

The important ingredients are:

1. A monochromatic wave in coordinate system A , is *not* a monochromatic wave in a different coordinate system, B .
2. A monochromatic wave in coordinate system A can be expressed as a sum (integral) over monochromatic wavefunctions in system B .
3. Such a superposition generally includes negative frequency wavefunctions in system B .

4. Since a quantum field can be expanded as a sum (integral) over positive frequency monochromatic wavefunctions whose coefficients are annihilation operators plus a superposition of negative frequency wavefunctions whose coefficients are creation operators.
5. This implies that a creation operator in coordinate system A is a superposition of both creation and annihilation operators in coordinate system B and vice versa.
6. This, in turn, means that the particle content of a given Fock-space state is different in the two coordinate systems.

3 The Unruh Effect

The mixing between the creation and annihilation operators in the two frames has consequences for the number densities of particles in a given energy range in the two frames:

$$a_k^\dagger(\omega)a_k(\omega) \neq b_l^\dagger(\Omega)b_l(\Omega).$$

In particular, for the vacuum state in frame A , $|0\rangle_A$, the expectation value of $a_k^\dagger(\omega)a_k(\omega)$ vanishes (for all ω and $\mathbf{k}$) whereas the expectation value of $b_l^\dagger(\Omega)b_l(\Omega)$, does *not* vanish. In other words this state $|0\rangle_A$, is *not* a vacuum state in frame B but has a non-zero particle content.

Expanding the creation and annihilation operators, $b_l^\dagger(\Omega)$, $b_l(\Omega)$ in frame B in terms of creation and annihilation operators, $b_l^\dagger(\Omega)$, $b_l(\Omega)$ in frame B in terms of creation and annihilation operators, $a_k^\dagger(\omega)$, $a_k(\omega)$ in frame a , and using

$${}_A\langle 0|a_{\mathbf{k}}(\omega)a_{\mathbf{k}'}^\dagger(\omega')|0\rangle_A = (2\pi)\delta(\omega - \omega')\delta(\mathbf{k} - \mathbf{k}'),$$

we find that the vacuum in frame A has a non-zero number density in frame B

$${}_A\langle 0|b_l^\dagger(\Omega)b_l(\Omega)|0\rangle_A = \sum_{\mathbf{k}} \int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} |\beta_{\mathbf{k},1}(\omega, \Omega)|^2. \quad (21)$$

The non-zero value of the expectation value of the number density in the B -frame for the vacuum state in the A -frame is known as the “Unruh effect”.

The RHS of (21) simplifies considerably if the ratio

$$R(\Omega) \equiv \left| \frac{\alpha_{\mathbf{k},1}(\omega, \Omega)}{\beta_{\mathbf{k},1}(\omega, \Omega)} \right| \quad (22)$$

depends only on Ω . This arises from the fact that the creation and annihilation operators in both frames obey the same commutation relations. Starting with the commutation relations

$$[b_l(\Omega), b_{l'}^\dagger(\Omega')] = (2\pi) 2\Omega \delta(\Omega - \Omega') \delta(l - l')$$

Using the expansions (19) and the fact that creation/ annihilation operators commute among themselves, this leads to

$$\begin{aligned} & \int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} \int_0^\infty \frac{d\omega'}{(2\pi)(2\omega')} \sum_{\mathbf{k}, \mathbf{k}'} (\alpha_{\mathbf{k}, \mathbf{l}}^*(\omega, \Omega) \alpha_{\mathbf{k}', \mathbf{l}'}(\omega', \Omega') - \beta_{\mathbf{k}, \mathbf{l}}^*(\omega, \Omega) \beta_{\mathbf{k}', \mathbf{l}'}(\omega', \Omega')) \left[a_{\mathbf{k}(\omega)} a_{\mathbf{k}'}^\dagger(\omega') \right] \\ &= (2\pi) 2\Omega \delta(\Omega - \Omega') \delta(\mathbf{l} - \mathbf{l}') \end{aligned} \quad (23)$$

Applying the commutation relation (14) and performing the integral over ω' and sum over $\mathbf{k}'$ to absorb the δ -functions, this reduces to

$$\int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} \sum_{\mathbf{k}} (\alpha_{\mathbf{k}, \mathbf{l}}^*(\omega, \Omega) \alpha_{\mathbf{k}, \mathbf{l}'}(\omega, \Omega') - \beta_{\mathbf{k}, \mathbf{l}}^*(\omega, \Omega) \beta_{\mathbf{k}, \mathbf{l}'}(\omega, \Omega')) = (2\pi) 2\Omega \delta(\Omega - \Omega') \delta(\mathbf{l} - \mathbf{l}'), \quad (24)$$

and finally, using (22), we end up with

$$\sum_{\mathbf{k}} \int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} \beta_{\mathbf{k}, \mathbf{l}}^*(\omega, \Omega) \beta_{\mathbf{k}, \mathbf{l}'}(\omega, \Omega') = \frac{1}{(|R(\Omega)|^2 - 1)} (2\pi) 2\Omega \delta(\Omega - \Omega') \delta(\mathbf{l} - \mathbf{l}'), \quad (25)$$

3.1 Rindler Space

As a concrete example, we consider the transformation from Minkowski space coordinates, x^μ to Rindler space coordinates y^μ

$$\begin{aligned} v \equiv x^0 + x^3 &= \frac{1}{a} e^{y^3} e^{ay^0} \equiv \frac{1}{a} e^{av^*} \\ u \equiv x^0 - x^3 &= -\frac{1}{a} e^{y^3} e^{-ay^0} \equiv -\frac{1}{a} e^{-au^*} \\ \mathbf{x}_T \equiv (x^1, x^2) &= \mathbf{y}_T, \end{aligned} \quad (26)$$

where u^* and v^* are the light-cone coordinates in Rindler space.

For small relative velocities ($|y^3| \ll |y^0|, \ll 1/a$) this is a transformation to a uniformly accelerated frame (acceleration a in the z -direction) for which

$$\begin{aligned} x^0 &\approx y^0 \\ x^3 &\approx \tilde{y} + \frac{1}{2}a(y^0)^2, \end{aligned}$$

where

$$\tilde{y} = y^3 - \frac{1}{a}$$

Note that the entire range of coordinates is mapped into the restricted range in Minkowski space:

$$0 < v < \infty, \quad -\infty < u < 0.$$

The lines $u = 0, v = 0$ determine a horizon. The signal from any event outside this allowed range outside this range will never reach an observer in the accelerated frame.

The metric in Minkowski space is

$$ds^2 = (dx^0)^2 - (dx^3)^2 - (d\mathbf{x}_T)^2 = du dv - (d\mathbf{x}_T)^2$$

The wave equation is

$$\left(\frac{\partial^2}{\partial u \partial v} - \frac{\partial^2}{(\partial \mathbf{x}_T^2)^2} \right) \phi(x) = 0 \quad (27)$$

For the moment, we restrict ourselves to waves moving in the longitudinal direction (along

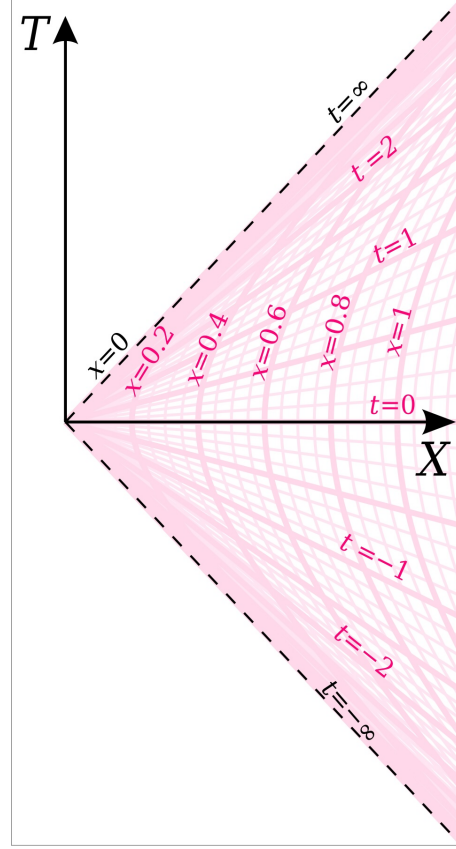


Figure 1: Rindler space coordinates (in red). The hyperbolae are paths of constant y^3 . The dashed lines are the horizon. The light signal from any event outside this horizon cannot reach the accelerating observer in Rindler space.

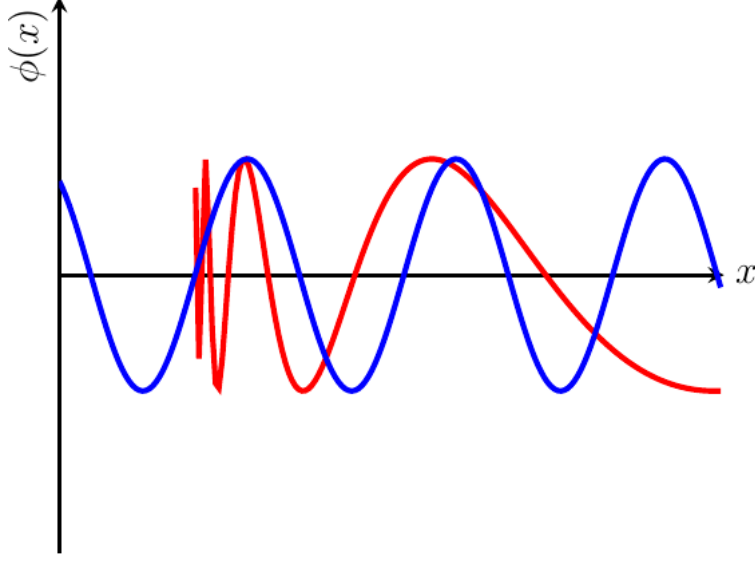


Figure 2: Monochromatic wave in the coordinates of an inertial observer(blue) and in the (Rindler) coordinates of an observer in an accelerating frame

the x^3 axis, for which the monochromatic solutions are either

$$\begin{aligned}\phi_{\omega,\text{in}} &= e^{-i\omega v} \text{ for ingoing waves, and} \\ \phi_{\omega,\text{out}} &= e^{-i\omega u} \text{ for outgoing waves.}\end{aligned}\tag{28}$$

In the Rindler frame, the metric is

$$ds^2 = e^{2ay^3} \left((dy^0)^2 - (dy^3)^2 \right) - (dy^1)^2 - (dy^2)^2,$$

For waves moving in the longitudinal; direction, the wave equation is simply

$$\left(\frac{\partial^2}{(\partial y^0)^2} - \frac{\partial^2}{(\partial y^3)^2} \right) \tilde{\phi}(y) \equiv \frac{\partial}{\partial u^*} \frac{\partial}{\partial v^*} \tilde{\phi}(u^*, v^*) = 0$$

This has ingoing monochromatic solutions

$$\tilde{\phi}_{\Omega,\text{in}}(v^*) = e^{-i\Omega v^*}$$

and outgoing monochromatic solutions

$$\tilde{\phi}_{\Omega,\text{out}}(u^*) = e^{-i\Omega u^*}$$

In order to perform the Klein-Gordon overlap integrals for the ingoing waves, we need to write $\tilde{\phi}_{\text{in},\Omega}$ as functions the (light-cone) Minkowski space coordinates (u, v) using the inverse of (26).

. We choose to write these in terms of light-cone coordinates i

$$\tilde{\phi}_{\text{in},\Omega} = e^{-i\Omega/a \ln(av)} = (av)^{-i\Omega/a}.$$

Using the expression (8) for the Klein-Gordon overlap integrals, and recalling that the limits on v in Rindler space are 0 to ∞ , we have (after integrating by parts)

$$\left(\phi_{\text{in},\omega}, \tilde{\phi}_{\text{in},\Omega}\right)_{KG} = i \int_0^\infty dv \left(e^{i\omega v} \partial_v (a v)^{-i\Omega/a} - (a v)^{-i\Omega/a} \partial_v e^{i\omega v} \right) = 2\omega \int_0^\infty dv e^{i\omega v} (a v)^{i\Omega/a} \quad (29)$$

We do *not* need to evaluate this integral, but simply note that in order for the integrand to vanish at the end point, $v \rightarrow \infty$ we require ω to have an infinitesimal positive imaginary part, i.e. the integral is analytic in the upper ω plane, $\Im m\{\omega\} > 0$. Secondly we note that changing integration variable to $z = \omega v$, we get

$$\left(\phi_{\text{in},\omega}, \tilde{\phi}_{\text{in},\Omega}\right)_{KG} = \left(\frac{\omega}{a}\right)^{-i\Omega/a} I(\Omega/a), \quad (30)$$

where $I(\Omega/a)$ is independent of ω .

Now from (20) we see that in order to determine (the modulus of) the ratio of $\alpha(\omega, \Omega)$ and $\beta(\omega, \Omega)$ we need to reverse the (real part of) ω . We need to do this by rotating Ω *anti-clockwise* so that it remains in the analytic lower half-plane, i.e.

$$\omega \rightarrow e^{i\pi} \omega.$$

We see from (30) that this introduces a factor

$$(e^{i\pi})^{-i\Omega/a} = e^{-\Omega\pi/a},$$

into $\beta(\omega, \Omega)$ so that we have the ratio

$$|R(\Omega)| \equiv \left| \frac{\alpha(\omega, \Omega)}{\beta(\omega, \Omega)} \right| = e^{\pi\Omega/a},$$

The integral of $|\beta_{\omega,\Omega}|^2$ over all ω diverges. This manifests itself in the appearance of $\delta(0)$, when setting $\Omega = \Omega'$ in (25). This problem is circumvented by constructing wavepackets out of the convolution of monochromatic waves with sharply peaked smoothing function, F , of width Δ . With this convolution, the number density $N_{\text{in}}(\Omega)$ of incoming photons (moving along the longitudinal axis) with frequencies between $\Omega - \Delta/2$ and $\Omega + \Delta/2$,

$$N_{\text{in}}(\Omega)\Delta = \int_0^\infty \frac{d\omega}{(2\pi)(2\omega)} |\bar{\beta}_{\text{in}}(\omega, \Omega)|^2, \quad (31)$$

where

$$\bar{\beta}_{\text{in}}(\omega, \Omega) = \int \frac{d\Omega'}{(2\pi)(2\Omega')} F_T(\Omega, \Omega') \beta_{\text{in}}(\omega, \Omega') \quad (32)$$

The function F_T is a normalized narrow peaked smoothing function, which transforms an infinite length purely monochromatic wave into a finite length wavepacket. and $\bar{\beta}_{\text{in}}(\omega, \Omega)$ is the smoothed β coefficient

$$\bar{\beta}_{\text{in}}(\omega, \Omega) = \int \frac{d\Omega'}{(2\pi)(2\Omega')} F_T(\Omega, \Omega') \beta_{\text{in}}(\omega, \Omega') \quad (33)$$

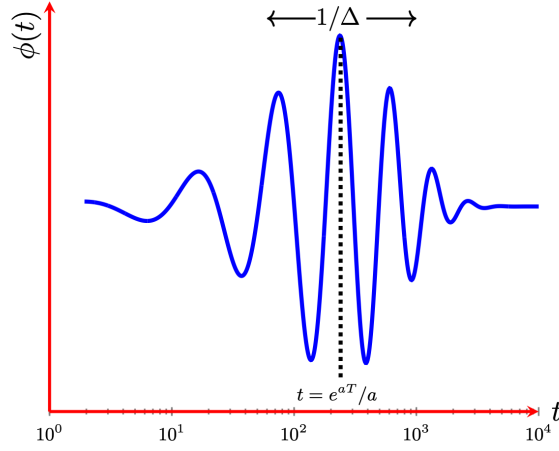


Figure 3: Wavepacket consisting of a monochromatic waves with a gaussian envelope of with $1/\Delta$.

A suitable smoothing function, $F_T(\Omega, \Omega')$ could be the Gaussian function

$$F_T(\Omega, \Omega') = \frac{\sqrt{2\pi}}{\Delta} (2\Omega') \exp \left(-\frac{(\Omega - \Omega')^2}{2\Delta^2} + i(T(\Omega - \Omega')) \right), \quad (34)$$

where T is the centre of the wavepacket.

The monochromatic wavefunction $\tilde{\phi}_\Omega$ is smoothed into a function $\bar{\Phi}_\Omega$, where

$$\bar{\Phi}_\Omega(y) = \int \frac{d\Omega'}{(2\pi)(2\Omega')} F_T(\Omega, \Omega') \left(ay^3 e^{ay^0} \right)^{i\Omega'/a} \quad (35)$$

Inserting (34) into (32), performing the Gaussian integral, and expressing it in terms of light cone coordinates using (26) we get

$$\bar{\Phi}_\Omega(v) = \exp \left(-\left(\frac{1}{a} \ln(av) - T \right)^2 \frac{\Delta^2}{2} \right) (av)^{i\Omega/a} \quad (36)$$

This is a monochromatic wave with a Gaussian envelope of width $1/\Delta$, and centred at $v = e^{aT}/a$.

In order to construct the smoother β coefficient $\bar{\beta}(\omega, \Omega)$, we need to form a convolution of the coefficients $\alpha(\omega, \Omega)$ and $\beta(\omega, \Omega)$ with the smoothing function F . We make the approximation that the smoothing function is sufficiently sharply peaked that the change in $I\Omega/a$ over the range of support the smoothing function F is negligible and so in analogy with (36)

$$(\phi_{in,\omega}, \bar{\phi}_{in,\Omega}) \approx I(\Omega/a) \left(\frac{\omega}{a} \right)^{i\Omega/a} \exp \left(-\left(\frac{1}{a} \ln\left(\frac{\omega}{a}\right) - T \right)^2 \frac{\Delta^2}{2} \right) \quad (37)$$

With this smoothing, the integral of $|\bar{\beta}(\omega, \Omega)|^2$ over ω now converges and we see that it is peaked at

$$\omega = ae^{aT}.$$

and from (31)

$$N_{in}(\Omega)\Delta\Omega = \frac{1}{(e^{2\pi\Omega/a} - 1)}. \quad (38)$$

This is the Planck factor for blackbody radiation with temperature

$$T_{\text{Unruh}} = \frac{a}{2\pi} \equiv \frac{\hbar a}{2\pi k_B c}, \quad (\text{in general units}). \quad (39)$$

This is a very small temperature. For example an observer whose acceleration is equal to the acceleration due to the Earth's gravity at its surface, experiences a temperature of 4×10^{-20} K!

Note that a Bogoliubov transformation transforms between two different coordinate systems and *not* between systems with different geometries. Minkowski space and Rindler space both have all components of the Riemann tensor set to zero so that there is no curvature. However, it transforms to a coordinate system with uniform acceleration (for small velocities) and therefore acts as a good approximation to the coordinate system an observer in a uniform gravitational field.

Clearly we get the same result for the density of states for outgoing particles in the longitudinal direction – this is obtained by interchanging u and v .

For waves in a direction which moves at an angle θ to the x^1 direction we show that the ratio of the coefficients is unchanged.

For the Minkowski space ingoing waves, the wavefunction can be written as

$$\phi_{in,\theta,\mathbf{n},\omega}(u, v, \mathbf{x}_T) = \frac{e^{i\omega\mathbf{k}_T \cdot \mathbf{x}_T}}{\sqrt{2\pi}} e^{-i\omega(\cos^2 \theta/2)v} e^{-i\omega(\sin^2(\theta/2)u},$$

where $\sin \theta \equiv |\mathbf{k}_T|/\omega$.

In Rindler space the ingoing wave equation is

$$\left[\frac{1}{ay^3} \frac{\partial^2}{(\partial y^0)^2} - ay^3 \frac{\partial}{\partial y^3} \left(y^3 \frac{\partial}{\partial y_1} \right) - \frac{\partial^2}{\partial \mathbf{y}_T^2} \right] \tilde{\phi}_{in,\theta,\Omega}(y) = 0 \quad (40)$$

This has solution

$$\tilde{\phi}_{in,\mathbf{k}_T,\Omega}(y) = \frac{e^{i\Omega\mathbf{k}_T \cdot \mathbf{y}_T}}{\sqrt{2\pi}} e^{-i\Omega y^0} \mathcal{I}_{-i\Omega/a} \left(\frac{|\mathbf{k}_T| e^{y^3}}{a} \right), \quad (41)$$

where $\mathcal{I}_\nu(x)$ is the modified Bessel function of order ν .

Rewriting this in terms of the Minkowski-space coordinates, we have

$$\tilde{\phi}_{in,\theta,\Omega}(u, v, \mathbf{x}_T) = \frac{e^{i\Omega \mathbf{k}_T \cdot \mathbf{x}_T}}{\sqrt{2\pi}} \left(\frac{v}{u}\right)^{-i\Omega/2a} \mathcal{I}_{-i\Omega/a}(|\mathbf{k}_T| \sqrt{uv}), \quad (42)$$

The Klein-Gordon overlap is expressed in terms of the behaviours of the two wavefunctions in the variable v , at some fixed value of u . We are at liberty to select this value of u to be (close to) zero - i.e. at the horizon of the Rindler space. In this limit, using the small argument expansion of modified Bessel functions, we have

$$\mathcal{I}_{-i\Omega/a}(|\mathbf{k}_T| \sqrt{uv}) \sim (uv)^{-i\Omega/2a},$$

so that

$$\lim_{u \rightarrow 0} \{\phi_{in,\mathbf{k}_T,\Omega}(u, v, \mathbf{x}_T)\} = \frac{e^{i\Omega \mathbf{k}_T \cdot \mathbf{x}_T}}{\sqrt{2\pi}} e^{-i\omega \cos^2(\theta/2)v} \quad (43)$$

and

$$\lim_{u \rightarrow 0} \{\tilde{\phi}_{in,\mathbf{l}_T,\Omega}(u, v, \mathbf{x}_T)\} \sim \frac{e^{i\Omega \mathbf{l}_T \cdot \mathbf{x}_T}}{\sqrt{2\pi}} v^{-i\Omega/a} \quad (44)$$

The Klein-Gordon overlap integral is (up to a factor)

$$\begin{aligned} \left(\phi_{in,\mathbf{k}_T,\Omega}, \tilde{\phi}_{in,\mathbf{l}_T,\Omega}\right)_{KG} &= \int \frac{d^2 \mathbf{x}_T}{(2\pi)} e^{-i(\mathbf{k}_T + \mathbf{l}_T) \cdot \mathbf{x}_T} \\ &\times \int dv \left(e^{i\omega \cos^2(\theta/2)v} \partial_\nu v^{-i\Omega/a} - v^{-i\Omega/a} \partial_\nu e^{i\omega \cos^2(\theta/2)v} \right) \end{aligned} \quad (45)$$

The integration of the transverse directions generates a transverse momentum-conserving δ -function, $\delta^2(\mathbf{k}_T - \mathbf{l}_T)$, meaning that there is no mixing of creation and annihilation operators for particles with different transverse momenta (the components of the metric in the transverse directions are the same in Minkowski and Rindler space).

For the integration over v , we have precisely the same argument as in the forward case, except that we make the change of variable $v = \omega \cos^2(\theta/2)z$ and (30) is replaced by

$$\left(\phi_{in,\omega}, \tilde{\phi}_{in,\Omega}\right)_{KG} = (\omega)^{i\Omega/a} (\cos(\theta/2))^{2i\Omega/a} I(\Omega/a), \quad (46)$$

Again, this is analytic in the upper ω plane and reversing the real part of ω introduces the same Boltzmann factor as for the forward case.

4 Hawking Radiation

4.1 Black Holes

In this section, we work in natural units, $G = \hbar = c = k_B = 1$.

A black hole is a concentration of mass, M , which generates such a curvature of space-time that in purely classical mechanics any particle of matter that falls behind a given horizon of radius R_H can never emerge again.

For a black hole with zero electric charge and zero angular momentum, the metric is given by the Schwarzschild

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (47)$$

In 1974 Hawking showed that in semi-classical quantum theory (meaning that gravity is treated classically without attempting to quantise it, whereas all other particles are treated quantum mechanically), this classical picture was not correct and that black holes did indeed radiate blackbody radiation with a (Hawking) temperature which was inversely proportional to the mass, M , of the black hole.

There have been many attempts to explain this radiation qualitatively, including an explanation by Hawking himself in terms of a vacuum fluctuation close to the horizon, producing a virtual particle antiparticle pair, one of which can tunnel over the horizon. In general, these arguments are misleading.

Although Hawking radiation is a separate phenomenon from the Unruh effect, and cannot be explained in terms of the equivalent acceleration of an observer in the gravitational field of a black hole, its explanation lies in the fact that creation operators for particles of a given energy (monochromatic waves) in one coordinate system are superpositions of *both* creation and annihilation operators in another coordinate systems (and similarly for annihilation operators).

4.2 Tortoise coordinates

In the Schwarzschild metric, the wave equation for spherically symmetric waves (no angular dependence or zero angular momentum) is

$$\left[r \left(1 - \frac{2M}{r}\right)^{-1} \partial_t^2 - \frac{1}{r} \partial_r \left(1 - \frac{2M}{r}\right) r^2 \partial_r \right] \phi(r, t) = 0 \quad (48)$$

$$\text{Let } \phi(r, t) = \frac{1}{r} f(r, t),$$

The wave equation becomes

$$\left[\left(1 - \frac{2M}{r}\right)^{-1} \partial_t^2 - \left(1 - \frac{2M}{r}\right) \partial_r^2 - \frac{2M}{r^2} \left(\partial_r - \frac{1}{r}\right) \right] f(r, t) = 0 \quad (49)$$

If we now change the radial variable r to the “tortoise coordinate”

$$r^* = r + 2M \ln \left| \frac{r}{2M} - 1 \right| \quad (50)$$

In these coordinates the metric becomes

$$ds^2 = \left(1 - \frac{2M}{r} \right) (dt^2 - dr^{*2}) - r^2 d\Omega^2. \quad (51)$$

The radial light-like geodesics ($ds = 0$) obey the relations $t = \pm r^*$.

The metric can also be written

$$ds^2 = \left(1 - \frac{2M}{r} \right) dv^2 - 2dvdr - r^2 d\Omega^2, \quad (52)$$

with $v = t + r^*(r)$, the advanced time in Schwarzschild space.

The wave equation, (49), becomes

$$(\partial_t^2 - \partial_{r^*}^2) f(r, t) = - \left(1 - \frac{2M}{r} \right) \frac{2M}{r^3} f(r, t) \quad (53)$$

The RHS vanishes close to the horizon and at large distances and may be set to zero without affecting the solution in the regions of interest.

In terms of tortoise light cone coordinates $v^* \equiv (t + r^*)$ and $u^* \equiv (t - r^*)$ this has (approximate) solutions

$$\begin{aligned} f_{\Omega, in} &= e^{-i\Omega v^*} \text{ for ingoing waves} \\ f_{\Omega, out} &= e^{-i\Omega u^*} \text{ for outgoing waves} \end{aligned} \quad (54)$$

It is important to note that although these are monochromatic waves in Schwarzschild space, they are *not* monochromatic waves in Minkowski space.

4.3 Vaidya Space

Vaidya space is a space-time which at sufficiently early times is Minkowski in the neighbourhood of an observer who is at (any) distance r_0 from the origin. A collapsing shock-wave on the light surface $v \equiv t + r = 0$ passes the observer at time $-t = r_0$ and then shrinks to form a black hole centred at the origin, with surface gravity (inversely proportional to the black-hole mass) $\kappa = 1/4M$. After the shock-wave has passed the observer, the space in the neighbourhood of the observer is Schwarzschild. See Fig.(4).

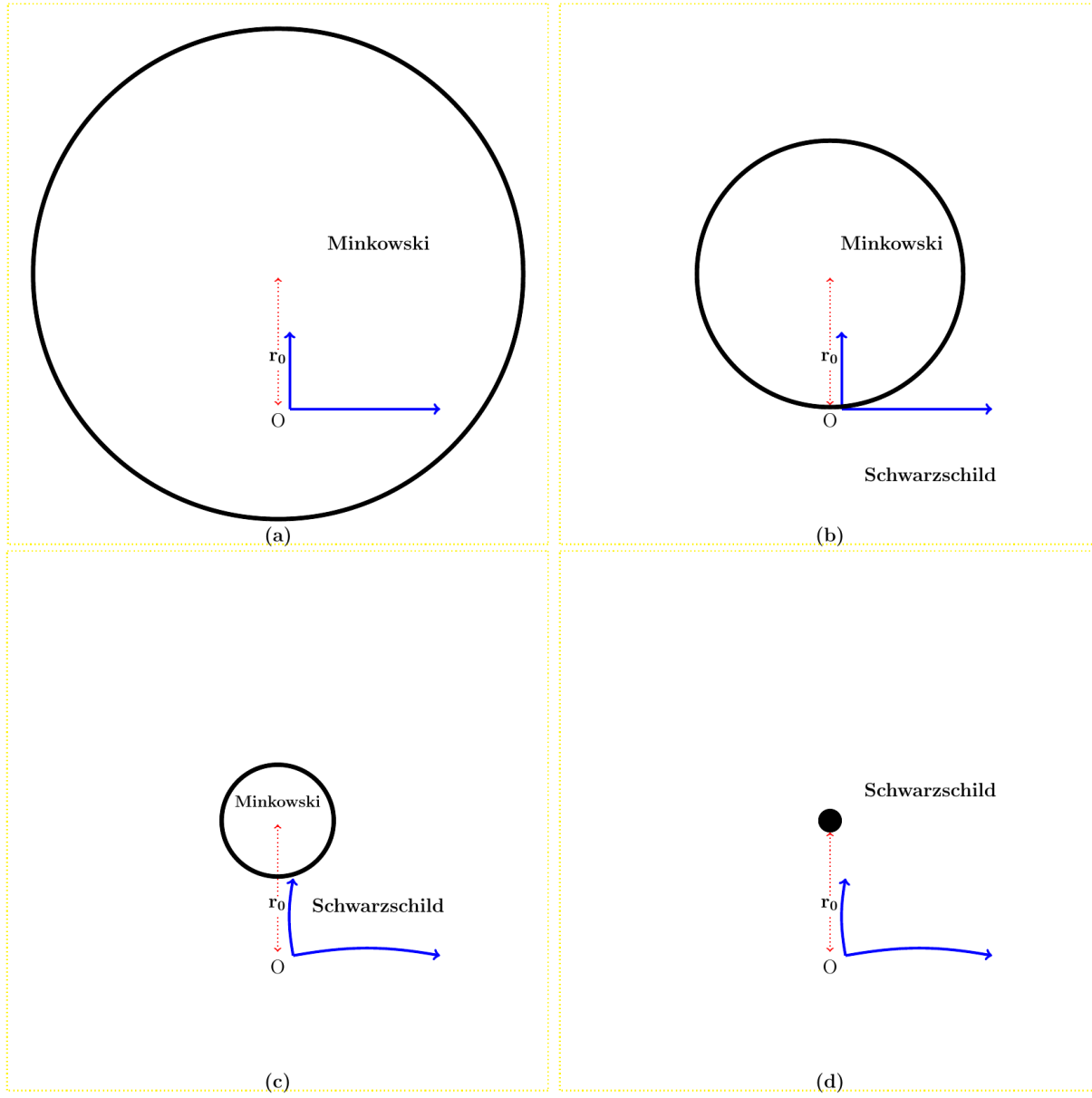


Figure 4: Observer O at a fixed distance, r_0 , from the black-hole centre. (a) Observer is inside collapsing shell – observer’s metric is Minkowski, (b) Shell crosses observer – observer’s metric changes from Minkowski to Schwarzschild. (c) Shell has passed the observer – observer’s metric is Schwarzschild, (d) Shell forms a black hole at the origin ($r = 0$) at time $t = 0$. The black hole is stationary in the frame of the observer.

The metric which describes such a space can be written

The metric which describes such a space can be written

$$ds^2 = \left(1 - \frac{2M}{r}\theta(v)\right) dv^2 - 2dv dr - \left(1 - \frac{2M}{r}\theta(v)\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (55)$$

where

$$\begin{aligned} v &\equiv t + r, \quad (v < 0) \\ &\equiv t^* + r^*, \quad (v > 0) \end{aligned} \tag{56}$$

Note that the time coordinate is *not* the same in the two regions.¹

An outgoing monochromatic wave in the Schwarzschild region has a wavefunction

$$e^{-i\Omega u^*},$$

where $u^* = v - 2r^*$. This is not a monochromatic solution in the Minkowski region, but it is possible to expand it as a superposition of such Minkowski-space outgoing monochromatic waves and this implies that u^* can be expressed as a function of u ,

In order to find such a function we express the radial component r as a function of v and u

$$r(v, u) = \frac{1}{2}(v - u)$$

and also as a function of v and u^*

$$r^*(v, u^*) = r(v, u^* + 2M \ln \left| \frac{r(v, u^*)}{2M} - 1 \right| = \frac{1}{2}(v - u^*)$$

At the shock-wave, $v = 0$ the value of r must have the same value in both formulations, in order to avoid a radial discontinuity at the shock-wave,

$$\text{Setting } r(v = 0, u) = r(v = 0, u^*),$$

we find

$$u^*(u) = u - 4M \ln \left(\frac{|v_H - u|}{4M} \right), \tag{57}$$

where $v_H \equiv -4M$ is the largest value of an ingoing light-surface, v , which can be reflected at the origin and transmitted out to infinite distance. Any light signal with $v > v_H$ is trapped by the collapsing shock-wave and cannot escape to infinity.

This space is often depicted by a Penrose diagram – (see Fig.5).

¹The θ -function at $v = 0$ should be smoothed out to avoid a singularity in the wave equation at $v = 0$.

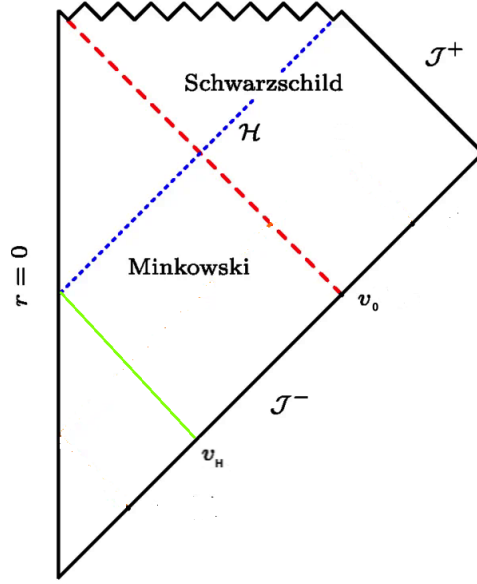


Figure 5: Penrose diagram: The horizontal axis is the radial distance and the vertical axis is time. The advanced and retarded light-cone coordinates v and u , are at 45° to the horizontal and vertical axes respectively. The line $v = 0$ partitions the diagram into a Minkowski space region and a Schwarzschild space region. $\mathcal{J}^-$ is the ingoing light surface for $u \rightarrow -\infty$ and $\mathcal{J}^+$ is the outgoing light surface for $v \rightarrow +\infty$. The line marked $\mathcal{H}$ is the path of the last light-ray, whose ingoing light surface is $v = v_H$, that can be reflected from the origin and transmitted to infinity.

At late time, a monochromatic wavefunction is

$$\frac{1}{r} e^{-i\Omega u^*(u)} = \frac{1}{r} e^{-i\Omega u} \left(\frac{|v_H - u|}{4M} \right)^{4iM\Omega}. \quad (58)$$

However, for early advanced time $v \leq v_H$, the wave can be reflected from the origin and so we need to add a term which depends on v such that the wavefunction is finite at the origin. When this is accounted for, we end up with a monochromatic wavefunction

$$\tilde{\Phi}_\Omega(u, v) = \frac{1}{r} \left(e^{-i\Omega u^*(u)} - e^{-i\Omega u^*(v)} \right). \quad (59)$$

Note that at $r = 0$, where $u = v$, the term in parenthesis vanishes, thereby cancelling out the $1/r$ singularity. This wavefunction is shown schematically for different times in Fig.6

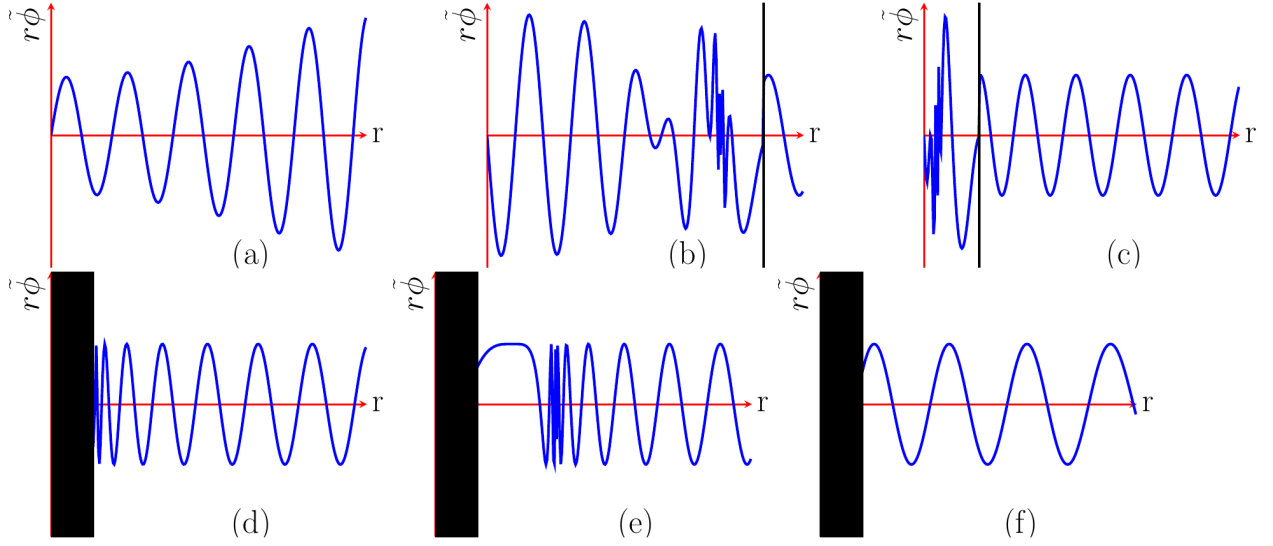


Figure 6: The wavefunction $\tilde{\phi}$ as a function of radial distance at different times
(a) Early time - the wave is a Minkowski-space standing wave,
(b) The shock-wave (black vertical line) comes in causing distortion of the standing waves to its left and distorted outgoing waves to the right,
(c) The shock-wave pushes the distorted wave toward the horizon,
(d) The black hole is formed, waves to the right of the horizon (the right edge of the black rectangle) are outgoing ,
(e) A distorted wave from the shock-wave moves outward from the black hole's horizon
(f) At late time a monochromatic wave is emitted from the horizon.

4.4 Bogoliubov Transformations

Creation and annihilation operators in Schwarzschild space are Bogoliubov transforms of the operators in Minkowski space, with coefficients

$$\alpha(\omega, \Omega) = \left(\phi_\omega, \tilde{\phi}_\Omega \right)_{KG}.$$

We choose to perform the Klein-Gordon overlap integral in the light-cone v at some fixed value of u . Note that in order for r to be positive, the lower limit of the integral over v is u

$$\left(\phi_\omega, \tilde{\phi}_\Omega \right)_{KG} = \int r^2 \int d\Omega \int_u^\infty dv \phi_\omega^*(u, v) \overset{\leftrightarrow}{\partial} \tilde{\phi}_\Omega(u, v)$$

This integral can be performed at any fixed value of u . However, it is most convenient to select $u \rightarrow -\infty$, i.e. the past infinity light surface $\mathcal{I}^-$, for which the lower limit of the integral over v is $-\infty$ and all the u -dependent terms vanish (when folded with a smoothing function the resulting finite-size wavepacket vanishes at sufficiently negative u). The wavefunction $\tilde{\phi}_\Omega(u \rightarrow -\infty, v)$ is only non-zero for $v < v_H$, so the range of the v integration is $-\infty < v < v_H$. It is important to note that this incoming (i.e. v -dependent) part of the wavefunction

which at sufficiently late times describes an outgoing monochromatic is the part which leads to non-zero Bogoliubov coefficients indicating that a vacuum in early Minkowski space has a non-zero particle density in late Schwarzschild space.

Performing the angular integral and integrating by parts, discarding the surface terms, we get

$$\left(\phi_\omega, \tilde{\phi}_\Omega\right)_{KG} = 8\pi\omega \int_{-\infty}^{v_H} dv e^{i(\omega-\Omega)v} \left(\frac{(v_H - v)}{4M}\right)^{4iM\Omega}$$

We note that the integrand converges as $v \rightarrow -\infty$ provided ω has a (small) negative imaginary part. The integral is there for analytic in the lower ω -plane and has a branch-point at $\omega = \Omega$.

Now we change the variable v

$$v \rightarrow i \frac{z}{(\omega - \Omega)},$$

to obtain

$$\left(\phi_\omega, \tilde{\phi}_\Omega\right)_{KG} = \omega I(\Omega) (\omega - \Omega)^{-4iM\Omega-1}, \quad (60)$$

where

$$I(\Omega) = 8\pi(-4Mi)^{-4iM\Omega} \int_0^\infty e^{-z} z^{4Mi\Omega} dz.$$

In order to obtain (the modulus of) the coefficient $\beta(\omega, \Omega)$ we need to reverse the sign of ω . Since the above integral is only analytic in the lower ω plane, this needs to be effected by the substitution

$$\omega \rightarrow e^{-i\pi}\omega,$$

and we find

$$\left|\frac{\beta(\omega, \Omega)}{\alpha(\omega, \Omega)}\right| = (e^{-i\pi})^{-4Mi\Omega} \frac{(\omega - \Omega)}{(\omega + \Omega)} = e^{-4M\pi\Omega} \frac{(\omega - \Omega)}{(\omega + \Omega)} \quad (61)$$

4.5 Wavepackets

Confining ourselves to purely monochromatic wavefunctions encounters the problem of a divergent integral over ω of the integrand $|\beta_{\omega,\Omega}|^2$. This results from the fact that a purely monochromatic wave has an infinite length. The problem is obviated by constructing localized wavepackets, albeit with a large width. This is achieved by convoluting the monochromatic waves over their frequencies with a sharply peaked smoothing function $F_{U_0}(\Omega, \Omega')$.

A suitable smoothing function with a peak of width Δ could be the Gaussian function

$$F_{U_0}(\Omega, \Omega') \equiv \frac{\sqrt{2\pi}}{\Delta} (2\Omega') e^{-(\Omega-\Omega')^2/2\Delta^2} e^{-i(\Omega-\Omega')U_0}, \quad (62)$$

where U_0 is the centre of the wavepacket.

The convolution of this smoothing function with a monochromatic wavefunction (in Schwarzschild space), $\tilde{\phi}_\Omega$ generates a wavepacket, $\bar{\phi}_\Omega$ with width $\sim 1/\Delta$:

$$\bar{\phi}_\Omega = \int \frac{d\Omega'}{(2\pi)(2\Omega')} F_{U_0}(\Omega, \Omega') \tilde{\phi}_{\Omega'}. \quad (63)$$

For example, at late retarded time $u^* = (t^* - r^*)$ in Schwarzschild coordinates, the monochromatic wave is $e^{-i\Omega u^*}$. After convolution with the smoothing function F_{U_0} , we obtain

$$\bar{\phi}_\Omega(u^*) = e^{-i\Omega u^*} e^{(-\Delta^2(U_0 - u^*)^2/2)} \quad (64)$$

This is a monochromatic wave with a Gaussian envelope centred at $u^* = U_0$ and width $1/\Delta$.

The coefficient $\alpha(\omega, \Omega)$ is now smoothed to

$$\bar{\alpha}(\omega, \Omega) = \int \frac{d\Omega'}{(2\pi)(2\Omega')} F_{U_0}(\Omega, \Omega') I(4M\Omega') (4Mi(\omega - \Omega'))^{4Mi\Omega' - 1} \quad (65)$$

If we assume that the smoothing function is sufficiently narrow so that the changes in $I(4M\Omega)$ and $\omega - \Omega$ are negligible over the range of support of F_{U_0} , then this may be approximated by

$$\bar{\alpha}(\omega, \Omega) \approx \alpha(\omega, \Omega) \exp\left(-(U_0 - v_H - 4M \ln(4M(\omega - \Omega)))^2 \Delta^2/2\right) \quad (66)$$

The exponential envelope renders the integral of $|\bar{\beta}(\omega, \Omega)|^2$ finite, and the integrand is peaked at

$$\omega - \Omega = \frac{1}{4M} e^{(U_0 - v_H)/4M}$$

An observer a long distance from the horizon of the black hole and at late times observes wavepackets with $U_0 \gg M$, for which $\omega \gg \Omega$, so that the ratio of $\alpha(\omega, \Omega)$ to $\beta(\omega, \Omega)$ becomes independent of ω and is approximated by

$$R(\Omega) \equiv \left| \frac{\bar{\alpha}(\omega, \Omega)}{\bar{\beta}(\omega, \Omega)} \right|_{\omega \gg \Omega} \approx e^{4M\pi\Omega}. \quad (67)$$

At a large distance from the black-hole horizon, the number of spherically symmetric particles radiated with frequency between $\Omega - \Delta/2$ and $\Omega + \Delta/2$, $\mathbf{k}_T$ is therefore

$$N_{\mathbf{k}_T, \lambda}(\Omega) \Delta = \frac{1}{|R(\Omega)|^2 - 1} = \frac{1}{(e^{8\pi M \Omega} - 1)}. \quad (68)$$

This is the exponential suppression factor for the Planck radiation formula with a (Hawking) temperature

$$T_H = \frac{1}{8\pi M} \equiv \frac{\hbar c^3}{8\pi G M k_B} \quad (\text{in general units}). \quad (69)$$

The various approximations that have been made in this derivation and the restriction (introduced for simplicity) to spherically symmetric wavefunctions, leads to a numerator which differs from that in the formula for blackbody radiation. This means that the radiation spectrum is not strictly that of blackbody radiation. This correction is known as the “grey-body” effect. It does not, however, alter the temperature dependent term in (68).

We note that a black hole has *negative* thermal capacity since an increase in the energy, M , of the black hole leads to a decrease in its temperature.

The Hawking temperature is the temperature of radiation at large distances from the horizon. As one goes closer to the horizon, the radiation is blue-shifted by the gravitational field of the black hole and at a distance r the radiation temperature, $T(r)$, is

$$T(r) = T_H \left(1 - \frac{2M}{r}\right)^{-1/2}, \quad (70)$$

and tends to infinity as one approaches the horizon. One can think of this blue-shift in the language of the Unruh effect – it arises from the fact that a co-moving observer close to the horizon is equivalent to an observer who is accelerating in order to avoid falling across the horizon and this acceleration generates radiation at a temperature which is proportional to the rate of acceleration.

A black hole of $1 M_\odot$ has a Hawking temperature of 6×10^{-8} K – much colder than the microwave background (CMB) temperature so at present such black holes will continue to absorb photons from the (CMB) and continue to grow. A black hole with a temperature greater than the CMB would have a radius less than $\sim 70 \mu\text{m}$.

4.6 Black Hole Entropy

If we add an energy dM to a black hole, its entropy increases by

$$dS = \frac{dM}{T} = 8\pi M dM.$$

The total entropy of a black hole of mass M is therefore

$$S_{BH}(M) = 8\pi \int_0^M M' dM' = 4\pi M^2 \quad (71)$$

This is usually written in terms of the surface area, Σ , of the black hole.

$$\Sigma = 4\pi r_S^2,$$

where $r_S = 2M$ is the Schwarzschild radius of the black hole.

$$S = \frac{\Sigma}{4} = \frac{k_B c^3 \Sigma}{4G\hbar} \quad (\text{in general units}). \quad (72)$$

Jacob Bekenstein produced a qualitative derivation of this entropy, by arguing that when a particle falls into a black hole at least one bit of information (the answer to the question “does the particle exist?” is lost – or rather it is deposited on the horizon of the black hole, since quantum information is never lost. This increases the entropy of the black hole by

$$dS = \ln 2.$$

The smallest mass, dM , of a particle that can fall into a black hole must have a Compton radius which is smaller than the Schwarzschild radius of the black hole’s horizon:

$$\frac{1}{dM} \lesssim r_S \text{ or } dM \gtrsim \frac{1}{r_S}$$

and the corresponding increase in the Schwarzschild radius

$$dr_S \sim \frac{2}{r_S},$$

with a corresponding an increase in area

$$d\Sigma = 4\pi r_S dr_S \sim 8\pi$$

So the number of bits of information that can be deposited on a black hole with surface area Σ is $\sim \Sigma/8\pi$ corresponding to an entropy

$$S = \ln 2 \frac{\Sigma}{8\pi}$$

This is lower than the value from Hawking’s calculation, but gives the correct dependence on the surface area of the black hole and (almost) the right order of magnitude.

4.7 Black Hole Evaporation

A black hole is therefore a blackbody radiation cavity with temperature T_H . Such a cavity radiates at a rate given by Stefan’s law. The rate of radiation, P , from the horizon surface of the black hole is given by

$$P = \sigma T_H^4 \Sigma, \tag{73}$$

where

$$\sigma = \frac{\pi^2}{60},$$

is the Stefan-Boltzmann constant.

Once again, this is an approximation as the Stefan-Boltzmann constant refers to black-body radiation and so we are neglecting any grey-body effects here.

The surface area of the black hole horizon is $16\pi M^2$. In order to conserve energy, this radiated energy comes from a reduction of the energy of the black hole which therefore loses mass at a rate of

$$-\frac{dM}{dt} = P = \frac{\pi^2}{60} \left(\frac{1}{8\pi M} \right)^4 (16\pi M^2) = \frac{1}{15360\pi M^2} \quad (74)$$

with solution

$$M(t) = M(0) \left(1 - \frac{t}{t_L} \right)^{1/3}, \quad (75)$$

with

$$t_L = 5120\pi M(0)^3 = \frac{5120\pi G^2 M(0)^3}{\hbar c^4} \text{ (in general uniots)}$$

For a black hole with mass $\sim 1M_\odot$, t_L is of order 10^{67} years.